Switching Cost Models as Hypothesis Tests*

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Abstract

We relate models based on costs of switching beliefs (e.g., due to inattention) to hypothesis tests. Specifically, for an inference problem with a penalty for mistakes and for switching the inferred value, a band of inaction is optimal. We show this band is equivalent to a confidence interval, and therefore to a two-sided hypothesis test.

Keywords: inference; switching cost; inferential expectations, hypothesis test.

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1 Introduction

This paper provides a new micro-foundation for two-sided hypothesis tests. Agents receive sequential information and conduct inference which penalizes adjustments to the estimator and deviations from the classical Bayesian estimate. We show that, to a first-order approximation for small adjustment costs, the resulting estimator has a band of inaction with width proportional to the

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Bayesian estimator's standard deviation. This makes it equivalent to a confidence interval and therefore to a two-sided hypothesis test.

Our result locates belief formation models based on hypothesis tests, such as Menzies and Zizzo's (2009) inferential expectations model, within a wider literature on switching costs due to sticky belief adjustment. Switching costs may arise, say, from 'menu costs', transactions in illiquid markets, cognitive effort in attention and observation or the consultation of experts (Caplin and Spulber, 1987; Alvarez et al., 2017; Magnani et al., 2016; Carroll, 2003).

State-dependent belief adjustments describe how new information about the underlying economic state X_t is incorporated. In our model, agents passively observe until new information exceeds a threshold, depending on the uncertainty of the estimated state, and only then readjust their policy. This infrequent adjustment is similar to models of inattention and portfolio choice (Abel et al., 2013; Huang and Liu, 2007).

A concrete example for our analysis is portfolio choice with partial information. Here, X_t represents the unknown expected returns which are estimated from time-series data. If $\hat{X}_t$ denotes the Bayesian estimate of X_t, the optimal portfolio then is typically of the form $h(\hat{X}_t)$. With transaction costs, this ideal portfolio cannot be implemented and instead has to be replaced by an approximation $h(\Theta_t)$, where Θ_t is an alternative estimate of X_t that only changes infrequently. The optimal Θ_t is in turn identified by our tradeoff between switching costs and inefficiency costs due to deviations from the optimal estimator.

Key to our approach is the use of asymptotic approximation methods, to allow closed-form solutions which are valid when costs are small. This typically yields an approximate 'no-action region', within which agents accept deviations from the no-cost optimum (Korn, 1998; Lo et al., 2004). Our specific contribution is to link switching cost models to hypothesis tests using the results of Altarovici et al. (2015). Their purpose is to describe trade within a financial market; we propose that their asymptotic approximation can also be applied to a wide class of recursive estimation problems.

2 The model

We base our setting on a Kalman–Bucy filter (Kalman and Bucy, 1961), as this has a wide variety of applications (see Bain and Crisan, 2009).

We write X for a hidden process, which we seek to estimate using observations Y. We suppose X and Y satisfy[1]

$$\begin{cases} dX_t = F_t X_t dt + \sqrt{Q_t} dW_t, & X_0 \sim N(\hat{X}_0, P_0), \\ dY_t = A_t X_t dt + \sqrt{R_t} dB_t, & Y_0 = 0, \end{cases}$$

[1]For simplicity, we assume here that X and Y are both scalar processes. The Kalman filter, and the results of Section 3, can also be obtained with multivariate X and Y, at a corresponding increase in notational complexity. In this case regions of inaction emerge, rather than intervals of inaction.

where W and B are independent Brownian motions. Here F, A, Q and R are deterministic functions, R and A are nonzero, and $(\hat{X}_0, P_0)$ are the mean and variance of our initial estimate of X_0. We write $\mathcal{F}_t$ for the information available[2] from observing Y up to time t.

For these dynamics Kalman–Bucy filtering shows that, conditional on our observations $\{Y_s\}_{0\leq s\leq t}$, the hidden state X_t has a normal distribution:

$$X_t|\mathcal{F}_t \sim N(\hat{X}_t, P_t).$$

The values of $(\hat{X}_t, P_t)$ have joint dynamics

$$\begin{cases} d\hat{X}_t = F_t\hat{X}_t dt + K_t d\hat{V}_t, \\ dP_t/dt = 2F_tP_t + Q_t - R_tK_t^2, \end{cases} \tag{1}$$

with initial values $(\hat{X}_0, P_0)$, where $K_t = P_tA_t/R_t$ denotes the *Kalman gain* process, and $d\hat{V}_t = dY_t - A_t\hat{X}_t dt$ defines the *innovations process* $\hat{V}$, which is a martingale under $\{\mathcal{F}_t\}_{t\geq 0}$, with quadratic variation $d\langle\hat{V}\rangle_t = R_t dt$. Observe that $\hat{X}_t$ is in general random, while P_t is a deterministic function of time.

Example 1 (Bayesian estimation of a constant average drift). A natural example of the Kalman–Bucy filter is when Y has a constant drift $X_t \equiv X_0$, which we estimate in a Bayesian manner. As $(Y_{t+h} - Y_t)/h \approx N(X_t, 1) = N(X_0, 1)$ for small h, this gives the continuous-time analogue of a Bayesian estimation problem for an unknown mean X_0 with normal errors, with prior $N(\hat{X}_0, P_0)$ leading to posterior $N(\hat{X}_t, P_t)$.

In this example, we have $F, Q \equiv 0$ and $A, R \equiv 1$. Then $X \equiv X_0$ is a (random) constant and $K_t = P_t$, so we obtain a closed-form solution to our filtering equations:

$$\begin{aligned} \frac{dP_t}{dt} &= -R_tK_t^2 = -P_t^2 && \Rightarrow && P_t = \frac{1}{1/P_0 + t}, \\ d\hat{X}_t &= K_t d\hat{V}_t = \frac{1}{1/P_0 + t} d\hat{V}_t && \Rightarrow && \hat{X}_t = w_t\frac{Y_t}{t} + (1 - w_t)\hat{X}_0 \end{aligned}$$

where $w_t = t/(t + 1/P_0)$ weights our estimate between the estimate from observations Y_t/t and the prior estimate $\hat{X}_0$. Observe that if $P_0 \approx \infty$ (i.e. we have a diffuse prior), then the posterior variance P_t collapses like $1/t$, as we would expect from a standard observation problem and $\hat{X}_t \approx Y_t/t$ is the classic ('frequentist') estimate for the drift of Y.

2.1 The cost of an estimator

We suppose that, over a fixed time period $[0, T]$, our agent estimates X_t with an approximation Θ_t of $\hat{X}_t$. She has initial wealth z, from which she continuously pays:

[2] Formally, $\mathcal{F}_t = \sigma(Y_s; s \leq t)$ is the filtration generated by Y.

- Monetary costs $\rho(\hat{X}_t - \Theta_t)$ due to tracking error relative to the optimal filter estimate. We assume ρ is convex, smooth and minimized at $\rho(0) = 0$.
- A cost λ whenever Θ_t changes.

For a differentiable utility function U, our agent wishes to optimize her utility of expected wealth

$$J(t, z, \Theta; \lambda) = E\Big[U(Z_T)\Big|\mathcal{F}_t\Big] = E\Big[U\Big(z - \int_t^T \rho\big(\hat{X}_t - \Theta_t\big) - \lambda \sum_{t \leq s \leq T} I_{\{\Delta\theta_s \neq 0\}}\Big)\Big|\mathcal{F}_t\Big] \tag{2}$$

over piecewise constant adapted processes Θ. As $\hat{X}$ is a Markov process, there exists a value function

$$v(t, \hat{X}_t, z, \Theta_t; \lambda) = \sup_{\Theta' : \Theta_t = \Theta'_t} J(t, z, \Theta'; \lambda).$$

Like in Korn (1998), Lo et al. (2004) and Altarovici et al. (2015), the value function can be expanded[3] in powers of λ. If λ is small, by ignoring higher order terms, we obtain an approximation to v, and hence to the optimal choice of Θ.

3 Dynamic programming

As is usual in dynamic decision making, we now seek to find an equation describing the value function v.

With fixed adjustment costs, it will be optimal to leave Θ unchanged until $\hat{X}_t - \Theta_t$ is sufficiently large. Write $\mathfrak{K}$ for the region where Θ remains fixed. A standard dynamic programming argument yields a partial differential equation for the value function $v(t, \hat{x}, z, \theta; \lambda)$. The optimal filter $\hat{X}$ without adjustment costs has the diffusive dynamics (1). We seek to find the optimal times to change the value of Θ. By the martingale principle of optimality[4], the value function evaluated along the state variables $(t, \hat{X}_t, Z_t, \Theta_t)$ is a martingale for the optimal Θ, and a supermartingale otherwise. Applying Itô's lemma, we can identify the drift of the random process $v(t, \hat{X}_t, z, \Theta_t; \lambda)$, which should never be positive, and is zero whenever it is not optimal to change Θ_t. In other words, writing

$$\Sigma_t = R_t K_t^2 = P_t^2 A_t^2 / R_t, \tag{3}$$

we have

$$0 \geq \partial_t v - (\partial_z v)\rho\big(\hat{x} - \theta\big) + (\partial_{\hat{x}} v)F\hat{x} + \frac{1}{2}\Sigma_t\, \partial_{\hat{x}\hat{x}} v, \tag{4}$$

with equality on $\mathfrak{K}$ (when it is not optimal to change θ).

[3] Corresponding verification theorems could be derived using stability results for viscosity solutions as in Altarovici et al. (2015) or martingale methods, cf. Feodoria (2016).

[4] This is a form of the dynamic programming principle for stochastic control problems.

Considering the possibility of changing θ, we know that our value function can never be lower than it would be after the optimal change in θ, so

$$v(t, z, \hat{x}, \theta; \lambda) \geq \sup_{\theta'} v(t, z - \lambda, \hat{x}, \theta'; \lambda), \tag{5}$$

with equality on the complement $\mathfrak{K}^c$ (when it is optimal to change θ). Combining these inequalities, we obtain the dynamic programming equation (or Bellman equation)

$$\begin{aligned} 0 = \min \Big\{ &- \partial_t v + (\partial_z v)\rho\big(\hat{x} - \theta\big) - (\partial_{\hat{x}} v)F\hat{x} - \frac{1}{2}\Sigma_t\, \partial_{\hat{x}\hat{x}} v, \\ & v(t, z, \hat{x}, \theta; \lambda) - \sup_{\theta'} v(t, z - \lambda, \hat{x}, \theta'; \lambda) \Big\}, \end{aligned} \tag{6}$$

with terminal value $v(T, z, \hat{x}, \theta; \lambda) = U(z)$. The difficulty is that the set $\mathfrak{K}$ needs to be determined as part of the solution (we have a 'free boundary' problem).

3.1 Asymptotic analysis

We will now analyze the behaviour of v when λ is small. In particular, we expand (4) and (5) in powers of λ, and by considering the first terms in our expansion, we are able to find an approximate closed-form solution.

When $\lambda = 0$, v is easy to calculate – one can use $\Theta_t = \hat{X}_t$ to achieve $v(t, z, \hat{x}; 0) \equiv U(z)$. We expect that the optimal strategy[5] will involve switching whenever $|\hat{x} - \theta| = O(\lambda^{1/4})$, resulting in a cost of $O(\lambda^{1/2})$. Writing

$$\xi := \lambda^{-1/4}(\hat{x} - \theta),$$

this suggests the ansatz

$$v(t, z, \hat{x}, \theta; \lambda) = U(z) - \lambda^{1/2}\phi(t, z) - \lambda\psi(t, z, \hat{x}, \xi) + o(\lambda) \tag{7}$$

where $\phi(T, z) = 0$ and $\inf_\xi \psi(t, z, \hat{x}, \xi) = \psi(t, z, \hat{x}, 0) = 0$ for all $(t, z, \hat{x})$. See Muhle-Karbe et al. (2017) for further discussion. Recalling our assumptions on ρ,

$$\rho(\hat{x} - \theta) = \rho(\lambda^{1/4}\xi) = \lambda^{1/2}\gamma\xi^2 + o(\lambda^{1/2}),$$

where

$$\gamma = \frac{\partial_{xx}\rho(0)}{2} > 0.$$

We now use this ansatz to expand our dynamic programming equation. First considering the behaviour when we do not choose to switch, we substitute the ansatz (7) into our drift condition (4), to obtain

$$0 \leq \lambda^{1/2}\Big(\partial_t\phi - \gamma\xi^2 U' + \frac{1}{2}\Sigma_t\, \partial_{\xi\xi}\psi\Big) + o(\lambda^{1/2}) \tag{8}$$

[5] This asymptotic behaviour comes from analyzing, over long horizons, how often the boundary of an interval will be hit by a random walk, averaging out the cost paid, then optimizing over the width of the interval chosen. This can be verified using the (mathematically rigorous) scaling arguments of Altarovici et al. (2015) and Lo et al. (2004), which also apply in our setting, mutatis mutandis.

with equality on $\mathfrak{K}$ (when switching is not optimal). Considering the possibility of switching, from our optimality condition (5) we have

$$
\begin{aligned}
0 &\le v(t,z,\hat{x},\theta;\lambda) - \sup_{\theta'} v(t,z-\lambda,\hat{x},\theta';\lambda) \\
&= U(z) - \lambda^{1/2}\phi(t,z) - \lambda\psi(t,z,\hat{x},\xi) \\
&\qquad - U(z-\lambda) + \lambda^{1/2}\phi(t,z-\lambda) + \inf_{\xi'} \psi(t,z-\lambda,\hat{x},\xi') + o(\lambda) \\
&= \lambda U'(z) - \lambda\psi(t,z,\hat{x},\xi) + o(\lambda)
\end{aligned}
\tag{9}
$$

with equality on $\mathfrak{K}^c$ (when switching is optimal).

Combining (8) and (9), the leading-order terms for small λ in each region in turn lead to the following approximate version of the dynamic programming equation (6):

$$
0 = \min\Big\{\partial_t\phi - U'(z)\gamma\xi^2 + \frac{1}{2}\Sigma_t\,\partial_{\xi\xi}\psi, \quad U'(z) - \psi(t,z,\hat{x},\xi)\Big\}. \tag{10}
$$

Following Atkinson and Wilmott (1995), we propose[6] a solution of the form

$$
\psi(t,z,\hat{x},\xi) = \begin{cases} U'(z)\Big(-1 + (M\xi^2-1)^2\Big) & \text{on } \mathfrak{K} = \{\xi : \xi^2 < 1/M\}, \\ -U'(z) & \text{on } \mathfrak{K}^c = \{\xi : \xi^2 \ge 1/M\}, \end{cases} \tag{11}
$$

where $M > 0$ is to be determined. On $\mathfrak{K}$ we have

$$
\partial_{\xi\xi}\psi = 4M(3M\xi^2 - 1)U'(z)
$$

so

$$
\begin{aligned}
0 &= \partial_t\phi - U'(z)\gamma\xi^2 + \frac{1}{2}\Sigma_t\big(4M(3M\xi^2-1)U'(z)\big) \\
&= \Big(\partial_t\phi - 2U'(z)\Sigma_t M\Big) + U'(z)\Big(6\Sigma_t M^2 - \gamma\Big)\xi^2.
\end{aligned}
\tag{12}
$$

This has to hold for all $\xi \in \mathfrak{K}$, so the coefficient of ξ^2 must equal zero. We solve for M and simplify using (3),

$$
M = \sqrt{\frac{\gamma}{6\Sigma_t}} = P_t^{-1}\sqrt{\frac{\gamma}{6A_t^2/R_t}}. \tag{13}
$$

This gives the approximately optimal no-switching region, and hence the following result:

$$
\mathfrak{K} = \Big\{|\xi| \le \Big(\frac{\Sigma_t}{\gamma/6}\Big)^{1/4}\Big\} = \Big\{|\hat{x} - \theta| \le \sqrt{P_t}\Big(\frac{A_t}{\sqrt{R_t}}\Big)^{1/2}\Big(\frac{6\lambda}{\gamma}\Big)^{1/4}\Big\}. \tag{14}
$$

[6] This is the smallest family of polynomials satisfying our assumptions which are smooth across the boundary.

From (12), substituting the value of M, we can also solve for ϕ:

$$\phi(t,z) = -U'(z)\sqrt{2\gamma/3}\int_t^T \sqrt{\Sigma_s}ds = -U'(z)\sqrt{2\gamma/3}\int_t^T P_s\frac{A_s}{\sqrt{R_s}}ds. \tag{15}$$

By construction, with this choice of ϕ and ψ, our ansatz (7) satisfies the approximate dynamic programming equation (10), and hence the original equation (6) up to an error of order $o(\lambda^{1/2})$.

To summarize our results, with the choices (15), (11), (13) and (14) for ϕ, ψ, M and $\mathfrak{K}$, we obtain the following result[7]:

Theorem 1. *At the leading order $O(\lambda^{1/2})$, for small adjustment costs λ, the estimator $\{\Theta_t\}_{t\geq 0}$ which maximizes the expected utility (2) is given by the rule:*

- *If*

$$|\hat{X}_t - \Theta_{t-}| \geq \sqrt{P_t}\Big(\frac{A_t}{\sqrt{R_t}}\Big)^{1/2}\Big(\frac{6\lambda}{\gamma}\Big)^{1/4}$$

  then set $\Theta_t = \hat{X}_t$.

- *Leave $\Theta_t = \Theta_{t-}$ otherwise.*

4 Interpretation as hypothesis testing

We now explore the connection with hypothesis testing. Consider testing the hypothesis

$$\mathrm{H}_0 : X_t = \Theta_{t-} \qquad \text{vs.} \qquad \mathrm{H}_1 : X_t \neq \Theta_{t-}$$

on the basis of observation of $\{Y_s\}_{s\leq t}$. Recall that the true state X_t is estimated by $\hat{X}_t$, with error variance P_t, so the standard two-sided hypothesis test rejects H_0 whenever Θ_{t-} lies outside the confidence interval

$$\Big(\hat{X}_t - c\sqrt{P_t}, \hat{X}_t + c\sqrt{P_t}\Big) \tag{16}$$

where c is the usual Gaussian critical value (for example, $c \approx 1.96$ for a 95% confidence level). We can therefore make the following connection with our optimal estimation problem with switching costs:

Theorem 2. *The approximately optimal choice of Θ given by Theorem 1 can be equivalently expressed as:*

- *If we reject the hypothesis $\mathrm{H}_0 : X_t = \Theta_{t-}$ (using the confidence interval approach) at a critical level c_t, then set $\Theta_t = \hat{X}_t$.*
- *Leave $\Theta_t = \Theta_{t-}$ otherwise.*

The critical level[8] is given by $c_t = (A_t/\sqrt{R_t})^{1/2}(6\lambda/\gamma)^{1/4}$.

[7]For clarity, we here write $\Theta_{t-} := \lim_{s\uparrow t}\Theta_s$ for the value of Θ before any change at t.

[8]Recall that $\gamma = \partial_{xx}\rho(0)/2$ relates to the cost of deviating from the Kalman–Bucy estimate.

Corollary 1. *If the infinitesimal signal-noise ratio $A_t/\sqrt{R_t}$ in our Kalman filter is constant, the optimal switching problem is equivalent to a hypothesis test with a constant confidence level determined by the ratio of cost coefficients λ/γ.*

Example 2. Suppose we are in the setting of Example 1, so $F = Q = 0$, $A = R = 1$ and $X_t = X_0$ is the (constant) unknown drift of our observations Y. For simplicity, take a diffuse prior, so $P_0 \approx \infty$. Recall that the Kalman filter estimate of X_t is $\hat{X}_t \approx Y_t/t$ and our estimation variance is $P_t \approx 1/t$. Using our result, the (approximately) optimal time to change Θ_t is when

$$\frac{|\Theta_t - Y_t/t|}{1/\sqrt{t}} > c$$

for $c = (6\lambda/\gamma)^{1/4}$. Here the left hand side is the usual two-sided test statistic for testing $\mathrm{H}_0 : X_t = \Theta_t$. In particular, we observe the $O(t^{-1/2})$ convergence of the width of the no-switching region, which agrees with the convergence of a confidence interval.

Remark 1. If the signal-noise ratio $A_t/\sqrt{R_t}$ is not constant, the critical value c_t in Theorem 2 will vary through time. Our results imply that in periods of lower-quality data the agent switches more frequently, or equivalently, uses a test with lower confidence level.

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