Analytical solution to one-dimensional consolidation in unsaturated soil due to time-dependent exponential temperature and external step loading

Solution analytique à la consolidation unidimensionnelle dans un sol non saturé en raison de la température exponentielle dépendant du temps et de la charge de l'étape externe

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ABSTRACT: Recent experimental studies demonstrate that temperature changes may significantly influence the deformation of unsaturated soils. Thus, there is an essential need to develop a predictive framework for the unsaturated consolidation capturing non-isothermal effects. This paper introduces analytical solutions to predict the one-dimensional (1D) consolidation of unsaturated soil deposit while incorporating the time-dependent exponential temperature variation. The one-way drainage boundary system and the uniform initial condition are adopted for the mathematical derivation. In this study, governing equations under the non-isothermal condition are first obtained. Then, Fourier sine series and the Laplace transform technique are used to solve these governing equations and obtain the final solutions. This study highlights the combined effects of time-dependent exponential temperature and an external step loading on the excess pore pressures at various depths. It is predicted that the effects of exponential temperature on the dissipation process would be much attenuated at a lower depth.

RÉSUMÉ: Des études expérimentales récentes démontrent que les changements de température peuvent influencer de façon significative la déformation des sols non saturés. Il existe donc un besoin essentiel de développer un cadre prédictif pour la consolidation non saturée capturant des effets non isothermes. Cet article présente des solutions analytiques pour prédire la consolidation unidimensionnelle (1D) du dépôt de sol non saturé tout en incorporant la variation de température exponentielle dépendante du temps. Le système de limite de drainage unidirectionnel et l'état initial uniforme sont adoptés pour la dérivation mathématique. Dans cette étude, les équations gouvernantes sous l'état non isotherme sont d'abord obtenues. Ensuite, la série de Fourier sine et la technique de transformée de Laplace sont utilisées pour résoudre ces équations et obtenir les solutions finales. Cette étude met en évidence les effets combinés de la température exponentielle dépendant du temps et d'une charge de marche externe sur les pressions de pores en excès à diverses profondeurs. On prévoit que les effets de la température exponentielle sur le processus de dissipation seraient très atténués à une profondeur inférieure.

KEYWORDS: 1D consolidation, unsaturated soil deposit, analytical solution, excess pore pressures, settlement.

1. INTRODUCTION

Consolidation of a natural soil deposit has been a primary geotechnical interest for several decades. This phenomenon involves the gradual dissipation of excess pore pressures from the void spaces due to an external applied load and eventually the soil volume would reduce considerably. Unsaturated soil foundation is mostly found in arid and semi-arid climatic regions, where attract large population and significant civil developments. It should be noted that a typical unsaturated soil mainly consists of soil skeleton (solid phase), water (liquid phase) and air (gaseous phase), thus, the compression process would involve simultaneous flows of pore-air and pore-water. The consolidation-related problems in unsaturated soils are usually considered to be nonlinear and require cumbersome evaluations to predict. Likewise, the inclusion of pore-air pressure has brought about a great challenge in estimating the consolidation due to complex stress state variables. These theoretical shortcomings result in a greater need for comprehensive models to evaluate the consolidation characteristics of an unsaturated soil deposit near the ground surface.

Several profound studies on unsaturated consolidation problems were initiated in early 1960s (e.g. Scott 1963). The research of interest continued to grow progressively until it reached its culmination in late 1970s when Fredlund and Hasan (1979) introduced a set of nonlinear inhomogeneous partial differential equations (PDEs) for the 1D consolidation describing the coupled flows of air and water phases. Dakshanamurthy and Fredlund (1981) later developed the continuity equation of air flow by capturing the thermal gradients in an unsaturated soil element and finally obtained the modified 1D consolidation equations. This novel concept has rendered a new framework for unsaturated consolidation problems under the non-isothermal condition.

The past two decades have witnessed notable investments on the consolidation theory of unsaturated soils. This has been evidenced by a significant increase in both analytical and numerical methods to predict the consolidation behaviour of unsaturated soils (Ho et al. 2015; Ho and Fatahi 2015; Ho and Fatahi 2016). Among original analytical research, Qin et al. (2008) adopted the Laplace transform and Cayley-Hamilton method to obtain a final solution while assuming all soil properties to be constant during the consolidation process. Shan et al. (2012) and Zhou et al. (2014), on the other hand, converted the inhomogeneous governing flow equations into the traditional homogeneous forms and then proposed solutions using the separation of variables technique. The above mentioned solutions and numerous others, however, only predict the consolidation behaviour under the isothermal condition. This theoretical drawback may reduce the reliability of models since the temperature change has been a crucial factor characterising the consolidation, in addition to the external loading (Alsherif and McCartney 2015).

In this paper, analytical solutions are presented to predict the 1D consolidation of unsaturated soil deposit induced by the time-dependent exponential temperature variation and an external step loading. The governing flow equations that capture the temperature change are adopted from Dakshanamurthy and Fredlund (1981). Fourier sine series and Laplace transformation techniques are employed to obtain final solutions. The combined effects of the exponential temperature and external step loading on the excess pore pressure dissipation rates and soil deformation will be discussed hereafter.

2. ANALYTICAL SOLUTION

Unsaturated soils are usually complex in nature and lack homogeneity in particle sizes, making it more difficult in predicting the consolidation characteristics. To achieve the closed-form analytical solution, some conventional assumptions are made as follows:

- The soil stratum is homogeneous;
- The soil skeleton and pore-water are incompressible;
- The air and water flows are continuous and independent;
- The air flow satisfies Fick's law whereas the water flow follows Darcy's law;
- Air diffusion through water is neglected;
- Soil deformation only occurs in the vertical direction (z-direction); and
- Soil properties (e.g. permeability coefficients, degree of consolidation, porosity etc.) and consolidation coefficients with respect to the air phase and water phase are assumed to be constant during the consolidation process.

The assumption of constant soil properties during the consolidation may not be strictly accurate for some applications. Particularly, under an external load, degree of saturation and porosity would vary during consolidation while the permeability coefficients for air and water phases could be expressed as functions of degree of saturation. In many existing literature, these properties are assumed to remain unchanged throughout the consolidation to alleviate the difficulty in obtaining the final solution. Furthermore, it may be acceptable to assume the properties to be constant during a transient compression process for a particular stress range.

2.1. *Governing equations capturing non-isothermal condition*

Dakshanamurthy and Fredlund (1981) proposed a set of governing equations describing the coupled flows of pore-air and pore-water in an unsaturated soil element as follows:

$$\begin{cases} u_{a,t} + C_a u_{w,t} + C_\Theta \Theta_{,t} + c_v^a u_{a,zz} = 0 \\ u_{w,t} + C_w u_{a,t} + c_v^w u_{w,zz} = 0 \end{cases} \tag{1}$$

where u_a and u_w are the excess pore-air and pore-water pressures; $u_{a,t}$ and $u_{w,t}$ are the first order of PDEs of excess pore-air and pore-water pressures with respect to time, respectively; $u_{a,zz}$ and $u_{w,zz}$ are the second order of PDEs of excess pore-air and pore-water pressures with respect to depth, respectively; and $\Theta_{,t}$is the first order of PDE of temperature. Additionally, the consolidation coefficients, as presented in Eq. 1, are expressed as:

$$C_a = \frac{1}{\left[\left(\frac{m_1^a}{m_2^a}-1\right)-\frac{n(1-S_r)}{m_2^a(u_a^0+u_{atm})}\right]}; \qquad C_\Theta = \frac{1}{\Theta_0\left[\frac{m_2^a\left(\frac{m_1^a}{m_2^a}-1\right)}{n(1-S_r)}-\frac{1}{u_a^0+u_{atm}}\right]};$$

$$c_v^a = \frac{k_a R \Theta_0}{gM}\frac{1}{\left[m_2^a(u_a^0+u_{atm})\left(\frac{m_1^a}{m_2^a}-1\right)-n(1-S_r)\right]};$$

$$C_w = \left(\frac{m_1^w}{m_2^w}-1\right); \text{ and } \qquad c_v^w = \frac{1}{m_2^w}\left(\frac{k_w}{\gamma_w}\right). \tag{2}$$

where m_1^a and m_1^w are the coefficients of air and water volume change with respect to the change of net stress (kPa^{-1}) , respectively; and m_2^a and m_2^w are the coefficients of air and water volume change with respect to the change of suction (kPa^{-1}), respectively; k_a and k_w are the permeability coefficients of air and water phases (m/s), respectively; g is the gravitational constant (9.81 m/s^2); u_a^0 is the initial pore-air pressure (kPa); u_{atm} is the atmospheric pressure (kPa); R is the universal air constant (8.31 $J/(mol.K)$); $\Theta_0 = \theta^\circ + 273$, is the absolute temperature (K); θ° is the average temperature of the soil profile (25°C); M is the molecular mass of air (i.e. 0.03 kg/mol); n is the porosity; S_r is the degree of saturation; and γ_w is the water unit weight (9.81 kN/m^3).

2.2. *Boundary and initial conditions*

Figure 1 illustrates a single soil stratum that consists of an infinite width and a measurable thickness, denoted H. Based on the conventional assumptions provided for the 1D consolidation theory, the air and water phases flow independently and continuously along the soil thickness. This study only considers the one-way drainage boundary system, in which the top surface of the soil stratum is permeable to air and water while the base is impermeable. The boundary condition can be mathematically presented as follows:

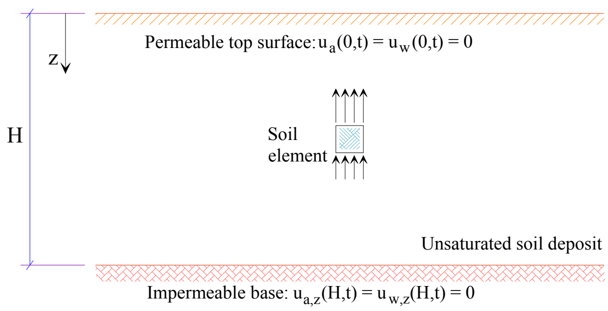


Figure 1. One-way drainage boundary system of unsaturated soil

$$\begin{cases} u_a(0,t) = u_w(0,t) = 0 \\ u_{a,z}(H,t) = u_{w,z}(H,t) = 0 \end{cases} \tag{3}$$

Assume that the instantaneous compression results in uniformly distributed excess pore-air and pore-water pressures along the domain $z \in (0, H)$. Thus, the initial condition is:

$$\begin{cases} u_a(z,0) = u_a^0 \\ u_w(z,0) = u_w^0 \end{cases}, \qquad z \in (0, H) \tag{4}$$

where u_a^0 and u_w^0 are initial excess pore-air and pore-water pressures.

2.3. *Excess pore pressures and consolidation settlement*

The general solutions for Eq. 1 can be obtained using Fourier sine functions as below:

$$\begin{cases} u_a(z,t) = \sum_{i=0}^{\infty} T_a(t)\ \sin(\mu_i z) \\ u_w(z,t) = \sum_{i=0}^{\infty} T_w(t)\ \sin(\mu_i z) \end{cases} \tag{5}$$

Based on the boundary condition provided in Eq. 3, the eigenvalue $\mu_i = (2i+1)\pi/(2H)$ $(i = 0, 1, 2, \ldots)$ is finally obtained. Also, the general thermal equation is introduced as functions of depth z and time t, giving:

$$\Theta(z,t) = \sum_{i=0}^{\infty} \vartheta^i(t)\ \sin(\mu_i z) \tag{6}$$

where $\vartheta^i(t) = \int_0^H \Theta(z,t)\sin(\mu_i z)dz \,/\, \int_0^H \sin^2(\mu_i z)dz$. (7)

Now, substituting Eqs. 5 – 7 into Eq. 1 would yield in:

$$\begin{cases} T_{a,t}(t) + C_a T_{w,t}(t) + C_\Theta \vartheta^i_{,t} - c_v^a(\mu_i)^2 T_a(t) = 0 \\ T_{w,t}(t) + C_w T_{a,t}(t) - c_v^w(\mu_i)^2 T_w(t) = 0 \end{cases} \tag{8}$$

Taking Laplace transform and then rearranging Eq. 8 would lead to the following matrix form:

$$T = \mathbf{M} + \mathbf{N} \tag{9}$$

where $T = \begin{Bmatrix} \overline{T}_a(s) \\ \overline{T}_w(s) \end{Bmatrix}$;

$$\mathbf{M} = \begin{Bmatrix} \frac{T_a(0)(C_wC_a-1)s + c_v^w[C_aT_w(0)+T_a(0)](\mu_i)^2}{(C_wC_a-1)s^2+(c_v^w+c_v^a)(\mu_i)^2 s - c_v^w c_v^a(\mu_i)^4} \\ \frac{T_w(0)(C_wC_a-1)s + c_v^a[C_wT_a(0)+T_w(0)](\mu_i)^2}{(C_wC_a-1)s^2+(c_v^w+c_v^a)(\mu_i)^2 s - c_v^w c_v^a\ (\mu_i)^4} \end{Bmatrix};\ \text{and}$$

$$\mathbf{N} = \begin{Bmatrix} \frac{C_\Theta[s - c_v^w(K)^2][s\bar{\vartheta}^i(s) - \vartheta^i(0)]}{(C_wC_a-1)s^2+(c_v^w+c_v^a)(\mu_i)^2 s - c_v^w c_v^a(\mu_i)^4} \\ -\frac{C_wC_\Theta[s\bar{\vartheta}^i(s)-\vartheta^i(0)]s}{(C_wC_a-1)s^2+(c_v^w+c_v^a)(\mu_i)^2 s - c_v^w c_v^a(\mu_i)^4} \end{Bmatrix}. \tag{10}$$

where $\overline{T}_a(s)$, $\overline{T}_w(s)$ and $\bar{\vartheta}^i(s)$ $(i = 0, 1, 2, \ldots)$ are Laplace transformed functions with the subjugate variable s. Referring to the initial condition presented in Eq. 4, the terms $T_a(0)$ and $T_w(0)$ can be expressed using the orthogonality of Fourier sine series:

$$\begin{cases} T_a(0) = \zeta_i u_a^0 \\ T_w(0) = \zeta_i u_w^0 \end{cases} \tag{11}$$

where $\zeta_i = 2/(\mu_i H)$. Besides, this study adopts the thermal equation that decreases linearly with depth and varies exponentially with time:

$$\Theta(z,t) = 273 + [\theta^\circ + A(1 - e^{-bt})]\left(1 - \xi\frac{z}{H}\right) \tag{12}$$

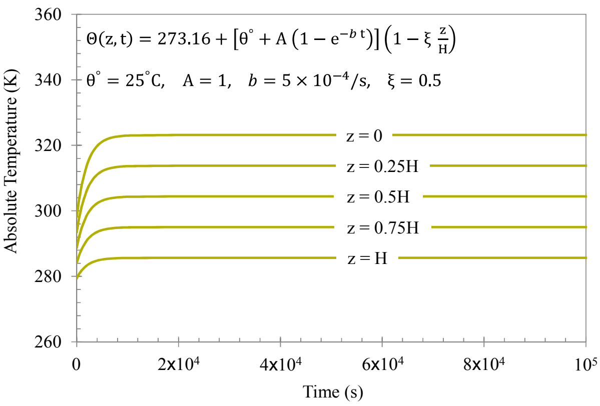


Figure 2. Time-dependent exponential temperature variation at different depths

where b is the thermal parameter presented in the exponential thermal equation (s^{-1}); A is the dimensionless parameter presented in the exponential thermal equation; ξ, which ranges from 0 to 1, is the gradient that controls the linear distribution of temperature throughout the soil profile. It should be noted that Eq. 12 presents a simplified simulation of temperature variation that may be only applicable to short-term laboratory investigations. The time-dependent exponential temperature variation at various depths is depicted in Figure 2.

Combining Eqs. 9 – 12 and then taking the Laplace inverse to obtain $T_a(t)$ and $T_w(t)$. The final solutions predicting excess pore-air and pore-water pressures in response to the exponential temperature change are presented as follows:

$$\begin{cases} u_a(z,t) = \sum_{i=0}^{\infty}\left\{\frac{\zeta_i[\Omega\ (e^{\alpha_1^i t} - e^{\alpha_2^i t}) + \Psi\ (e^{\alpha_1^i t} + e^{\alpha_2^i t})]}{2\eta} + \Xi_a^i\right\}\sin(\mu_i z) \\ u_w(z,t) = \sum_{i=0}^{\infty}\left\{\frac{\zeta_i[\Omega'\ (e^{\alpha_1^i t} - e^{\alpha_2^i t}) + \Psi'(e^{\alpha_1^i t} + e^{\alpha_2^i t})]}{2\eta} + \Xi_w^i\right\}\sin(\mu_i z) \end{cases} \tag{13}$$

where

$\eta = [(c_v^w - c_v^a)^2 + 4c_v^w c_v^a C_w C_a]^{\frac{1}{2}}$;

$\Omega = (c_v^a - c_v^w)u_a^0 - 2c_v^w C_a u_w^0$; $\Psi = \eta u_a^0$;

$\Omega' = (c_v^w - c_v^a)u_w^0 - 2c_v^a C_w u_a^0$; $\Psi' = \eta u_w^0$;

$$\Xi_a^i = \Upsilon^i\left\{\frac{e^{\alpha_1^i t}\ \beta_1^i}{(\alpha_1^i + b)(\alpha_1^i - \alpha_2^i)} + \frac{e^{\alpha_2^i t}\ \beta_2^i}{(\alpha_2^i + b)(\alpha_2^i - \alpha_1^i)} + \frac{e^{-bt}[(\beta_1^i - \alpha_1^i) - b]}{(\alpha_1^i + b)(\alpha_2^i + b)}\right\};$$

$$\Xi_w^i = \Upsilon'^i\left\{\frac{e^{\alpha_1^i t}\ \alpha_1^i}{(\alpha_1^i + b)(\alpha_2^i - \alpha_1^i)} + \frac{e^{\alpha_2^i t}\ \alpha_2^i}{(\alpha_2^i + b)(\alpha_1^i - \alpha_2^i)} + \frac{b e^{-bt}}{(\alpha_1^i + b)(\alpha_2^i + b)}\right\};$$

$\Upsilon^i = \frac{bAC_\Theta\iota^i}{C_wC_a - 1}$; $\Upsilon'^i = \frac{bAC_wC_\Theta\iota^i}{C_wC_a - 1}$;

$\alpha_1^i = \frac{1}{2}\left(\frac{c_v^w + c_v^a + \eta}{1 - C_wC_a}\right)(\mu_i)^2$; $\alpha_2^i = \frac{1}{2}\left(\frac{c_v^w + c_v^a - \eta}{1 - C_wC_a}\right)(\mu_i)^2$;

$\beta_1^i = \left[\frac{1}{2}\left(\frac{c_v^w + c_v^a + \eta}{1 - C_wC_a}\right) - c_v^w\right](\mu_i)^2$; $\beta_2^i = \left[\frac{1}{2}\left(\frac{c_v^w + c_v^a - \eta}{1 - C_wC_a}\right) - c_v^w\right](\mu_i)^2$;

$$\iota^i = \frac{2\ [\mu_i H - \xi\ (-1)^i]}{(\mu_i H)^2}. \tag{14}$$

Note that the terms $\Xi_a^i = \Xi_w^i = 0$ when the temperature becomes constant. The settlement of unsaturated soil layer can be determined by:

$$S(t) = (m_2^s - m_1^s)\left[\int_0^H u_a dz - Hu_a^0\right] - m_2^s\left[\int_0^H u_w dz - Hu_w^0\right] \tag{15}$$

Eq. 15 estimates the time-dependent settlement of unsaturated soil deposit under the exponential temperature variation. It is noteworthy that $m_1^s = m_1^a + m_1^w$ and $m_2^s = m_2^a + m_2^w$.

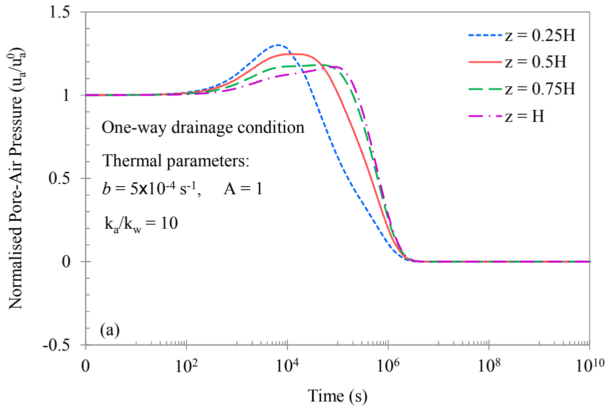
3. RESULTS AND DISCUSSION

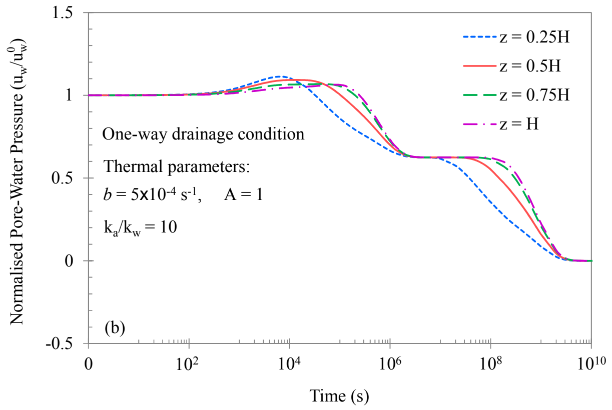
In this section, variations of the excess pore-air and pore-water pressures and the consolidation settlement due to the time-dependent exponential temperature and external step loading q are investigated. The adopted soil properties are as follows:

$m_1^s = -2.5 \times 10^{-4}\ \text{kPa}^{-1}$;	$m_2^s = 0.4m_1^s$;
$m_1^w = 0.2m_1^s$;	$m_2^w = 4m_1^w$;
$n = 0.50$;	$S_r = 80\%$;
$k_w = 10^{-10}\ \text{m/s}$;	$k_a/k_w = 10$;
$H = 10\text{m}$;	$R = 8.31\ \text{J/(mol. K)}$;
$M = 0.03\ \text{kg/mol}$;	$u_{atm} = 100\ \text{kPa}$;
$\Theta = (\theta^\circ + 273.16)\ \text{K}$;	$\theta^\circ = 25°\text{C}$;
$u_a^0 = 20\ \text{kPa}$;	$u_w^0 = 40\ \text{kPa}$;
$q = 100\ \text{kPa}$.	(16)

Soil properties provided in Eq. 16 are used to obtain consolidation coefficients for the air phase (C_a, C_Θ and c_v^a) and for the water phase (C_w and c_v^w). According to Fredlund et al. (2012), the constant load q (i.e. 100kPa) applied to the unsaturated ground surface would generate an initial excess pore-air pressure u_a^0 of 20kPa and an initial excess pore-water pressure u_w^0 of 40kPa.

Figures 3a and 3b present changes in normalised pore-air (u_a/u_a^0) and pore-water (u_w/u_w^0) pressures at various depths, respectively, due to the combined effects of time-dependent exponential temperature variation and external step loading q. It is observed that the exponential increase in temperature induces moderate increases in excess pore-air and pore-water





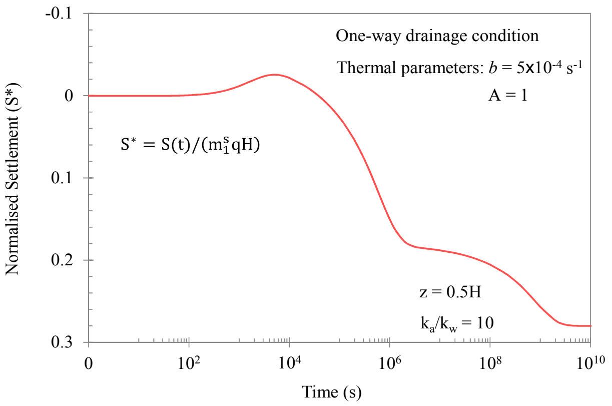


Figure 4. Settlement of unsaturated soil due to the time-dependent exponential temperature and external step loading

pressures at the early stages of consolidation. During these stages, it takes shorter time for excess pore pressures near the ground surface to attain the peak values as the temperature approaches the asymptote. It is also noteworthy that the peak values may reduce with depth. In other words, the effect of exponential temperature is much attenuated at lower depth. Since there are no temperature changes after about 10^4s, excess pore pressures would eventually dissipate similar to those curves under the isothermal condition.

Figure 4 shows the normalised settlement of unsaturated soil stratum induced by combined effects of temperature change and external step loading (i.e. $S^* = S(t)/(m_1^s qH)$). There is a slight increase in the settlement curve during the early stages of consolidation as the result of the exponential increase in temperature. At later stages, the soil settlement gradually increases resembling to that under the isothermal condition because the temperature reaches the asymptote and remains unchanged afterwards.

4. CONCLUSIONS

This paper presents an analytical solution to predict the 1D consolidation of unsaturated soil deposit induced by the time-dependent exponential temperature and external step loading. The governing equations of flow, incorporating the non-isothermal condition, were first introduced. The mathematical procedure adopted the one-way drainage system and uniform initial condition. Fourier sine series and Laplace transform technique were employed to obtain the closed-form analytical solutions.

For the graphical presentation, excess pore-air and pore-water pressure dissipation rates and consolidation settlement were investigated. It was predicted that both pressure curves increase at the first stages of consolidation process. Once the temperature remains constant, the excess pore pressures tend to dissipate gradually similar to those under the isothermal condition. In addition, the effect of exponential temperature is less obvious at lower depths.

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