J. Group Theory, Ahead of Print
DOI 10.1515/jgth-2018-0042

On a theorem of Avez

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Communicated by George Willis

Abstract. For each symmetric, aperiodic probability measure μ on a finitely generated group G, we define a subset A_μ consisting of group elements g for which the limit of the ratio $\mu^{*n}(g)/\mu^{*n}(e)$ tends to 1. We prove that A_μ is a subgroup, is amenable, contains every finite normal subgroup, and $G = A_\mu$ if and only if G is amenable. For non-amenable groups we show that A_μ is not always a normal subgroup and can depend on the measure. We formulate some conjectures relating A_μ to the amenable radical.

1 Introduction

Let μ be a symmetric, aperiodic probability measure μ on a finitely generated group G whose support generates G. Let e denote the identity element of G, and let μ^{*n} denote the n-fold convolution of the measure, so that $\mu^{*n}(g)$ is the probability that an n-step random walk induced by μ starting at e ends at g. Avez [2] showed that when G is amenable,

$$\lim_{n\to\infty} \frac{\mu^{*n}(g)}{\mu^{*n}(e)} = 1 \quad \text{for all } g \in G.$$

In this paper, we extend Avez' result in the following way: For an arbitrary finitely generated group G, we consider the set, which we call A_μ, of all $g \in G$ for which the limit of the ratio $\mu^{*n}(g)/\mu^{*n}(e)$ tends to 1. Avez' result says that if G is amenable, then $A_\mu = G$. We prove that when G is non-amenable, A_μ is a proper, amenable subgroup. Moreover, A_μ contains every finite normal subgroup, so contains the elliptic radical (the largest normal, locally finite subgroup of G), and so is non-trivial in many cases. We compute A_μ for some examples and show that, in general, it is not a normal subgroup and may depend on the measure. We close by formulating some conjectures relating A_μ to the amenable radical.

This work is part of PhD work of the second author [13]; more details and applications can be found therein. Other relevant work that motivates the present paper includes [3, 7–10, 12, 14].

Research supported by Australian Research Council grant FT110100178.

2 Preliminaries

In this article, $\mathbb{Z}_+$ denotes the positive integers. Recall that a probability measure μ on a group G is *symmetric* if $\mu(x) = \mu(x^{-1})$ for all $x \in G$. The *support* of μ is the set $\{x \in G \mid \mu(x) > 0\}$, which we denote by $\operatorname{supp}(\mu)$. The *convolution* $\mu * \tau$ of two measures μ, τ on a discrete group is

$$\mu * \tau(y) = \sum_{x \in G} \mu(x)\tau(x^{-1}y).$$

The distribution of a n-step random walk induced by μ is the n-fold convolution power of μ, which we denote by μ^{*n}. The *period* of a measure μ is

$$\gcd\{n \in \mathbb{Z}_+ \mid \mu^{*n}(e) > 0\}.$$

The measure μ is said to be *aperiodic* if it has period 1. Note that for a symmetric measure, the period can only take the values 1 or 2.

A function $\zeta\colon G \to \mathbb{R}$ on a finitely generated group G is an ℓ^2*-function*, or $\zeta \in \ell^2(G)$, if $\sum_{g \in G} |\zeta(g)|^2$ is finite. The corresponding inner product is

$$\langle \zeta, \iota \rangle_2 = \sum_{g \in G} \zeta(g)\iota(g),$$

and the norm is $\|\zeta\|_2 = \sqrt{\langle \zeta, \zeta \rangle}$, as usual. The action of the group G on $\ell^2(G)$ defined by $g \cdot \zeta(x) = \zeta(g^{-1}x)$ for all $x \in G$ is called the *left regular representation* of the group.

Observe that

$$\begin{aligned} \mu^{*2n}(g) &= \sum_{x \in G} \mu^{*n}(x)\mu^{*n}(x^{-1}g) \\ &= \sum_{x \in G} \mu^{*n}(x)\mu^{*n}(g^{-1}x) \\ &= \sum_{x \in G} \mu^{*n}(x)(g \cdot \mu^{*n}(x)) \\ &= \langle \mu^{*n}, g \cdot \mu^{*n} \rangle, \end{aligned} \tag{2.1}$$

and so $\mu^{*2n}(e) = \langle \mu^{*n}, \mu^{*n} \rangle = \|\mu^{*n}\|_2^2$.

The notion of amenability has many characterizations. Here we use the following:

Theorem 2.1 ([5, 11]). *G is amenable if and only if there is a sequence f_n of probability measures on G such that $\|g \cdot f_n - f_n\|_2 \to 0$ for every $g \in G$.*

3 Defining A_μ

Definition 3.1. Let G be a finitely generated group, and let μ be a symmetric, aperiodic probability measure on G whose support generates G. We define

$$A_{G,\mu} = \left\{ g \in G \;\middle|\; \lim_{n\to\infty} \frac{\mu^{*n}(g)}{\mu^{*n}(e)} = 1 \right\}.$$

When it is understood which group is being used, the set will be referred to as A_μ.

The definition is clearly motivated by Avez' result: when G is amenable, we have $A_\mu = G$. A similar construction based on Theorem 2.1 would be the set of all $g \in G$ for which $\|g \cdot f_n - f_n\|_2$ tends to 0 with respect to some fixed sequence f_n of probability measures on G. An obvious choice for such a sequence would be $\xi_n = \frac{\mu^{*n}}{\|\mu^{*n}\|_2}$. It turns out that this construction coincides with A_μ.

Proposition 3.2. *Let G be a finitely generated group and μ a symmetric, aperiodic probability measure on G whose support generates G. Then*

$$A_\mu = \{g \in G \mid \|g \cdot \xi_n - \xi_n\|_2 \to 0\}.$$

Proof. By equation (2.1), we have

$$\frac{\mu^{*2n}(g)}{\mu^{*2n}(e)} = \frac{\langle \mu^{*n}, g \cdot \mu^{*n}\rangle}{\|\mu^{*n}\|_2^2} = \langle \xi_n, g \cdot \xi_n\rangle.$$

Observe that

$$\begin{aligned}
\|g \cdot \xi_n - \xi_n\|_2^2 &= \sum_{x\in G} (g \cdot \xi_n - \xi_n)^2(x) \\
&= \sum_{x\in G} (g \cdot \xi_n)^2(x) - 2\sum_{x\in G} (g \cdot \xi_n)(x)\xi_n(x) + \sum_{x\in G} (\xi_n)^2(x) \\
&= \|g \cdot \xi_n\|_2^2 - 2\langle g \cdot \xi_n, \xi_n\rangle + \|\xi_n\|_2^2 \\
&= 2 - 2\langle g \cdot \xi_n, \xi_n\rangle
\end{aligned}$$

since $\xi_n, g \cdot \xi_n$ are unit vectors. Thus $\|g \cdot \xi_n - \xi_n\|_2$ approaches 0 if and only if $\langle g \cdot \xi_n, \xi_n\rangle = \frac{\mu^{*2n}(g)}{\mu^{*2n}(e)}$ approaches 1. □

Corollary 3.3. *G is amenable if and only if $G = A_\mu$.*

Proof. This follows immediately from Theorem 2.1 and Proposition 3.2. □

The following observation will be useful.

Lemma 3.4. *Let μ be a symmetric, aperiodic probability measure on G whose support generates G. For any fixed $k \in \mathbb{Z}_+$, we have $A_\mu = A_{\mu^{*k}}$.*

4 Algebraic properties of A_μ

We now show that more than being some peculiar collection of elements, the sets A_μ have algebraic structure. Throughout this section, we consider G a finitely generated group and μ a symmetric, aperiodic probability measure on G whose support generates G, and $\xi_n = \frac{\mu^{*n}}{\|\mu^{*n}\|_2}$.

Theorem 4.1. *A_μ is a subgroup.*

Proof. Let $g, h \in G$. We have

$$\begin{aligned}\|gh \cdot \xi_n - \xi_n\|_2 &= \|g \cdot (h \cdot \xi_n - \xi_n) + g \cdot \xi_n - \xi_n\|_2 \\ &\leq \|g \cdot (h \cdot \xi_n - \xi_n)\|_2 + \|g \cdot \xi_n - \xi_n\|_2 \\ &= \|h \cdot \xi_n - \xi_n\|_2 + \|g \cdot \xi_n - \xi_n\|_2\end{aligned}$$

since the ℓ^2-norm is invariant under translation. Since $g, h \in A_\mu$ the right-hand side limits to 0, so $gh \in A_\mu$. Clearly, $e \in A_\mu$ and A_μ is closed under inverses since μ is symmetric. □

In [13], a slightly stronger statement is given, which gives some structural information about the cosets of A_μ.

Theorem 4.2. *A_μ is amenable.*

The idea of our proof is to give a sequence of probability measures on A_μ that are almost invariant under the action of A_μ. Proposition 3.2 says that we have such a sequence in $\ell^2(G)$, which we modify to obtain a sequence in $\ell^2(A_\mu)$.

Proof. Choose a set $I = \{s_1, s_2, \ldots\}$ of right coset representatives for A_μ, which is countable since G is finitely generated. For $n \in \mathbb{Z}_+, s \in I$, define $\phi_{n,s}\colon G \to \mathbb{R}$ by

$$\phi_{n,s}(x) = \begin{cases} \xi_n(x) & \text{if } x \in A_\mu s, \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\xi_n = \sum_{s \in I} \phi_{n,s}.$$

Since A_μ is a subgroup, translation by $k^{-1} \in A_\mu$ on the left preserves the right cosets. Hence

$$k \cdot \xi_n = \sum_{s \in I} k \cdot \phi_{n,s}.$$

We will now construct a sequence of unit vectors in $\ell^2(A_\mu)$ that are almost invariant. For $n \in \mathbb{Z}_+, s \in I$, define $\psi_{n,s}\colon A_\mu \to \mathbb{R}$ by $\psi_{n,s}(h) = \phi_{n,s}(hs) = \xi_n(hs)$, where $h \in A_\mu$. Then $\psi_{n,s} \in \ell^2(A_\mu)$ since $\sum_{h\in A_\mu} \psi_{n,s}(h) = \sum_{h\in A_\mu} \xi_n(hs)$ is finite. We also have that the norm of $\psi_{n,s}$ in $\ell^2(A_\mu)$ is equal to the norm of $\phi_{n,s}$ in $\ell^2(G)$. We denote this norm by $a_{n,s}$. Note $\sum_{s\in I}(a_{n,s})^2 = \|\xi_n\|_2^2 = 1$.

Putting all this together, we have

$$\begin{aligned}
\|k\cdot\xi_n - \xi_n\|_2^2 &= \Big\|\sum_{s\in I} k\cdot\phi_{n,s} - \sum_{s\in I}\phi_{n,s}\Big\|_2^2 \\
&= \Big\|\sum_{s\in I}(k\cdot\phi_{n,s} - \phi_{n,s})\Big\|_2^2 \\
&= \sum_{x\in G}\sum_{s\in I}[(k\cdot\phi_{n,s} - \phi_{n,s})(x)]^2 \\
&= \sum_{s\in I}\sum_{x\in G}[(k\cdot\phi_{n,s} - \phi_{n,s})(x)]^2 \\
&= \sum_{s\in I}\sum_{y\in A_\mu}[(k\cdot\phi_{n,s} - \phi_{n,s})(ys)]^2 \\
&\qquad\qquad (\text{since } \phi_{n,s} \text{ is zero outside the } s\text{-coset}) \\
&= \sum_{s\in I}(a_{n,s})^2 \sum_{y\in A_\mu}\left[\left(k\cdot\frac{\phi_{n,s}}{a_{n,s}} - \frac{\phi_{n,s}}{a_{n,s}}\right)(ys)\right]^2.
\end{aligned}$$

Now if, for all $s \in I$, we have

$$\sum_{y\in A_\mu}\left[\left(k\cdot\frac{\phi_{n,s}}{a_{n,s}} - \frac{\phi_{n,s}}{a_{n,s}}\right)(ys)\right]^2 \geq \epsilon,$$

then the above equation becomes

$$\|k\cdot\xi_n - \xi_n\|_2^2 \geq \epsilon\sum_{s\in I}(a_{n,s})^2 = \epsilon.$$

Therefore, $\|k\cdot\xi_n - \xi_n\|_2^2 < \epsilon$ implies there exists s such that

$$\sum_{y\in A_\mu}\left[\left(k\cdot\frac{\phi_{n,s}}{a_{n,s}} - \frac{\phi_{n,s}}{a_{n,s}}\right)(ys)\right]^2 < \epsilon.$$

Since $\|k\cdot\xi_n - \xi_n\|_2^2$ limits to zero for every $k \in A_\mu$, there exists a sequence s_n for which

$$\sum_{y\in A_\mu}\left[\left(k\cdot\frac{\phi_{n,s_n}}{a_{n,s_n}} - \frac{\phi_{n,s_n}}{a_{n,s_n}}\right)(ys)\right]^2 \to 0 \quad \text{for every } k \in A_\mu.$$

Rewriting in terms of corresponding functions in $\ell^2(A_\mu)$,

$$\sum_{y\in A_\mu}\left[\left(k\cdot\frac{\psi_{n,s_n}}{a_{n,s_n}}-\frac{\psi_{n,s_n}}{a_{n,s_n}}\right)(y)\right]^2=\left\|k\cdot\frac{\psi_{n,s_n}}{a_{n,s_n}}-\frac{\psi_{n,s_n}}{a_{n,s_n}}\right\|_2^2\to 0,$$

so $\frac{\psi_{n,s_n}}{a_{n,s_n}}$ supplies a sequence of almost invariant unit vectors in $\ell^2(A_\mu)$, and A_μ is amenable. □

That A_μ is an amenable subgroup does not preclude it being trivial for all non-amenable G, nor does it guarantee that A_μ reflects any of the underlying structure of G. The next result shows that in many cases, A_μ is an interesting non-trivial subgroup.

Recall that the *elliptic radical* of a finitely generated group G is the largest normal, locally finite subgroup of G (see, for example, [4]). It is the group generated by all finite normal subgroups of G and is contained in the amenable radical, the largest amenable normal subgroup. We now prove a result which implies that the elliptic radical is contained in A_μ.

Theorem 4.3. *A_μ contains every finite normal subgroup of G. In particular, the elliptic radical is contained in A_μ.*

Proof. Let F be a finite normal subgroup of G. Since the support of μ generates G, F is finite and μ is aperiodic, we have $\mu^{*|F|}(f)$ is non-zero for all $f\in F$. Setting $\kappa=\mu^{*|F|}$, we have $F\subseteq \operatorname{supp}(\kappa)$, and $A_\mu=A_\kappa$ by Lemma 3.4.

Let $S=\operatorname{supp}(\kappa)$. Then each walk $(g_0=e,g_1,\dots)$ induced by κ corresponds uniquely to a sequence $((h_0,f_0),(h_1,f_1),\dots)$, where $h_0=f_0=e$, $h_i\in\langle S\setminus F\rangle$, $f_i\in F$ and $g_i=h_if_i$ defined by the following process: if $g_n=g_{n-1}x$, $x\in\operatorname{supp}(\kappa)$, then

$$(h_n,f_n)=\begin{cases}(h_{n-1},f_{n-1}x), & x\in F,\\ (h_{n-1}x,x^{-1}f_{n-1}x), & x\in S\setminus F.\end{cases}$$

Define the measure $\phi\colon\langle S\setminus F\rangle\to\mathbb{R}$ by

$$\phi(x)=\begin{cases}\kappa(F), & x=e,\\ \kappa(x), & x\in S\setminus F.\end{cases}$$

Then ϕ^{*n} is the distribution of the first coordinate after n steps.

The process on the second coordinate is a Markov chain on the state space F, where each move corresponds either to a right multiplication by $x\in F$ or to a conjugation by some element of $x\in S\setminus F$, each with probability $\kappa(x)$. Let τ_n denote the distribution of the second coordinate after n steps. We will prove that τ_n approaches the uniform distribution of F using standard Markov chain theory (see for example [6] for further information).

Let $\Pr(f \to g)$ be the probability of moving from state f to g in one step of the Markov chain. If the step from f to g is a conjugation by x (i.e., $g = x^{-1} f x$), then $f = xgx^{-1}$, so $\Pr(f \to g) = \Pr(g \to f)$ since κ is symmetric. Otherwise, the step is induced by right multiplication, and clearly $\Pr(f \to g) = \Pr(g \to f)$ (since $g = fx$ only if $f = gx^{-1}$). It follows that the Markov chain satisfies the *detailed balance* condition for the uniform measure $\pi = \frac{1}{|F|}$, i.e.,

$$\pi(f) \Pr(f \to g) = \pi(g) \Pr(g \to f),$$

and so π is a stationary distribution on F, that is,

$$\pi(f) = \sum_{y \in F} \pi(y) \Pr(y \to f).$$

Since $e \in \operatorname{supp}(\kappa)$, the Markov process on F is *aperiodic*, and since F is finite, the process is *irreducible*. Then, by the fundamental theorem of Markov chains (see, for example, [6, Theorem 3.12]), τ_n converges to the unique stationary distribution π.

Now consider an n-step walk of the walk motivated by κ, which ends at some $f \in F$. We have

$$\kappa^{*n}(f) = \sum_{g \in F} \phi^{*n}(g).\tau_n(g^{-1} f)$$

since to end at f, we must have first coordinate g and second coordinate $g^{-1} f \in F$. Then

$$\begin{aligned}\lim_{n\to\infty} \frac{\kappa^{*n}(f)}{\kappa^{*n}(e)} &= \lim_{n\to\infty} \frac{\sum_{g\in F} \phi^{*n}(g)\tau_n(g^{-1}f)}{\sum_{g\in F} \phi^{*n}(g)\tau_n(g^{-1})} \\ &= \lim_{n\to\infty} \frac{\sum_{g\in F} \phi^{*n}(g)\pi(g^{-1}f)}{\sum_{g\in F} \phi^{*n}(g)\pi(g^{-1})} = 1\end{aligned}$$

since π is the uniform distribution on F, and so $F \subseteq A_\kappa = A_\mu$. □

Theorem 4.3 is notable for two reasons. Firstly, it shows that, whenever a finitely generated group contains a finite normal subgroup, A_μ is non-trivial. Secondly, this result is independent of μ.

5 Examples

Recall that non-abelian free groups have no non-trivial amenable normal subgroups. That is, the amenable radical is trivial.

Lemma 5.1. *Let F_d be the free group of rank $d \geq 2$ with free basis generators including a, b, and let μ be a symmetric, aperiodic measure whose support generates F_d satisfying $\mu(e) > 0$ and $\mu(a) = \mu(b) > 0$. Then $A_{F_d,\mu}$ is trivial.*

Proof. Let $u \in \{a^{\pm 1}, b^{\pm 1}\}^+$. If $u \in A_\mu$, then by interchanging $a^{\pm 1}$ with $b^{\pm 1}$, we obtain a word v that also lies in A_μ by symmetry of the measure with respect to the generators a, b. If u, v are not powers of the same element, in which case they generate a free group of rank 2, and since A_μ is an amenable subgroup, it must be trivial. Otherwise, if u, v generate a cyclic group, choose instead to replace $a^{\pm 1}$ by $b^{\mp 1}$. □

Lemma 5.2. *Suppose G, H are finitely generated groups with symmetric, aperiodic probability measures ϕ and ψ respectively whose supports generate G and H respectively. Recall that the product measure μ on $G \times H$ is defined by*

$$\mu(x, y) = \phi(x)\psi(y).$$

Then

$$A_{G\times H,\mu} = A_{G,\phi} \times A_{H,\psi}.$$

Proof. To prove this, we first note that $\mu^{*n}(x, y) = \phi^{*n}(x)\psi^{*n}(y)$. This may be shown inductively. It is true for $n = 1$ by definition, and

$$\mu^{*n}(x, y) = \phi^{*n}(x)\psi^{*n}(y)$$

implies

$$\begin{aligned}
\mu_{n+1}(x, y) &= \sum_{(g,h)\in G\times H} \mu^{*n}(g, h)\mu(g^{-1}x, h^{-1}y) \\
&= \sum_{g\in G} \sum_{h\in H} [\phi^{*n}(g)\psi^{*n}(h)][\phi(g^{-1}x)\psi(h^{-1}y)] \\
&= \sum_{g\in G} \sum_{h\in H} [\phi^{*n}(g)\phi(g^{-1}x)][\psi^{*n}(h)\psi(h^{-1}y)] \\
&= \sum_{g\in G} \phi^{*n}(g)\phi(g^{-1}x) \sum_{h\in H} \psi^{*n}(h)\psi(h^{-1}y) \\
&= \phi_{n+1}(x)\psi_{n+1}(y).
\end{aligned}$$

Thus

$$\lim_{n\to\infty} \frac{\mu^{*n}(g,h)}{\mu^{*n}(e_G,e_H)} = \lim_{n\to\infty} \frac{\phi^{*n}(g)\psi^{*n}(h)}{\phi^{*n}(e_G)\psi^{*n}(e_H)}$$
$$= \lim_{n\to\infty} \frac{\phi^{*n}(g)}{\phi^{*n}(e_G)} \lim_{n\to\infty} \frac{\psi^{*n}(h)}{\psi^{*n}(e_H)}$$

from which the result follows. □

Example 5.3. Let F_d be the free group of rank $d \geq 2$ with free basis generators including a, b, and let ϕ be a symmetric, aperiodic measure whose support generates F_d satisfying $\phi(e) > 0$ and $\phi(a) = \phi(b) > 0$. Let H be an amenable group with good measure ψ, and let μ be the product measure on $F_d \times H$. Then $A_{F_d \times H,\mu} = H$, which is exactly the amenable radical of $F_d \times H$.

In light of these examples and the fact that A_μ contains the elliptic radical, one might ask whether A_μ is in fact always the amenable radical. If so, this would imply for one thing that the set A_μ is invariant under choice of measure. It turns out that this is not the case – in the next section, we give an example where the amenable radical is trivial but A_μ is not. Moreover, we show that A_μ depends on the choice of measure.

6 Dependence on the measure

Proposition 6.1. *Let G be a finitely generated group with a finite subgroup F. Then there exists a symmetric, aperiodic probability measure ϕ on G whose support generates G such that $F \subset A_\phi$.*

Proof. Take $\psi = \pi_F * \mu * \pi_F$, where π_F is the uniform measure on F. Then

$$\phi(x) = \frac{1}{|F|^2} \sum_{f_1, f_2 \in F} \mu(f_1 x f_2),$$

which is symmetric, $\phi(e) \geq \frac{1}{|F|^2}\mu(e) > 0$ and $\mathrm{supp}(\phi) \supseteq \mathrm{supp}(\mu)$. For $f \in F$, $x \in G$, we also have $\phi(fx) = \phi(x)$, so

$$\psi^{*n}(f) = \sum_{g\in G} \psi(g)\psi^{*n-1}(g^{-1}f)$$
$$= \sum_{x\in G} \psi(f^{-1}g)\psi^{*n-1}(g^{-1}f)$$
$$= \psi^{*n}(e),$$

so $F \subset A_\psi$. □

Corollary 6.2. *There exists a finitely generated group G and symmetric, aperiodic probability measures μ, τ on G whose support generates G so that $A_{G,\mu} \neq A_{G,\tau}$.*

Proof. Consider the free product $G = \langle a \mid a^2 = 1\rangle * \langle b \mid b^3 = 1\rangle$. By Proposition 6.1, there are measures μ, τ so that $a \in A_\mu$ and $b \in A_\tau$. If $A_\mu = A_\tau$, then $A_\mu = \langle a, b\rangle = G$, which is a contradiction since G is not amenable. Other examples are readily constructed from free products of finite groups. □

The same example also gives the following:

Corollary 6.3. *There exists a finitely generated group G and a symmetric, aperiodic probability measure μ on G whose support generates G so that A_μ is not equal to the amenable radical.*

Proof. Since $C_2 * C_3$ contains finite subgroups, we may use the arguments from Proposition 6.1 to construct a measure μ for which A_μ is non-trivial. However, $C_2 * C_3$ has a trivial amenable radical. This follows from the fact that it is C^*-simple [1], or by considering the action of the group on a tree. If N is a normal amenable subgroup of a group acting on a tree, then by normality and the Tits alternative, it fixes all vertices in G/A (or G/B), or all edges, or a G-orbit of ends. Since the G-action on the space of ends is minimal, this implies in all three cases that N is trivial. □

In particular, A_μ is not always normal.

7 Connection to the amenable radical

In all cases considered, A_μ always *contains* the amenable radical. If this were true for all measures μ, the next results would give a way to directly link the amenable radical with random walk distributions.

Lemma 7.1. *Let G be a finitely generated group, and let μ be a symmetric, aperiodic probability measure on G whose support generates G. Define a measure $\mu_g \colon G \to \mathbb{R}$ by $\mu_g(x) = \mu(g^{-1}xg)$ for each $x \in G$. Then μ_g is a symmetric, aperiodic probability measure on G whose support generates G, and $A_{\mu_g} = gA_\mu g^{-1}$.*

Proof. We have

$$\mu_g(x^{-1}) = \mu(g^{-1}x^{-1}g) = \mu(g^{-1}xg) = \mu_g(x),$$

$\mu_g(e) = \mu(e) > 0$ and $\operatorname{supp}(\mu_g) = g^{-1}\operatorname{supp}(\mu)g = G$.

For $y \in G$,

$$\begin{aligned}
{\mu_g}^{*2}(y) &= \sum_{x \in G} \mu_g(x)\mu_g(x^{-1}y) \\
&= \sum_{x \in G} \mu(g^{-1}xg)\mu(g^{-1}x^{-1}yg) \\
&= \sum_{h \in G} \mu(g^{-1}h)\mu(h^{-1}yg) \\
&= \mu^{*2}(g^{-1}yg).
\end{aligned}$$

Using an inductive argument, it is clear that

$${\mu_g}^{*n}(y) = \mu^{*n}(g^{-1}yg).$$

Now

$$\begin{aligned}
x \in A_{\mu^g} &\iff \lim_{n\to\infty} \frac{{\mu_g}^{*n}(x)}{{\mu_g}^{*n}(e)} = 1 \\
&\iff \lim_{n\to\infty} \frac{\mu^{*n}(g^{-1}xg)}{\mu^{*n}(e)} = 1 \\
&\iff g^{-1}xg \in A_\mu \\
&\iff x \in gA_\mu g^{-1}.
\end{aligned}$$

□

Proposition 7.2. *Let G be a finitely generated group. If $x \in G$ does not belong to the amenable radical, then for any symmetric, aperiodic probability measure μ on G whose support generates G, there exists $g \in G$ such that $x \notin A_{\mu_g}$.*

Proof. Suppose for contradiction that $x \in G$ belongs to A_μ for every symmetric, aperiodic probability measure μ on G whose support generates G. Then for some fixed μ, by the previous lemma, we have $x \in A_{\mu_g} = gA_\mu g^{-1}$ for all $g \in G$. Thus

$$x \in \bigcap_{g \in G} gA_\mu g^{-1},$$

which is a normal amenable subgroup; hence x belongs to the amenable radical. □

Corollary 7.3. *Let $\mathcal{A}_G$ denote the amenable radical of G, and let $\mathcal{M}_G$ be the set of all symmetric, aperiodic probability measures on G whose support generates G. If $\mathcal{A}_G \subseteq A_\mu$ for every $\mu \in \mathcal{M}_G$, then $\mathcal{A}_G = \bigcap_\mu A_{\mu \in \mathcal{M}_G}$.*

We close by formulating two conjectures.

Conjecture 7.4. *Let G be a finitely generated group. Then for any symmetric, aperiodic probability measure μ on G whose support generates G, the subgroup A_μ contains the amenable radical.*

Even more desirable would be the following:

Conjecture 7.5. *Let G be a finitely generated group. Then there exists some μ such that A_μ is the amenable radical.*

Acknowledgments. The authors wish to thank Alain Valette, Sasha Fish, Vadim Kaimanovich, Narutaka Ozawa, Anne Thomas, George Willis and the anonymous reviewer for helpful suggestions, references and feedback.

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Received December 20, 2017; revised November 11, 2018.

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