Complete characterization of the ground-space structure of two-body frustration-free Hamiltonians for qubits

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The problem of finding the ground state of a frustration-free Hamiltonian carrying only two-body interactions between qubits is known to be solvable in polynomial time. It is also shown recently that, for any such Hamiltonian, there is always a ground state that is a product of single- or two-qubit states. However, it remains unclear whether the whole ground space is of any succinct structure. Here, we give a complete characterization of the ground space of any two-body frustration-free Hamiltonian of qubits. Namely, it is a span of tree tensor network states of the same tree structure. This characterization allows us to show that the problem of determining the ground-state degeneracy is as hard as, but no harder than, its classical analog.

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I. INTRODUCTION

Quantum spin models are simplified physical models for real materials but are believed to capture some of their key physical properties, which lie in the heart of modern condensed matter theory [1]. Ground states of strongly correlated spin systems are usually highly entangled, even if the system Hamiltonian carries only local interactions. So in general, finding the ground state of such a system is intractable with traditional techniques, such as mean-field theory.

In practical spin systems, different local terms in the Hamiltonian might also compete with each other, a phenomenon called frustration, which makes the system even more difficult to analyze [2]. However, frustration is not a necessary factor to cause ground-state entanglement. Frustration-free Hamiltonians can carry lots of interesting physics, ranging from gapped spin chains [3] to topological orders [4,5].

During recent years, the active frontier of quantum information science has brought new tools for study of quantum spin systems. In particular, local Hamiltonian problems are shown to be, in general, very hard, i.e., QMA-complete [6]. It is also realized that the study of k-local frustration-free Hamiltonians for qubits is closely related to the quantum k-satisfiability problem (Q-k-SAT) [7], which is the quantum analogy of the classical k-satisfiability (k-SAT), a problem that is of fundamental importance and has been extensively studied in theoretical computer science (see, e.g., [8]).

Spin models with two-body interactions are of the most physical relevance, as two-body interactions—in particular, of nearest neighbors or next nearest neighbors on certain types of lattices—are the strongest interaction terms in the real system Hamiltonian. Because two-level systems are the most common in nature, spin-1/2 (qubit) systems are of particular importance.

It is realized, however, that certain ground states of a two-body frustration-free (2BFF) Hamiltonian of qubits could be pretty trivial, with almost no entanglement at all. Algorithmically, the problem of finding the ground state of a 2BFF Hamiltonian of qubits is known to be solvable in polynomial time on a classical computer [7]. It has also been shown recently that for any such Hamiltonian, there is always a ground state that is a product of single- or two-qubit states; and if there is a genuine entangled ground state, the ground space must be degenerate [9]. There are also similar observations of the ground states in random or generic instances [10–13], saying that the entire ground space is of a trivial structure, which is almost always the fully symmetric space, with ground-space degeneracy $n+1$, where n is the number of qubits [10,11,14,15].

The main purpose of this work is to characterize the entire ground space in the most general setting. We improve the understanding of the ground space of 2BFF Hamiltonians of qubits by showing that it is always a span of tree tensor network states of the same tree structure. In other words, these states can be described as "being generated," from products of single qubit states, by applying the same series of isometries (from single qubit to two qubits). Tree-like networks of isometries have appeared before as a tool for constructing variational ground states for frustrated systems [16]; here we show that they provide exact ground states for 2BFF Hamiltonians.

As this characterization holds for the most general case, it implies that the problem of determining the ground-state degeneracy is as hard as, but no harder than, its classical analog. Putting this in the more formal language of computational complexity theory, our results give proof that computing the ground-space degeneracy of 2BFF Hamiltonian (#Q-2-SAT) is in a complexity class called #P [17]. On the other hand, the classical analog #2-SAT of #Q-2-SAT is #P-hard, therefore the #Q-2-SAT problem is #P-complete. This answers an open question raised in [11]. Therefore our results further bridge the studies in both quantum many-body physics and computer science.

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II. TWO-BODY FRUSTRATION-FREE HAMILTONIAN

Consider a system of n qubits labeled by the set $V = \{1,2,\ldots,n\}$. We are interested in 2BFF Hamiltonians $H = \sum H_J$ of the system. The Hamiltonian is called two-body if each term H_J acts nontrivially only on two qubits. The index J indicates the two qubits on which H_J acts. The Hamiltonian H is called frustration-free if its ground state also minimizes the energy of each term H_J simultaneously. Without loss of generality, we can assume throughout the paper that the smallest eigenvalue of each term H_J is 0 by shifting the energy spectrum. In this convention, the frustration-free Hamiltonian H itself will have zero ground energy. Specifically, we have

$$\mathcal{K}(H) = \bigcap [\mathcal{K}(H_J) \otimes \mathcal{H}_{\bar{J}}], \tag{1}$$

where $\mathcal{K}(H)$ is the ground space of H and $\mathcal{H}_{\bar{J}}$ is the Hilbert space of the qubits not in J. From this equation, one easily sees that it is the ground space of each term H_J, not the structure of excited states, that matters for the ground space of a frustration-free Hamiltonian H. Denote the projection to the ground space of H_J by Π_J. Since the ground space of Π_J and that of H_J are the same, it suffices to consider local terms to be projections Π_J for our purposes.

Closely related to the analysis of 2BFF qubit Hamiltonians is the quantum 2-SAT problem (Q-2-SAT) first considered by Bravyi [7]. Naturally generalizing classical 2-SAT, the Q-2-SAT problem asks whether, for a given set of two-qubit projections $\{\Pi_J\}$ of an n-qubit system, there is a global state $|\Psi\rangle$ such that $\Pi_J|\Psi\rangle = 0$ for all J. Apparently, we answer "yes" to the problem if and only if the Hamiltonian $\sum \Pi_J$ is frustration-free. It is known that Q-2-SAT is decidable in polynomial time on a classical computer [7]. The proof of the statement actually constructs a specific n-qubit state $|\Psi\rangle$ in the ground space of $\sum \Pi_J$ if there is any. Our techniques are similar to those used by Bravyi, but we show the stronger result that one can not only find one state in the ground space, but also represent the entire ground space in terms of a span of special states.

A simple relation between local operations and frustration-free Hamiltonians turns out to be useful. Let L_j be a nonsingular local operator acting on the jth qubit. Note that L_j might not be a physical operation, which is introduced for the convenience of discussion. Define local operations $L = \bigotimes_{j=1}^{n} L_j$, $L_J = \bigotimes_{j\in J} L_j$. The action of L on $H = \sum H_J$ is defined to be the map from $H = \sum H_J$ to

$$H^L = \sum L_J^\dagger H_J L_J. \tag{2}$$

Notice that H^L is still a two-body Hamiltonian for qubits. The relation between the ground space of H and H^L is

$$L^{-1}\mathcal{K}(H) = \mathcal{K}(H^L). \tag{3}$$

To see this, first let $|\Psi\rangle \in L^{-1}\mathcal{K}(H)$. This is equivalent to $L|\Psi\rangle \in \mathcal{K}(H)$ and, by Eq. (1), to $H_J L|\Psi\rangle = 0$ for all J. The last condition holds if and only if $|\Psi\rangle \in \mathcal{K}(H^L)$ by the nonsingularity of L_j's.

III. THE GROUND-SPACE STRUCTURE AND THE HOMOGENEOUS CASE

Given the 2BFF Hamiltonian $H = \sum H_J$, what can we say about the ground-space $\mathcal{K}(H)$? As argued previously, we only need to consider Hamiltonians of the form $H = \sum \Pi_J$ where Π_J's are projections onto $\mathcal{K}(H_J)^\perp$. We start our analysis by considering the rank of the projections Π_J.

First, if there is a Π_J of rank 3, the only possible state for the two qubits in J is $I - \Pi_J$ of rank 1, and this reduces to a problem on qubits in $V \setminus J$.

If there is a Π_J of rank 2, the state of qubits in J is restricted to a two-dimensional subspace. Let $|\psi_0\rangle_{a,b}$ and $|\psi_1\rangle_{a,b}$ be two orthogonal states that span the subspace, where a,b are the two qubits in J. One can encode qubits a and b by a single qubit d. For this purpose, we define an isometry U in the following form $U : |0\rangle_d \mapsto |\psi_0\rangle_{a,b}$, $|1\rangle_d \mapsto |\psi_1\rangle_{a,b}$. One concern regarding this encoding is what will happen to other terms of the Hamiltonian that represent interactions between a qubit in J and one outside J. Consider, for example, a local term $\Pi_{a,c}$ between qubit a and qubit c. In this case, one can view $\Pi_{a,c}$ as a term acting on a,b,c, and after the encoding, it is again an interaction of two qubits, namely, c and d. More formally, for terms like $\Pi_{a,c}$, one can replace it with $U^\dagger(\Pi_{a,c} \otimes I_b)U$. This procedure produces a set of constraints on $n-1$ qubits. It is easy to verify that a state $|\Psi\rangle$ is in the ground space of the reduced problem if and only if $U|\Psi\rangle$ is in the ground space of the original problem [7,9].

When there is no projection of rank higher than 1, we are dealing with the homogeneous case [7]. It turns out that the homogeneous case is the hardest and we discuss it in two separate sections. As we will see, the ground space of the homogeneous Hamiltonian (more precisely, the simplified homogeneous Hamiltonian defined later) is spanned by single-qubit product states. The above case analysis gives an explicit representation of the ground space of a general 2BFF qubit Hamiltonian, which is given by the following.

Main observation. The ground space is always a span of tree tensor network states of the same tree structure.

We illustrate this observation in Fig. 1. In this figure, every (blue) triangle is an isometry operation, where the input (the qubit to the left of the triangle) encodes the output (the two qubits to the right of the triangle). By these isometries, the ground space of the original Hamiltonian can be outputted when the input of the whole forest is the ground space of the simplified homogeneous Hamiltonian, which is spanned by a set of product states. In the language of tensor network states [18,19], one can also represent these states in terms of tree tensor networks after combining the input product states and the roots of trees in the forest.

Consider the Hamiltonian $H = \sum \Pi_J$, where Π_J's are rank 1 projections. One can visualize the interactions in H

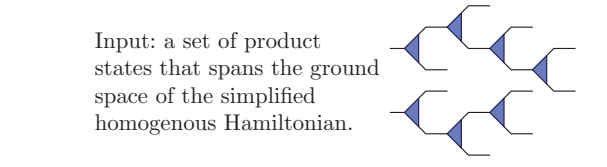

FIG. 1. (Color online) General structure of the ground space.

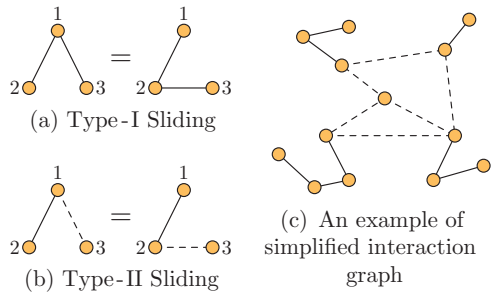

FIG. 2. (Color online) Simplification of the interaction graph.

by a graph G. The graph has n vertices corresponding to the qubits and two vertices are connected when there is a nontrivial interaction Π_J acting on them. We also distinguish two types of edges in the interaction graph. Let $\Pi = |\phi\rangle\langle\phi|$ be a projection. We use a solid edge in the graph when $|\phi\rangle$ is entangled and a dashed edge when $|\phi\rangle$ is a product state.

Given a general homogeneous Hamiltonian, the interaction graph will consist of both solid and dashed edges (see Fig. 2). The main technique is to simplify the interaction graph in hand without changing the ground space. Two sliding operations, as shown in Figs. 2(a) and 2(b), are used in the simplification. Type I sliding says that if we have entangled interactions between 1,2 and 1,3, we can change it to two entangled interactions between 1,2 and 2,3 without affecting the ground space. Type II sliding is of a similar spirit but involves both entangled and product interactions. We only prove the validity of type I sliding, as a similar argument holds for the type II sliding operation.

Let $\Pi_{12} = |\phi\rangle\langle\phi|$ and $\Pi_{13} = |\psi\rangle\langle\psi|$ be the two rank 1 operators acting on qubit 1,2 and 1,3. We find a local interaction Π_{23} acting on 2,3 such that $\Pi_{12} + \Pi_{23}$ has the same ground space as $\Pi_{12} + \Pi_{13}$. As $|\phi\rangle$ and $|\psi\rangle$ are entangled states, one can find local operations L_2 and L_3 acting on qubits 2 and 3, respectively, such that $|\phi\rangle = I_1 \otimes L_2|Y\rangle$ and $|\psi\rangle = I_1 \otimes L_3|Y\rangle$ where $|Y\rangle$ is the singlet state $(|01\rangle - |10\rangle)/\sqrt{2}$. The ground space of $\Pi_{12} + \Pi_{13}$ is therefore

$$\begin{aligned}
&\mathcal{K}(I \otimes L_2|Y\rangle\langle Y|_{12} I \otimes L_2^\dagger + I \otimes L_3|Y\rangle\langle Y|_{13} I \otimes L_3^\dagger) \\
&\quad = (L^\dagger)^{-1}\mathcal{K}(|Y\rangle\langle Y|_{12} + |Y\rangle\langle Y|_{13}) \\
&\quad = (L^\dagger)^{-1}\mathcal{K}(|Y\rangle\langle Y|_{12} + |Y\rangle\langle Y|_{23}) \\
&\quad = \mathcal{K}(\Pi_{12} + L_2 \otimes L_3|Y\rangle\langle Y|_{23} L_2^\dagger \otimes L_3^\dagger),
\end{aligned}$$

where the first equation uses Eq. (3), the second one is obtained by a direct calculation establishing that $\mathcal{K}(|Y\rangle\langle Y|_{12} + |Y\rangle\langle Y|_{13})$ is the symmetric subspace of the three qubits, and the last step employs Eq. (3) again. This validates the type I sliding operation.

Repeated applications of the two types of sliding operations can modify an arbitrary graph (a homogeneous Hamiltonian) with solid and dashed edges to the so-called simplified interaction graph (simplified homogeneous Hamiltonian). The simplified graph (denoted S) has a backbone (denoted B) of only dashed edges and several solid-edge tails attached to the backbone. An example of such a graph is shown in Fig. 2(c). This simplification can be done in two steps by first changing each connected component of solid edges into a tail and then sliding all dashed edges connected to a tail to one end of the tail. During the process of the sliding operations, it may happen that there is more than one edge between two vertices. If these multiple edges represent different constraints, one will essentially have a high-rank constraint and can deal with it using the isometry technique discussed in the previous section.

IV. THE SIMPLIFIED HOMOGENEOUS CASE

We now study the simplified homogeneous case. The interaction graph has a simple structure which intuitively suggests what the corresponding ground space may look like. We start by analyzing the backbone, where all the edges are dashed lines. We observe the following

Observation 1. For the interaction graph with dashed edges only, the ground space is spanned by orthogonal single-qubit product states.

To understand this observation, note that for an interaction graph with only dashed edges, it contains only product-state constraints $\Pi_{ij} = |\psi\rangle\langle\psi|$ with $|\psi\rangle = |\alpha_i\rangle \otimes |\alpha_j\rangle$. Note that if we consider the corresponding classical problem in this setting, all the constraints are product states of computational basis states $|0\rangle$ and $|1\rangle$. In this case the solution space is obviously spanned by product states in the computational basis. In the quantum case, however, the general product-state constraint $|\alpha_i\rangle \otimes |\alpha_j\rangle$ may not have $|\alpha_i\rangle$ and $|\alpha_j\rangle$ be the computational basis states.

However, given a single product-state constraint $|\alpha_i\rangle \otimes |\alpha_j\rangle$, one can always transform both $|\alpha_i\rangle$ and $|\alpha_j\rangle$ to some computational basis states by the local operation $L = L_i \otimes L_j$. Note that according to Eqs. (2) and (3), this local operation does not change the ground-space dimension and the product-state structure. Furthermore, if an n-vertex interaction graph with only dashed edges is one-dimensional (1D) in geometry (a straight line or a ring), then one can always find a local operation $L = \bigotimes_{i=1}^n L_i$ which transforms all the constraints simultaneously to product states of computational basis states. So a 1D backbone is essentially "classical."

It turns out that this simple 1D analysis is enough to reduce the n-particle problem to an $n-1$ particle problem, which leads to a proof by induction that the ground space of a Hamiltonian (denoted H_B) corresponding to an interaction graph with only dashed edges (the backbone) can be spanned by product states. Indeed, by properly choosing the invertible operators L_i, we can further show that the ground space can be spanned by orthogonal product states, with the intuitive idea that invertible operators can transform nonorthogonal states to orthogonal states.

Concretely, let us first examine several simple examples. The first example considers a chain of interactions as in Fig. 3(b). Let $|\alpha_j\rangle \otimes |\beta_j\rangle$ be the constraint on the jth edge. We will call it an alternating chain if $|\beta_{j-1}\rangle$ and $|\alpha_j\rangle$ are linearly independent for all j. It is easy to see that the solution space is $k+1$-dimensional for an alternating chain of k qubits. The second example, shown in Fig. 3(c), is called the alternating loop. As its name suggests, it is a loop where the two constraints on any vertex are linearly independent. Any alternating loop has solution space of dimension 2, namely, the span of $|00\cdots0\rangle$ and $|11\cdots1\rangle$ up to the local operation $L = \bigotimes_j L_j$ that maps $|\alpha_j\rangle$ and $|\beta_{j-1}\rangle$ to $|0\rangle$ and $|1\rangle$. The final example we consider is called the quasialternating loop.

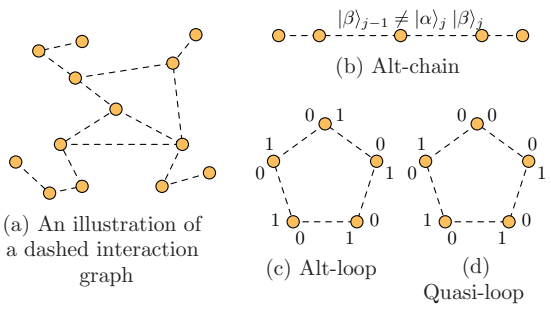

FIG. 3. (Color online) A dashed interaction graph, with three examples.

It is almost the same as the alternating loop except that there is one special vertex on the loop having the same constraint on the two edges adjacent to it. Figure 3(d) gives such an example where the top vertex is special. It is easy to see that the constraint on the special vertex of a quasialternating loop must be satisfied. In particular, for the loop in Fig. 3(d), the top vertex must be $|1\rangle$, as otherwise it will be impossible to satisfy all five constraints on the loop.

We now start the proof of Observation 1 by induction on n, the number of qubits. For $n = 1,2$, the observation is trivial. If there is a vertex a on which the constraints are the same up to global phases, let the constraints be $|0\rangle_a$, and more concretely, let the constraints on an edge that connects to a be of the form $|0\rangle_a|\alpha\rangle_b$ for some qubit b. We can write any state in the ground space as $|\Phi\rangle = |0\rangle_a|\Phi_0\rangle + |1\rangle_a|\Phi_1\rangle$. Obviously, $|\Phi_0\rangle$ and $|\Phi_1\rangle$ are both in the ground space of constraints not acting on a. Moreover, $|\Phi_0\rangle$ also needs to be orthogonal to $|\alpha\rangle_b$'s. By the induction hypothesis, both $|\Phi_0\rangle$ and $|\Phi_1\rangle$ are in a product span. Therefore, $|\Phi\rangle$ is also in a product span. On the other hand, if one cannot find any vertex whose constraints are the same, we can find either an alternating loop or a quasialternating loop in the graph. If a quasialternating loop is found, we know the state for the special vertex of the loop and can use the induction hypothesis on the remaining system. Otherwise, if an alternating loop is found, we can write any state in the ground space as

$$|\Phi\rangle = |00\cdots0\rangle|\Psi_0\rangle + |11\cdots1\rangle|\Psi_1\rangle, \tag{4}$$

up to local operations on the loop. If a constraint acts on two qubits on the loop, it can only restrict the loop to be exactly $|00\cdots0\rangle$ or $|11\cdots1\rangle$. The analysis is similar to the first case when a constraint $|\alpha\rangle_a|\beta\rangle_b$ acts on one qubit a on the loop and another qubit b outside of the loop. This completes the proof. Note that the local operations chosen here are determined by the constraints of alternating loops, and that one will never have two alternating loops giving different local operations for a single qubit; the orthogonality of the states up to local operations follows. We note that the orthogonality property only holds for the product constraints. The symmetric subspace, for example, is not a span of *orthogonal* product states up to local operations, although it is the span of $|00\rangle,|11\rangle,|{+}{+}\rangle$ where $|+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$.

We now move to discuss the entire simplified homogeneous case, with an interaction graph with both backbone and solid edge tails attached to the backbone. Based on Observation 1, we further have

Observation 2. For the simplified interaction graph, the ground space is spanned by single-qubit product states.

To understand this observation, denote the homogeneous Hamiltonian corresponding to a simplified interaction graph by H_S. Note that, given the orthogonal product states that span the solution space of H_B, ground states of H_S can be obtained in the following way. Recall that the ground space corresponding to each solid edge tail is essentially a symmetric space. Therefore, for any product state $\bigotimes_{i=1}^{|B|}|\alpha_i\rangle$ on the backbone, and the ith qubit connected to the solid tail Γ_i, the state $\bigotimes_{i=1}^{|B|}|\Gamma_i\rangle$ is a ground state of H_S, where $|\Gamma_i\rangle = |\alpha_i\rangle^{\otimes|\Gamma_i|}$. This is very straightforward, as the symmetric space associated with each solid edge tail simply enforces "copying" the state of the qubit in the backbone to the qubits on the attached tail.

More specifically, let $\mathcal{B}$ be the ground space of the dashed constraints in the backbone B, and $\mathcal{T}$ be the symmetric subspace confined by the tail of qubit set T, where $B \cap T$ has exactly one qubit a, through which the tail is attached to the backbone. We prove that $\mathcal{S} = \mathcal{T} \otimes \mathcal{H}_{B\setminus\{a\}} \cap \mathcal{B} \otimes \mathcal{H}_{T\setminus\{a\}}$ is again a product span. Write $\mathcal{B}$ as the direct sum

$$(\mathcal{B}_0 \otimes \mathcal{H}_a) \oplus \left(\bigoplus_{j=1}^{d} \mathcal{B}_j \otimes |\alpha_j^{\perp}\rangle_a\right),$$

where $|\alpha_j\rangle_a$'s are different dashed constraints on vertex a and d is the number of such $|\alpha_j\rangle_a$'s. For the basis of $\mathcal{B}_0$, all the constraints in the backbone are already satisfied, and therefore, qubit a can be any state. We say that qubit a is free in this case. For the basis of $\mathcal{B}_j$, qubit a has to be $|\alpha_j^{\perp}\rangle$ in order to satisfy all the constraints in the backbone. In this case, the state can only be extended to the tail by copying. In summary, $\mathcal{S}$ contains the space

$$(\mathcal{B}_0 \otimes \mathcal{T}) \oplus \left(\bigoplus_{j=1}^{d} \mathcal{B}_j \otimes |\alpha_j^{\perp}\rangle^{\otimes|T|}\right). \tag{5}$$

We need to show that this is actually everything in $\mathcal{S}$.

We first claim that the product basis for $\mathcal{B}_j$'s all together form a linearly independent set. By orthogonality (up to local operations), $\mathcal{B}_j$ and $\mathcal{B}_k$ are orthogonal if $|\alpha_j\rangle$ and $|\alpha_k\rangle$ are not. On the other hand, if $|\alpha_j\rangle$ and $|\alpha_k\rangle$ are orthogonal, the bases for $\mathcal{B}_j$ and $\mathcal{B}_k$ are linearly independent. Otherwise, we find a state $|\psi\rangle$ in both $\mathcal{B}_j$ and $\mathcal{B}_k$, meaning that $|\psi\rangle$ should be in $\mathcal{B}_0$, a contradiction. Now, for any state $|\Psi\rangle$ in $\mathcal{S}$, we can write it as $|\Psi\rangle = \sum_j |\Psi_j\rangle|\Phi_j\rangle$, where $|\Psi_j\rangle$'s are linearly independent product states spanning $\mathcal{B}$. Let $|\hat{\Psi}_j\rangle$ be the state on $B \setminus \{a\}$ when the state on B is $|\Psi_j\rangle$. One can also collect terms according to the state on $B \setminus \{a\}$, that is, $|\Psi\rangle = \sum_k |\hat{\Psi}_k\rangle \sum_l |\Psi\rangle_{k,l}^a|\Phi_{k,l}\rangle$. As shown previously, $|\hat{\Psi}_k\rangle$'s are linearly independent, and we know that $\sum_l |\Psi\rangle_{k,l}^a|\Phi_{k,l}\rangle$ is in T for each k. That is, state $|\psi\rangle$ is indeed in the space of Eq. (5). As the symmetric subspace can always be spanned by product states, we have finished the proof for the case of one tail. For multiple tails, the proof is essentially the same by an induction on the number of tails.

V. APPLICATION

One important application of our results it to analyze the computational complexity of the problem of counting

the ground-space degeneracy of 2BFF qubit Hamiltonians. Previously, this problem was considered to be hard due to the possible entanglement structure in the ground space [11]. Our complete characterization of the ground space shows that the ground space has a simple structure where entanglement plays no essential role. Therefore, determining the ground-state degeneracy is as hard as, but no harder than, its classical analog. In the more formal language of computational complexity theory, our results can lead to a proof that computing the degeneracy of a 2BFF Hamiltonian (#Q-2-SAT) is in a complexity class called #P [17]. Class #P contains functions f if there is a polynomial time algorithm A such that

$$f(x) = |\{y, A(x,y) \text{ accepts.}\}|,$$

where $|S|$ means the cardinality of set S, and y is usually called a proof to the verifier A.

In order to count the ground-state degeneracy of a 2BFF Hamiltonian, we should note the following two facts. First, as indicated by the ground-space structure in Fig. 1, the isometries will not change the dimension, so we only need to consider the simplified homogeneous case. Second, actually if one replaces the solid edges of the tails to be dashed edges forming alternating chains, the dimension of the ground space does not change either, as long as we choose the constraint of the tail on the vertex connecting to the backbone to be different from all other constraints $|\alpha_j\rangle$ of that vertex. To understand this, we need to review the extension of the product span with an intersection of symmetric subspaces. If the vertex in the intersection is free, we will have the whole symmetric subspace on the tail which is of dimension $k+1$, where k is the number of qubits in the tail. This coincides with the dimension of the alternating chain. If the vertex in the intersection is not free, we will have a unique extension in the tail, which again coincides with the case of an alternating chain.

It therefore suffices to count the dimension of any dashed graph. To show that it is in #P, one can choose the proof to the verifier to be the nondeterministic 0,1 choices in the cases of (i) an all-the-same-constraint vertex and (ii) an alternating loop.

On the other hand, the classical analog #2-SAT of #Q-2-SAT is #P-hard, therefore the #Q-2-SAT problem is #P-complete.

VI. SUMMARY

In this paper, we give a complete characterization of the ground space of 2BFF qubit Hamiltonians. The entire ground space is shown to be a span of tree tensor network states. This proof contains two major steps. First, we reduce the problem to the homogeneous case by isometries. Second, we reduce the homogeneous case to the simplified homogeneous case by two types of sliding operations which do not change the ground space, giving a much more intuitive picture to analyze the problem.

Our results sharpen the understanding of quantum spin-1/2 systems and, hopefully, provide ideas for the further research of quantum many-body systems, as well as further bridges between the studies in quantum manybody physics and computer science.

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