SUMS OF CERTAIN PRODUCTS OF FIBONACCI AND LUCAS NUMBERS-PART III

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ABSTRACT. For the Fibonacci numbers, the summation formula

$$\sum_{k=1}^{n} F_k^2 = F_n F_{n+1}$$

is well-known. Its charm lies in the fact that the right side is a product of terms from the Fibonacci sequence. In the earlier paper [5], the author presents similar formulas where, in each case, the right side consists of arbitrarily long products of an *even* number of distinct terms from the Fibonacci sequence. The formulas in question contain several parameters, and this contributes to their generality.

In this paper, we provide additional results of a similar nature where the right side consists of arbitrarily long products of an *odd* number of distinct terms from the Fibonacci sequence. Most of the results that we present apply to a sequence that generalizes both the Fibonacci and Lucas numbers.

1. INTRODUCTION

We begin by defining the integer sequences that occur in this paper. Let $a \geq 0$ and $b \geq 0$ be integers, with $(a, b) \neq (0, 0)$. Define, for all integers n, the sequences $\{H_n\}$ and $\{\overline{H}_n\}$ by

$$H_n = H_{n-1} + H_{n-2},\ H_0 = a,\ H_1 = b, \tag{1.1}$$

and

$$\overline{H}_n = H_{n-1} + H_{n+1}.$$

It is an easy exercise to show that

$$\overline{\overline{H}}_n = 5H_n, \tag{1.2}$$

a task that we leave to the interested reader. For $(a, b) = (0, 1)$, we have $\{H_n\}$=$\{F_n\}$, and $\{\overline{H}_n\}$=$\{L_n\}$, which are the Fibonacci and Lucas sequences, respectively.

Let α and β denote the two distinct real roots of $x^2 - x - 1 = 0$. Set $A = b - a\beta$ and $B = b - a\alpha$. Then the Binet forms for $\{H_n\}$ and $\{\overline{H}_n\}$ are, respectively,

$$H_n = \frac{A\alpha^n - B\beta^n}{\alpha - \beta}, \tag{1.3}$$

and

$$\overline{H}_n = A\alpha^n + B\beta^n. \tag{1.4}$$

The Binet forms for the Fibonacci and Lucas numbers are obtained from (1.3) and (1.4), respectively, by setting $A = B = 1$.

The motivation for this paper comes from our earlier paper [5], in which we present eight theorems giving formulas for sums of products of Fibonacci and Lucas numbers. For instance, in [5], formulas (2.1) and (2.3) are, respectively,

$$\sum_{k=1}^{n} F_{sk}F_{s(k+1)}\cdots F_{s(k+4m)}L_{s(k+2m)} = \frac{F_{sn}F_{s(n+1)}\cdots F_{s(n+4m+1)}}{F_{s(2m+1)}}, \tag{1.5}$$

$$\sum_{k=1}^{n} L_{sk}L_{s(k+1)}\cdots L_{s(k+4m)}F_{s(k+2m)} = \left[\frac{L_{sk}L_{s(k+1)}\cdots L_{s(k+4m+1)}}{5F_{s(2m+1)}}\right]_0^n, \tag{1.6}$$

where $s > 0$ is an even integer, and $m \geq 0$ is an integer. For convenience, in (1.6) we use familiar notation in which k is taken to be the dummy variable. For instance, $\left[F_{s(k+2m)}\right]_0^n$ is to be interpreted as the difference $F_{s(n+2m)} - F_{s(0+2m)}$.

As an instance of (1.5), take $s = 2$ and $m = 3$. Then (1.5) becomes

$$\sum_{k=1}^{n} F_{2k}F_{2(k+1)}\cdots F_{2(k+12)}L_{2(k+6)} = \frac{1}{377}F_{2n}F_{2(n+1)}\cdots F_{2(n+13)}. \tag{1.7}$$

Notice that the product on the right of (1.7) consists of an *even* number of distinct terms from the Fibonacci sequence. Indeed, in each of the formulas in [5], the product on the right side consists of an even number of distinct terms from the Fibonacci/Lucas sequences. In this paper, we supplement our results in [5] by presenting similar results in which the right side is a product that consists of an *odd* number of distinct terms from one of the sequences defined earlier in this section. As for [4] and [5], our motivation in this paper is to present closed forms for finite sums of products of terms from Fibonacci/generalized Fibonacci sequences, where the closed forms involve a single product or a difference of two products.

When writing [5], and its precursor [4], we were unaware that our presentation could be streamlined with the use of the sequences $\{H_n\}$ and $\{\overline{H}_n\}$. Had we been aware of this, we would have expressed (1.5) and (1.6) more generally as

$$\sum_{k=1}^{n} H_{sk}H_{s(k+1)}\cdots H_{s(k+4m)}\overline{H}_{s(k+2m)} = \left[\frac{H_{sk}H_{s(k+1)}\cdots H_{s(k+4m+1)}}{F_{s(2m+1)}}\right]_0^n, \tag{1.8}$$

with the same conditions on the parameters s and m. In (1.8), $\{H_n\} = \{F_n\}$ yields (1.5), while the use of (1.2) shows that $\{H_n\} = \overline{F}_n = \{L_n\}$ yields (1.6). In this paper, where possible, we express our results in terms of the sequences $\{H_n\}$ and $\{\overline{H}_n\}$.

In Section 2, we present the three simplest results that we discovered in the course of our research. In fact, these results are special cases of results that appear in our earlier paper [7]. We present the three results in question here only to complete the overall picture. In Section 3, we present our main results in the form of five theorems, and in Section 4 we give a proof of one of these theorems. In Section 5, we indicate certain key identities that are required for the proofs of the remaining theorems in Section 3.

2. The Product on the Right Side Has One Distinct Term

In this section, we present separately the summation formulas in which the product on the right side involves only one distinct term. In this regard, for s an integer, we have

$$\sum_{k=1}^{n} H_{s(2k-1)} = \begin{cases} \frac{1}{5F_s}\left[\overline{H}_{2sk}\right]_0^n, & s \neq 0 \text{ and } s \text{ even;} \\ \frac{1}{L_s}\left[H_{2sk}\right]_0^n, & s \text{ odd,} \end{cases} \tag{2.1}$$

and

$$\sum_{k=1}^{n}(-1)^k H_{s(2k-1)} = \begin{cases} \frac{1}{L_s}\left[(-1)^k H_{2sk}\right]_0^n, & s \text{ even;} \\ \frac{1}{5F_s}\left[(-1)^k \overline{H}_{2sk}\right]_0^n, & s \text{ odd.} \end{cases} \tag{2.2}$$

Furthermore, for any integer $s \neq 0$ we have

$$\sum_{k=1}^{n} H_{s(2k-1)}\overline{H}_{s(2k-1)} = \frac{1}{F_{2s}}\left[H_{2sk}^2\right]_0^n. \tag{2.3}$$

The summands in (2.1)–(2.3) are special cases of summands that occur in the recent paper [7]. More specifically, we have the following:

- In the summand that occurs in Theorem 2.1 of [7], replace W by H and put $m = -s$ to obtain the summand in (2.1) above;
- In the summand that occurs in Theorem 2.2 of [7], replace W by H and put $m = -s$ to obtain the summand in (2.2) above;
- In the summand that occurs in Theorem 2.5 of [7], replace W by H and put $(s, k, m) = (2s, -s, -s)$ to obtain the summand in (2.3) above.

In fact, in [7] the summands in question contain more parameters, and involve sequences more general than $\{H_n\}$ and $\{\overline{H}_n\}$. Furthermore, the corresponding finite sums depend upon the parity of n. We include (2.1)–(2.3) in this paper to highlight their relationship with the theorems that we present in the next section, thus rendering our presentation more complete.

We have discovered an alternating counterpart for (2.3). However, the closed form for the associated finite sum cannot be expressed in same manner as the right sides of (2.1)–(2.3), and so we omit this result. For a more general alternating result, see Theorem 2.6 in [7].

3. The Product on the Right Side Has $2m+1$ Distinct Terms, $m \geq 1$

In this section, we present closed forms for finite sums where the right side involves a product of $2m+1$ distinct terms, $m \geq 1$. In the first three theorems, we express our results in terms of the sequences $\{H_n\}$ and $\{\overline{H}_n\}$. As such, these results can be specialized to both the Fibonacci and Lucas numbers.

Theorem 3.1. *Let $s \neq 0$ and $m \geq 1$ be integers. Then*

$$\sum_{k=1}^{n} H_{2sk}\cdots H_{2s(k+2m-1)}\overline{H}_{s(2k+2m-1)} = \left[\frac{H_{2sk}\cdots H_{2s(k+2m)}}{F_{s(2m+1)}}\right]_0^n, \quad s \text{ even},$$

$$\sum_{k=1}^{n} H_{2sk}\cdots H_{2s(k+2m-1)}H_{s(2k+2m-1)} = \left[\frac{H_{2sk}\cdots H_{2s(k+2m)}}{L_{s(2m+1)}}\right]_0^n, \quad s \text{ odd}.$$

In Theorem 3.1, let $(s, m) = (1, 1)$ and take $H_n = F_n$. Then we obtain

$$\sum_{k=1}^{n} F_{2k}F_{2k+1}F_{2k+2} = \frac{1}{4}F_{2n}F_{2n+2}F_{2n+4}.$$

Theorem 3.2. *Let $s \neq 0$ and $m \geq 1$ be integers. Then*

$$\sum_{k=1}^{n}(-1)^k H_{2sk}\cdots H_{2s(k+2m-1)}H_{s(2k+2m-1)} = \left[\frac{(-1)^k H_{2sk}\cdots H_{2s(k+2m)}}{L_{s(2m+1)}}\right]_0^n, \quad s \text{ even},$$

$$\sum_{k=1}^{n}(-1)^k H_{2sk}\cdots H_{2s(k+2m-1)}\overline{H}_{s(2k+2m-1)} = \left[\frac{(-1)^k H_{2sk}\cdots H_{2s(k+2m)}}{F_{s(2m+1)}}\right]_0^n, \quad s \text{ odd}.$$

In Theorem 3.2, let $(s, m) = (1, 1)$ and take $H_n = F_n$. We then have

$$\sum_{k=1}^{n}(-1)^k F_{2k}L_{2k+1}F_{2k+2} = \frac{(-1)^n}{2}F_{2n}F_{2n+2}F_{2n+4}.$$

The sum that we present in our next theorem is independent of the parity of s.

Theorem 3.3. *Let $s \neq 0$ and $m \geq 1$ be integers. Then*

$$\sum_{k=1}^{n} H_{2sk}^2 \cdots H_{2s(k+2m-1)}^2 H_{s(2k+2m-1)}\overline{H}_{s(2k+2m-1)} = \left[\frac{H_{2sk}^2 \cdots H_{2s(k+2m)}^2}{F_{2s(2m+1)}}\right]_0^n.$$

Setting $(s, m) = (1, 1)$ and taking $H_n = F_n$ in Theorem 3.3, we obtain

$$\sum_{k=1}^{n} F_{2k}^2 F_{2k+2}^2 F_{4k+2} = \frac{1}{8}F_{2n}^2 F_{2n+2}^2 F_{2n+4}^2.$$

For the two theorems that follow, s is an odd integer. We could find no analogous results where s is even and where the right side is a product of an odd number of distinct terms. Furthermore, the results in Theorems 3.4 and 3.5 cannot be combined into a single result that is expressible in terms of the sequences $\{H_n\}$ and $\{\overline{H}_n\}$.

Theorem 3.4. *Let s be an odd integer and $m \geq 1$ be an integer. Then*

$$\sum_{k=1}^{n}(-1)^k F_{sk}^2 \cdots F_{s(k+2m-1)}^2 F_{s(2k+2m-1)} = \frac{(-1)^n F_{sn}^2 \cdots F_{s(n+2m)}^2}{F_{s(2m+1)}}.$$

In Theorem 3.4, let $(s, m) = (1, 3)$. We then have

$$\sum_{k=1}^{n}(-1)^k F_k^2 F_{k+1}^2 F_{k+2}^2 F_{k+3}^2 F_{k+4}^2 F_{k+5}^2 F_{2k+5} = \frac{(-1)^n}{13}F_n^2 F_{n+1}^2 \cdots F_{n+6}^2.$$

Theorem 3.5. *Let s be an odd integer and $m \geq 1$ be an integer. Then*

$$\sum_{k=1}^{n}(-1)^k L_{sk}^2 \cdots L_{s(k+2m-1)}^2 F_{s(2k+2m-1)} = \left[\frac{(-1)^k L_{sk}^2 \cdots L_{s(k+2m)}^2}{5F_{s(2m+1)}}\right]_0^n.$$

Noting that s must be odd, in Theorem 3.5, let $(s, m) = (3, 2)$. We then have

$$\begin{aligned}&\sum_{k=1}^{n}(-1)^k L_{3k}^2 L_{3k+3}^2 L_{3k+6}^2 L_{3k+9}^2 F_{6k+9}\\ &= \frac{1}{3050}\left((-1)^n L_{3n}^2 L_{3n+3}^2 L_{3n+6}^2 L_{3n+9}^2 L_{3n+12}^2 - 12418350465024\right).\end{aligned}$$

4. A Sample Proof

In order to demonstrate a method of proof for all the theorems in Section 3, we now prove Theorem 3.3. To this end, we require identities (30)–(33) in [1], which, in the notation of the present paper are, respectively,

$$\begin{aligned}H_{n+k} + H_{n-k} &= H_n L_k, \quad k \text{ even},\\ H_{n+k} - H_{n-k} &= H_n L_k, \quad k \text{ odd},\\ H_{n+k} + H_{n-k} &= \overline{H}_n F_k, \quad k \text{ odd},\\ H_{n+k} - H_{n-k} &= \overline{H}_n F_k, \quad k \text{ even}.\end{aligned} \tag{4.1}$$

For the result in Theorem 3.3, let $l_n = l_n(s,m)$ denote the sum on the left side, and let $r_n = r_n(s,m)$ denote the right side. Keeping in mind that $m \geq 1$, and with the use of all four cases in (4.1), we see that

$$\begin{aligned}
r_{n+1} - r_n &= \frac{H^2_{2s(n+1)} \cdots H^2_{2s(n+2m)} \left(H^2_{2s(n+2m+1)} - H^2_{2sn}\right)}{F_{2s(2m+1)}}, \\
&= \frac{H^2_{2s(n+1)} \cdots H^2_{2s(n+2m)} \left(H_{2s(n+2m+1)} - H_{2sn}\right)\left(H_{2s(n+2m+1)} + H_{2sn}\right)}{F_{2s(2m+1)}}, \\
&= \frac{H^2_{2s(n+1)} \cdots H^2_{2s(n+2m)} H_{s(2n+2m+1)} \overline{H}_{s(2n+2m+1)} F_{s(2m+1)} L_{s(2m+1)}}{F_{2s(2m+1)}}, \\
&= H^2_{2s(n+1)} \cdots H^2_{2s(n+2m)} H_{s(2n+2m+1)} \overline{H}_{s(2n+2m+1)}, \\
&= l_{n+1} - l_n,
\end{aligned}$$

and this is true for $n \geq 1$.

In a similar manner, we have

$$\begin{aligned}
r_1 &= \frac{H^2_{2s(1)} \cdots H^2_{2s(2m)} \left(H^2_{2s(2m+1)} - H^2_{2s(0)}\right)}{F_{2s(2m+1)}}, \\
&= \frac{H^2_{2s(1)} \cdots H^2_{2s(2m)} \left(H_{2s(2m+1)} - H_{2s(0)}\right)\left(H_{2s(2m+1)} + H_{2s(0)}\right)}{F_{2s(2m+1)}}, \\
&= \frac{H^2_{2s(1)} \cdots H^2_{2s(2m)} H_{s(2m+1)} \overline{H}_{s(2m+1)} F_{s(2m+1)} L_{s(2m+1)}}{F_{2s(2m+1)}}, \\
&= H^2_{2s(1)} \cdots H^2_{2s(2m)} H_{s(2m+1)} \overline{H}_{s(2m+1)}, \\
&= l_1.
\end{aligned}$$

Since $r_1 = l_1$, and $r_{n+1} - r_n = l_{n+1} - l_n$ for $n \geq 1$, we see that $r_n = l_n$ for $n \geq 1$, which proves Theorem 3.3.

5. Concluding Comments

In the previous section, to assist in the proof of Theorem 3.3, we make use of each of the identities in (4.1). Likewise, to prove Theorems 3.1 and 3.2, we proceed similarly, making use of the identities in (4.1). Specifically, to prove the first sum in Theorem 3.1, we make use of the fourth identity in (4.1), and to prove the second sum in Theorem 3.1 we make use of the second identity in (4.1). Two prove the first sum in Theorem 3.2, we make use of the first identity in (4.1), and to prove the second sum in Theorem 3.2 we make use of the third identity in (4.1).

To prove Theorems 3.4 and 3.5, the identities that we require are, respectively,

$$F^2_{sn} + F^2_{s(n+2m+1)} = F_{s(2m+1)} F_{s(2n+2m+1)}, \tag{5.1}$$

$$L^2_{sn} + L^2_{s(n+2m+1)} = 5F_{s(2m+1)} F_{s(2n+2m+1)}, \tag{5.2}$$

where m and n are integers, and s is an odd integer.

Identity (5.1) is a consequence of the identity

$$F^2_{n+a_1}/F_{a_1-a_2} + F^2_{n+a_2}/F_{a_2-a_1} = F_{2n+a_1+a_2}, \tag{5.3}$$

in which $a_1 \neq a_2$ are integers. Identity (5.3) can be found in [2]. In (5.3), replace (n, a_1, a_2) by $(sn, 0, s(2m+1))$ under the assumption that s is odd, to obtain (5.1). Identity (5.2) is the *dual* identity of (5.1) with respect to the variable n. For an explanation of the concept of a dual identity, see Dresel's original article [3]. Alternatively, for readers without access to [3], an abbreviated account of Dresel's main ideas, including the concept of a dual identity, is given in [6].

Finally, we mention that in the recent paper [8], Treeby derives certain special cases of some of the theorems that occur in [5]. The interested reader may wish to explore whether Treeby's method can be used to derive special cases of some of our theorems in Section 3 above.

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