Conformity and Influence*

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Monday 28th March, 2016

Abstract

To better understand trends, this paper models the behavior of decision-makers seeking conformity and influence in a connected population. Link formation is one-sided with information flowing from the target to the link's originator. A premium for leading ensures that a link's target benefits more from the link than its originator, and yet a leader serves the population by coordinating decisions. The desire for conformity drives the population to organize into a single hub with the leader at the center. Certain conditions support multiple leader structures as Nash as well. A strong desire to influence produces an equilibrium with an unlinked subpopulation.

Keywords: Opinion leadership, Social networks, Conformity, Non-cooperative games
(JEL Codes: C72, D83, D85)

"My goal is to acquire works that great museums letch after."[1]

1 Introduction

Apparently, in certain settings, decision-makers enjoy conformity and place a premium on preempting the popular choice. A natural tension exists between conformity and early adoption. Thornton (2009) observes that an avant-garde collector's reputation is based on his or her success in being an early collector of an emergent artist's works. At the same time the success of an aspiring artist is driven, in part, by the reputation of the collectors acquiring the artist's works. The buying and selling of art is not conducted anonymously. The scenario is such that

*I am indebted to Buz Brock, Richard Carson, Sven Feldmann, Virginie Masson, William Schworm, Anne van den Nouweland, Joel Watson, John Wooders, and Myrna Wooders for their comments and suggestions, as well as seminar and conference participants at the University of Melbourne, George Washington University, University of New South Wales, University of California San Diego, and the Australasian Economic Theory Workshop. Support was provided by an industry linkage grant jointly financed by pureprofile Pty. Ltd. and the Australian Research Council (LP-0990750) and by the University of Technology Sydney Business Faculty Reader program.

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[1]Avant-garde art collector David Teiger quoted in Thornton (2009) (p.100)

an individual benefits from acting in advance of a phenomenon, the emergence of which may be influenced by the individual's own actions.[2] Apparently, acting early is less risky for some than it is for others. Socially determined consequences such as those found among avant-garde collectors can be assigned to a variety of decision-makers, from designers and retailers, particularly in subjective consumer products, to real estate developers, foreign direct investors, media outlets, revolutionaries, politicians, and political supporters.[3]

The developed game captures the highlighted social aspect of decision making. The available actions capture the tension between early adoption and conformity. They also create pathways for exerting or responding to influence. Popularity arises from coordinating behavior made possible by the flow of information over personal contacts. The selective use of contacts is modeled with directed links that form an endogenous social network. Coordination manifests as a network of followers linked directly or indirectly to a leader. The premium earned by preempting the decision of others creates an asymmetry. The premium means that the benefit of a link is greater for the target than it is for the player forming the link.

Players are *ex ante* homogeneous such that prior to organizing they face the same payment opportunities, there are no explicit costs for either moving or waiting, and they have equal access to when and how to choose among the options. Thus, while the presence of a leader serves to make the followers better off, the leader's ability to do so is derived entirely from the followers' willingness to concede the leadership position to a single individual and adopt optimizing strategies accordingly. The conflict between cooperating with and competing for leadership impacts the social structure.

Actions manifest as a social structure in the form of directed links. Equilibrium in the developed environment is reflected in the social structure. A population organized around a single leader is the prevailing equilibrium. The decisions that sustain a second leader are not in the hands of the leader but in those of the followers. The environment, not individuals, determine whether equilibria with multiple leader-follower populations also exist.

[2] Of Teiger, Thornton (2009) comments, "He enjoys being a player in the power game of art, particularly at this level where patronage can have an impact on public consciousness." (p.100). Thornton also observes, "Unlike other industries, where buyers are anonymous and interchangeable, here, artists' reputations are enhanced or contaminated by the people who own their work." (p.88). Glazek (2014) profiles a patron who promotes emerging artists among collectors with little knowledge of art. Unresolved in the piece is whether the artists had no future among knowledgable collectors or whether the artists' career were poisoned by their affiliation with the patron.

[3] "Making a big bet on something before anyone else really grasps it. That is what success has in common in energy and in equities," political strategist Tim Phillips as cited in Confessore et al. (2015)

The present paper identifies equilibria for a simultaneous play game and the resulting social structures. Section 2 introduces a network structure, strategies, and payoffs. The included examples based on populations of two and three players illustrate that while a greater reward to leading undermines the interest in following, equilibrium always includes at least one follower. To identify how relative proximity to the leader alters behavior requires analysis with larger populations, an undertaking that starts in Section 3. A population of five players allows for multiple, and possible majority and minority, leaders. Section 4 considers the conditions necessary to allow multiple leaders to co-exist in equilibrium. Extensions of the model, including non-linear payoffs and possible best response cascades, are considered in Section 5.

Appendix A includes formal definitions of essential population and social structures. Appendix B includes a formal statement and proof of each proposition. Appendix C formally develops examples from Sections 2 and 4.2. Appendix D formally develops examples based on sequential play.

1.1 Related literature

The strategic complementarities found in Katz and Shapiro (1985) rewards adopting a popular choice. Classic evidence of social influence in individual decisions, even in the absence of physical complementarities, can be found in Whyle (1954), Katz and Lazarsfeld (1955) and Arndt (1967). Hill et al. (2006) and Dwyer (2007) exploit modern technology to consider social connections as they develop in mobile phone friend networks and online chats.

Some of the early examples exploring the influence of social networks model a bi-directional interaction between individual decision-making and global behavior, including Schelling (1971), Schelling (1973), and Katz and Shapiro (1985).[4] Cowan and Jonard (2004) document the impact of local and global connectivity on overall knowledge across a population.

The issues of conformity and influence arise in the social learning model of DeGroot (1974) and in the word-of-mouth communication of Ellison and Fudenberg (1995). The importance of network structure and individual behavior are further developed in such works as Golub and Jackson (2010), Acemoglu and Ozdaglar (2011), Acemoglu et al. (2013), Corazzini et al. (2012), Battiston and Stanca (2015), and Buechel et al. (2015), where conformity and influence are the consequence of the social learning environment. Arifovic et al. (2015) considers conformity as a

[4]Watts (2001) and Jackson and Watts (2002) offer useful literature reviews of works on social influence.

motivator shaping beliefs and network formation in the context of social learning. In the current investigation, players actively seek conformity and influence. They also operate in a setting in which the actions of the population entirely define the state. There is no underlying exogenous truth to be discerned from the opinions of one's neighbors. All uncertainty is intrinsic. These features alter the nature of information gathering. Combining information from various sources does not necessarily serve the individual's objectives.

Coordination in adoption imparts a positive peer effect in the Brock and Durlauf (2001) model of utility-driven conformity. The Ali and Kartik (2012) preference for complimentary actions motivates strategic exploitation of influence in the sequential decision making of the Banerjee (1992) observational learning model.[5] The benefits of early adoption appear in models such as the Pesendorfer (1995) early adoption of new fashion and in the Challet et al. (2001) model of investing.

Jackson and Wolinsky (1996), Watts (2001), Jackson and Watts (2002), and related works employ sequential decision-making to model endogenous network formation, allowing the individual decision-maker to take as given the links beyond her control. Seeking the rewards to connectivity, costly links are formed as a myopic best response to the current network structure. Bala and Goyal (2000) considers network formation as a process of one-sided link creation in the presence of inertia, so that only a fraction of the population simultaneously reset their links.

Haller and Sarangi (2005), Galeotti and Goyal (2010), Zhang et al. (2011), and Baetz (2015) characterize the endogenously determined equilibrium network structures as the product of a static model or of simultaneous linking decisions, agnostic on the issue of how a particular coordinating structure might arise. These works also explicitly model the beneficial interaction that gives rise to network connectivity rather than folding the benefit of interaction into a reward to connectivity. In equilibrium the population employs the identified network to achieve the model-specific ends. Like the current investigation, the settings generate asymmetry in outcomes from *ex ante* homogeneity.

Social connections form the foundation upon which the agents develop strategies to facilitate

[5]To adapt the Ali and Kartik (2012) observational learning model to the present setting, provide observational learning agents with the discretion to implement their choice at the time of their own choosing and offer a greater reward to those adopting the conforming choice early. Expanding the authors' own example, the model captures political contributions made to curry favor from the eventual winner. Allowing the candidate to place greater value in earlier contributors offers a counterweight to the information advantage gained from delay. Freeing contributors to choose the timing of a contribution increases intrinsic uncertainty, particularly when contributions can be made simultaneously, as the contributor cannot know the value of their contribution on subsequent decision-makers.

coordination. Were the population to seek conformity when repeatedly confronted with a new set of options, reliable social connections substitute for the inability to employ a consistent product-specific language as relied upon in the coordination game of Crawford and Haller (1990).

Multiple Nash equilibria exist in the present model. The asymmetry in the payoff means that the players have conflicting interests with regards to which equilibrium emerges. The two-player version of the current environment reflects the endogenous heterogeneity that can emerge in R&D and duopoly games, as in Reinganum (1985), Sadanand (1989), Hamilton and Slutsky (1990), Amir and Wooders (1998), and Tesoriere (2008). Such games may be played over two stages, but a parallel emerges in the decision regarding when the player wishes to act. Amir et al. (2010) generalizes the issue of symmetry breaking, as is the case when a leader and follower emerge. The general n player game retains the issues regarding asymmetry in outcome while introducing new strategy possibilities. It also introduces the possibility of best response cascades as in Dixit (2003) and Heal and Kunreuther (2010) as a way of refining the equilibrium set of structures.

2 Model

Let $N = \{1, \ldots, n\}$ be the set of players and let the $n \times n$ matrix g describe the *potential* directed links between players. If i can form a link to j then $g_{ij} = 1$ and $g_{ij} = 0$ otherwise. Let $g_{ii} = 1$ always. Write $N^d(i; g) = \{j \in N \backslash \{i\} | g_{ij} = 1\}$ for a set of players to which i can form a link. The link opportunities captured by $N^d(i; g)$ reflect the subset of the population that player i is able to directly observe based, for example, on physical proximity or personal contact. The absence of a potential link from i to j indicates that i does not have the opportunity to directly observe the action of j.

Let $O = \{O_1, O_2, \ldots, O_m\}$ be a set of $m \geq 2$ options or alternatives. To capture the absence of a common labeling of the alternatives, let K be a set of m labels for these alternatives and let the one-to-one function f_i, determined by nature, map player i's labels to alternatives, $f_i : K \to O$. Each player thus privately observes a set of labeled alternatives. For every i, j pair there is a one-to-one correspondence that is unknown to the players. Labels $\{k_A, k_B\} \in K$ correspond to the same alternative for players i and j if $k_A = f_i^{-1}(f_j(k_B))$.

Let a_i denote the action of player i. Players act simultaneously with each player choosing

(i) one of the m alternatives or (ii) to link to another player. In the former case, the player chooses $a_i = k_i \in K$ which corresponds to alternative $f_i(a_i) \in O$. If player i links to player j, then assign $a_i = j$. A player who chooses an alternative is said to *lead* while a player who links to another is said to *follow*. The set of actions for player i is $A_i = K \cup N^d(i;g)$. Write $a = (a_1, \ldots, a_n)$ for an action profile, where $a_i \in A_i$. Let $\mathscr{A}(g,m)$ be the set of all possible action profiles given g and m.

An action profile a induces an $n \times n$ matrix σ describing the *actual* links between the players as determined by their actions. If $a_i = j$ then $\sigma_{ij} = 1$ and if $a_i \in K$, such that i leads, then $\sigma_{ii} = 1$. Otherwise, $\sigma_{ij} = 0$. Thus, for the matrix σ, $\sigma \cdot \mathbf{1} = \mathbf{1}$, indicating that each player employs one and only one source to inform adoption, including possibly self-informed adoption. Imposing a single source is non-binding on the obtained solutions. Say that j is a **predecessor** of i if $\sigma_{ij} = 1$ or if there is a sequence of players $j_1, \ldots, j_\tau$ such that $\sigma_{ij} = \ldots = \sigma_{j_\tau j} = 1$. Write $N^P(i;\sigma)$ for the predecessors of i. Say that j is a **successor** of i if $\sigma_{ji} = 1$ or if there is a sequence of players $j_1, \ldots, j_r$ such that $\sigma_{jj_1} = \ldots = \sigma_{j_\tau i} = 1$. Write $N^S(i;\sigma)$ for the successors of i.

Let $N^L(\sigma) = \{i | a_i \in K\}$ denote the set of players who lead. A **leader** is an agent who leads and has a non-empty set of successors. It is possible to *lead* without being a *leader* since the player may have no successors. If player i leads and player j is a successor of i, this makes player i player j's leader. Note that each player i has at most one player who leads as a predecessor, that is $|N^L(\sigma) \cap N^P(i;\sigma)| \in \{0,1\}$ for each i. It is possible for a successor to be without a leader. Let L_i identify the predecessor of i who is a leader.

Example 1. Consider a population of twelve players arranged in a ring with each player able to link to her nearest neighbor on either side. For $m = 2$, the set of feasible action profiles includes, as an illustrative example, the action

$$a = (f_1^{-1}(O_1), f_2^{-1}(O_1), 2, 5, 6, f_6^{-1}(O_1), 6, 7, f_9^{-1}(O_2), 9, 12, 11).$$

Figure 1 includes graphical representations of g and the σ induced by a. Here, $N^L(\sigma) = \{1,2,6,9\}$ and for $i \in \{4,5,7,8\}$, $L_i = 6$. In Figure 1c, the leaders all occupy the root node of a tree capturing the tiers of a hierarchical social structure determined by σ with predecessors above successors. Player 1, without successors, occupies a trivial tree. Player 9 is the only

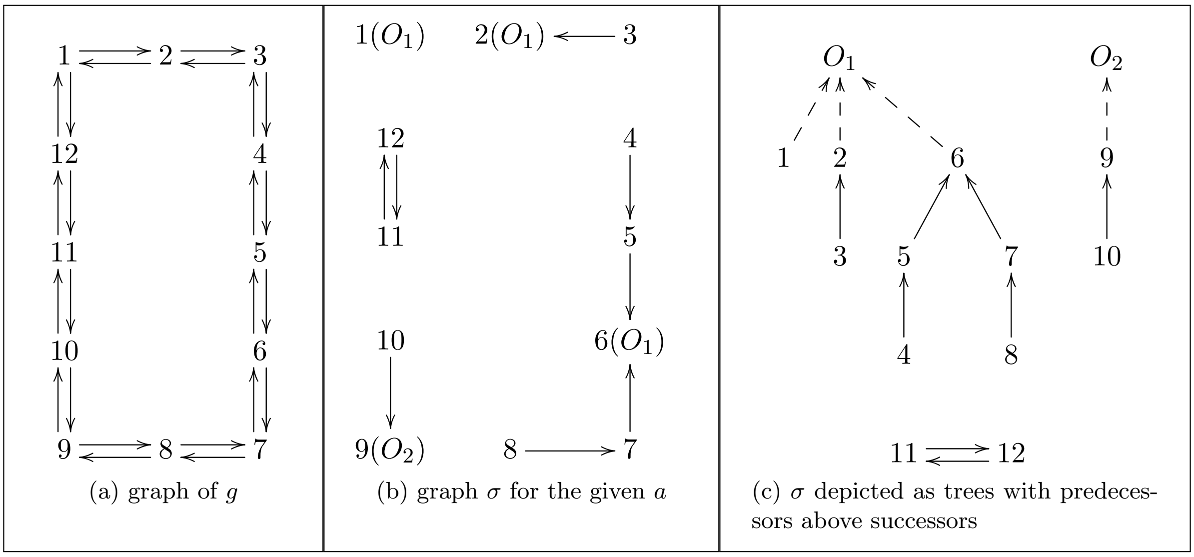


Figure 1: A g and feasible σ for a population of $n = 12$ players with $m = 2$ alternatives. The structure of g is a ring with $N^d(1; g) = \{12, 2\}$, $N^d(12; g) = \{11, 1\}$ and $N^d(i; g\} = \{i - 1, i + 1\}$ otherwise. Frame (1a) is a graphical depiction of the g matrix. Frame (1b) is a graphical depiction of the σ matrix resulting from the actions $a = (f_1^{-1}(O_1), f_2^{-1}(O_1), 2, 5, 6, f_6^{-1}(O_1), 6, 7, f_9^{-1}(O_2), 9, 12, 11)$. The choice by those who lead is included in parenthesis. Frame (1c) depicts the groupings implied by σ as trees (or "hierarchies") with predecessors positioned above successors and with the alternative above the trees. Dashed arrows indicate a leader's choice according to a. Followers 11 and 12, lacking a path to one of the alternatives, are placed at the bottom.

leading player to have selected O_2. Player 5 has 6 as a predecessor and 4 as a successor. Players 11 and 12 fail to adopt one of the alternatives as they are successors to each other and thus without a leader.

Figure 1c includes a route from each player to her adopted alternative through her predecessors and her leader's chosen alternative. Define the distance from player i to her adopted alternative as the number of players between i and the alternative. This distance helps to determine payoffs. Using d_i to denote player i's distance,

$$d_i = \begin{cases} 0 & \text{if } i \in N^L(\sigma) \\ 1 & \text{if } \sigma_{ij} = 1, j \in N^L(\sigma) \\ \tau + 1 & \text{if } \sigma_{ij_1} = \ldots = \sigma_{j_\tau j} = 1, j \in N^L(\sigma) \\ \infty & \text{otherwise.} \end{cases}$$

Use d_{ij} to denote the distance from successor i to predecessor j measured in the number of links connecting i to j. Observe that when $L_i = j$, $d_{ij} = d_i$.

Let $N^c(i;a)$ denote the set of "conforming" players adopting the same alternative as does player i (exclusive of i). Let $N^e(i;a)$ denote the set of "ensuing" adopters, a subset of $N^c(i;a)$ who are of greater distance from the alternative than is i.[6] Let $O_i \in O$ represent the alternative adopted by player i. Let k_i represent player i's label associated with alternative adopted so that $O_i = f_i(k_i)$. Let μ_i^c and μ_i^e represent the cardinality of the respective populations, $\mu_i^c = |N^c(i;a)|$ and $\mu_i^e = |N^e(i;a)|$.

The map from action profiles to payoffs rewards conformity to the popular alternative and for adopting the popular alternative in advance of other players.[7] The appendix develops an additively separable payoff function in μ_i^c and μ_i^e from a utility function valuing social interaction. Let $\pi(i;\sigma)$ represent player i's possible payoff based on the structure σ. The payoff for player i is

$$\pi_{NL}(i;\sigma) = \phi(\mu_i^c) + \psi(\mu_i^e) \tag{1}$$

[6]The notion of time and distance are isomorphic when adoption disseminates at a rate of one unit of time per link.

[7]Since payoffs depend on the popularity of the adopted option and the relative time to adoption, in practice, the information should eventually become available to the players. A report on the popularity of each alternative broken down by time is sufficient. Such information could be seen as emerging slowly over the network after all decisions have been made or as tabulated and published in a bulletin.

Player	μ_i^c	μ_i^e	π_i	Player	μ_i^c	μ_i^e	π_i	Player	μ_i^c	μ_i^e	π_i
1	7	5	22	5	7	2	13	9	1	1	4
2	7	5	22	6	7	5	22	10	1	0	1
3	7	2	13	7	7	2	13	11	0	0	0
4	7	0	7	8	7	0	7	12	0	0	0

Table 1: Payoff for $a = (f_1^{-1}(O_1), f_2^{-1}(O_1), 2, 5, 6, f_6^{-1}(O_1), 6, 7, f_9^{-1}(O_2), 9, 12, 11)$ using payoff parameters $r_c = 1$ and $r_e = 3$.

with $\phi(0) = \psi(0) = 0$ and where $\phi(\mu)$ and $\psi(\mu)$ are increasing and continuously twice differentiable. The first element of the payoff is the conformity component, much like the community effect of Blume and Durlauf (2001). The second element in (1) captures the reward to holding a distance advantage over the ensuing players. Let $\Pi(a) = (\pi_1, \ldots, \pi_n)'$ be the $n \times 1$ vector of payoffs according to a.

As a special case, let

$$\pi(i; \sigma) = r_c \mu_i^c + r_e \mu_i^e, \tag{2}$$

with non-negative reward coefficients r_c and r_e.[8] Table 1 reports the payoff to each player based on the action a from Example 1. As observed in Figure 1c, for player $i \in \{1, \ldots, 8\}$, $N^c(i; a) = \{1, \ldots, 8\} \backslash \{i\}$ so that $\mu_i^c = 7$. In addition, for $i \in \{3, 5, 7\}$, $N^e(i; a) = \{4, 8\}$, reflecting that all players of equal distance from O_1 benefit equally from the players who are of greater distance. Player 9, having chosen differently than the other leading players, benefits only from her successor, player 10. For players $i \in \{4, 8, 10, 11, 12\}$, $N^e(i; a) = \emptyset$. Players 11 and 12, failing to adopt a choice, receive no payoff nor do they contribute to the payoff of any other player.

2.1 Strategic behavior

For any two leading players i and j, uncertainty about other players' f_i cause players to believe $\Pr(f_i(a_i) = f_j(a_j)) = 1/m$. The uncertainty in whether two leaders will match alternatives means that there can be a random element to pure strategy payoffs. A couple of small-n examples illustrate the issues and outcomes inherent to the setting.

[8] The linear model generalizes to one in which instead of rewarding early adoption there is a cost or penalty to late adoption. Similar to the examples used in Brindisi et al. (2011) in which late adopters pay higher costs, payoffs become $\pi(i; \sigma) = b_c(\mu_i^c) - b_e(\mu_i^c - \mu_i^e)$ where b_c is the per member conformity payoff and b_e is the cost associated with each player who acts concurrent or in advance of player i on the same alternative. With $b_c = r_c + r_e$ and $b_e = r_e$, the two scenarios are isomorphic.

		Player 2	
		lead	follow
Player 1	lead	$\frac{1}{2}, \frac{1}{2}$	$r_e + 1, 1$
	follow	$1, r_e + 1$	0,0

Table 2: The relevant action-dependent expected payoff matrix and game of Example 2 with $n = 2$, $m = 2$, $r_c = 1$, $r_e > 0$.

Example 2. $n = 2$, $m = 2$, $\phi(\mu^c) = \mu^c$, $\psi(\mu^e) = r_e\mu^e$, $r_e > 0$, and $g = \underset{2\times 2}{\mathbf{1}}$. The game as presented in Table 2 excludes the inconsequential distinction between leading with "A" and leading with "B".

The Nash equilibrium strategy profile produces one leader and one follower. For the equilibrium with player 2 leading player 1, player 2 receives the higher payoff for being the leader. Player 1's lower payoff remains higher than the expected payoff that can be obtained from also leading. Player 2's selection of $a_2 \in \{$"A", "B"$\}$ has no impact on the realized payoffs in equilibrium nor does the choice effect expected payoffs in non-equilibrium play.[9] The symmetry of the game means that there is also an equilibrium with player 1 leading player 2. The players want to avoid the strategy profile in which both lead. They also want to avoid the outcome produced when each follows the other.[10]

Example 3. $n = 3$, $m = 2$, $\phi(\mu^c) = \mu^c$, $\psi(\mu^e) = r_e\mu^e$, $r_e > 0$, and $g = \underset{3\times 3}{\mathbf{1}}$. A larger population introduces the possibility of adopting a minority option. Table 3 reports the expected payoff matrix for the actions for players 1 and 2 based on player 3 leading.

The set of equilibrium structures depends on r_e. For $r_e \leq 2$, the equilibrium structures are those in which one player leads and the other two follow. If player 2 follows player 3, player 1's optimal strategy is to also follow player 3 and is indifferent between succeeding 3 directly or indirectly by succeeding 2. Player 1's indifference stems from being the most distant follower in either scenario. Off equilibrium, if player 2 had not followed player 3, player 1 is still sure to follow. If player 2 follows 1, 1 still follows 3 in order to gain the conformity reward while retaining a distance advantage over 2. If both players 2 and 3 are leading, player 1 is indifferent in choosing which to follow. Symmetry ensures the same structures regardless of

[9]See Appendix C for the full payoff matrix associated with each possible outcome and the proper normal-form game.

[10]The mixed strategy solution for this example has $\Pr_i(\text{lead}) = (1 + r_e)/(\frac{3}{2} + r_e)$. The value of the game in the mixed strategy solution is $v = (2 + 2r_e)/(3 + 2r_e)$. Since $v < 1$ for all $r_e \geq 0$, the value of the mixed strategy solution is always less than the follower's payoff in the pure strategy game.

		Player 2		
		lead	follow 1	follow 3
Player 1	lead	1, 1, 1	$\frac{3}{2}+r_e$, $\frac{3}{2}$, $1+\frac{r_e}{2}$	$1+\frac{r_e}{2}$, $\frac{3}{2}$, $\frac{3}{2}+r_e$
	follow 2	$\frac{3}{2}$, $\frac{3}{2}+r_e$, $1+\frac{r_e}{2}$	0, 0, 0	2, $2+r_e$, $2+2r_e$
	follow 3	$\frac{3}{2}$, $1+\frac{r_e}{2}$, $\frac{3}{2}+r_e$	$2+r_e$, 2, $2+2r_e$	2, 2, $2+2r_e$

Player 3 leads

Table 3: Example 3 relevant expected payoff table for $n = 3$, $k = 2$, $r_c = 1$, $r_e > 0$, $\sigma_{3,3} = 1$

which individual leads.

For $r_e > 2$ the equilibrium structures are those in which one player follows and the other two players lead. If player 2 follows player 3, player 1's optimal strategy is to lead. This strategy is motivated by the $1/m = 1/2$ probability that $f_1(a_1) = f_3(a_3)$ when both 1 and 3 lead. In case they match, player 1 gains the distance advantage over player 2. This gamble becomes worthwhile for sufficiently large r_e. Notice that player 1 only takes this gamble when player 2 follows player 3. Without a successor to player 3, there is no inducement for player 1 to gamble with leading. Regardless of r_e, if both 2 and 3 lead, player 1 chooses to follow.

For a network of directed links, a strongly connected network is one for which every player pair $\{i, j\}$ has either $g_{ij} = 1$ or there exists $j_1, \ldots, j_m$ such that $g_{ij_1} = \ldots = g_{j_m j} = 1$. As a consequence, for every $\{i, j\}$ pair there is a directed path from i to j. Let $G(n)$ be the universe of strongly connected networks based on a population size n. Both examples 2 and 3 are based on a g that is a complete graph (all players are able to link to any other player directly). For $n = 2$ the only strongly connected graph is the complete graph. There are 18 possible $g \in G(3)$ with five that are unique to a relabeling of the players. The benefit to coordinating on an alternative through imitation is the same for any $g \in G(3)$. The following can be demonstrated to be true for all $g \in G(3)$.

1. For $r_e \leq 2$, an action profile is a Nash equilibrium if and only if it produces one leader and two successors.

2. For $r_e > 2$, an action profile is a Nash equilibrium if and only if it produces two leading players and one successor.

3. The set of Nash equilibria include action profiles that produce i as the unique leader for all $i \in \{1, 2, 3\}$.

From 1 and 2 above, every equilibrium action produces one and only one non-trivial tree. From 3, it is always possible that any one of the players can hold the favorable position of leader.

To be developed in the following sections, the features found in the $n = 3$ population generalize for any size population occupying a strongly connected network. For linear reward, these features are

- Pure strategy Nash equilibria exist.
- A unique leader, possibly in the presence of other leading players, is among the equilibrium social structures.
- Any player $i \in N$ can be the equilibrium leader of the non-trivial tree.
- Whether the leader is the only agent who leads in equilibrium depends on a formula expressed primarily in terms of r_c/r_e and m. Above a threshold, the entire population can join to form a single tree. For r_c/r_e below the threshold, the size of the single non-trivial tree declines in r_c/r_e.

3 Single-leader equilibria

This section formally develops the behavior observed in the two examples of Section 2.1 while generalizing to a population of size n and a network of potential links $g \in G(n)$. A special case of the strongly connected graph is the complete graph. Considering the more general structure of potential links allows application more broadly to settings in which participants seeking to be involved in social phenomena do not necessarily have direct access to all members of the population. Limits on connectivity create scenarios of interest that cannot be addressed when considering a complete graph. A population of $n > 3$ makes feasible coexisting multiple non-trivial trees. Some additional aspects of the equilibrium actions only come to light when considering a larger n. There is, for example, a social structure considered in Section 4 that requires $n \geq 8$.

The equilibrium concept employed is that of a pure strategy Nash equilibrium. Section 3 culminates in establishing the condition to ensure that the set of Nash equilibria is non-empty and includes social structures, embodied in σ, consisting of just a single leader.

3.1 Hierarchies

The term **hierarchy** refers to a non-trivial tree. Let $h(i;g)$ be the set of σ given g such that $i \in N^L(\sigma)$ with a non-trivial tree of successors. Let $H(i;g)$ represent the set of σ given g such that $\{i\} = N^L(\sigma)$ with a successor population $N^S(i;\sigma) = N \backslash \{i\}$. As a reminder, $g \in G(n)$ is a strongly connected network of n players.

Lemma 1. *For every strongly connected g and for every $i \in N$ there exists a non-empty set of structures in which the entire population follows i and i leads.*

Note that the a generating $\sigma \in H(i;g)$ is generally not unique. In particular, $\sigma \in H(i;g)$ is independent of $a_i \in K$. In addition, there generally exists more than one path through which any follower j can link to leader i.

Analysis of a structure is facilitated by identifying populations as they relate to follower $j \in N^S(i;\sigma)$ within structure σ. Let $N^x(j;\sigma)$ be the set of successors of i who are of distance no greater than d_{ji}. Let $N^y(j;\sigma)$ be the set of successors of i with a distance greater than d_{ji} who are not successors of j. Recall that set $N^s(j;\sigma)$ identifies the population that succeeds player j. Let $\mu^x_j = \mu^x(j;\sigma) = |N^x(j;\sigma)|$, $\mu^y_j = \mu^y(j;\sigma) = |N^y(j;\sigma)|$, and $\mu^s_j = \mu^s(j;\sigma) = |N^S(j;\sigma)|$. The nodes of the left tree depicted in Figure 2 are labeled consistent with each player's position relative to j with $\mu^x_j = 1$, $\mu^y_j = 3$ and $\mu^s_j = 2$.[11] Observe that for any $\sigma \in H(i;g)$,

$$\begin{aligned} \mu^c_j &\equiv 1 + \mu^x_j + \mu^y_j + \mu^s_j = n-1 \text{ and} \quad (3)\\ \mu^e_j &\equiv \mu^y_j + \mu^s_j. \quad (4)\end{aligned}$$

For $\sigma \in H(i;g)$, let $h^-(i,\sigma;g)$ be the set of structures produced when some player $j \in N^S(i;\sigma)$ leads rather than follows. Let

$$\begin{aligned} A_{NL}(j;\sigma) &:= A_1 + A_2(\mu^y_j, \mu^s_j) + A_3(\mu^s_j) \quad (5)\\ A_1 &= (m-1)(\phi(n-1) - \phi(n-2))\\ A_2(\mu^y_j, \mu^s_j) &= m(\psi(\mu^y_j + \mu^s_j) - \psi(\mu^s_j))\\ A_3(\mu^s_j) &= (m-1)\phi(n-2) - \psi(n-2)] - [(m-1)\phi(\mu^s_j)) - \psi(\mu^s_j)\end{aligned}$$

[11] The possible second tree lead by i_B depicted on the right of Figure 2 is developed in Section 4.

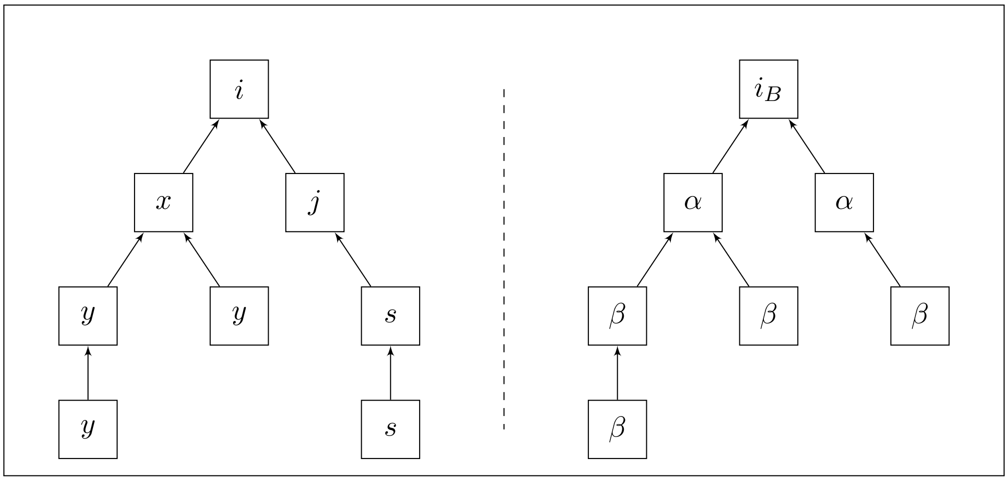


Figure 2: Labeled positions in relation to player j. In j's own tree are $x \in N^x(j;\sigma)$, $y \in N^y(j;\sigma)$, and $s \in N^S(j;\sigma)$. In the presence of a second tree (considered in Section 4) are $\alpha \in N^\alpha(j;\sigma)$ and $\beta \in N^\beta(j;\sigma)$.

and let

$$B_{NL}(n,m) := (m-1)\frac{\phi(n-1)}{\psi(n-2)} - 1. \tag{6}$$

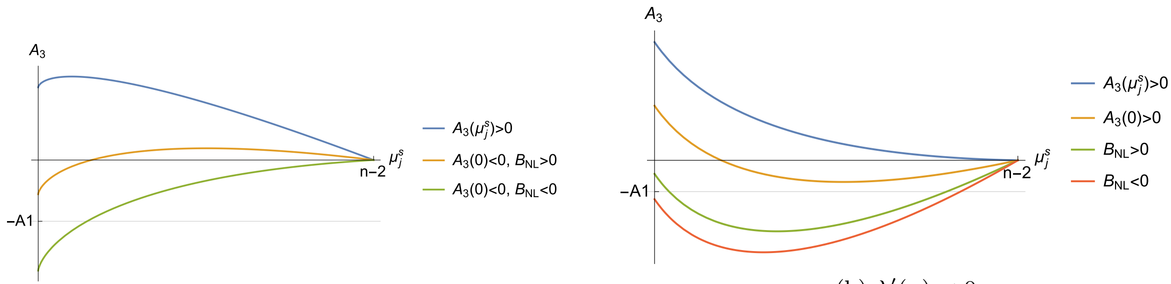
As a reminder, both $\phi'(\mu) > 0$ and $\psi'(\mu) > 0$. Let $\lambda(\mu) = \phi(\mu)/\psi(\mu)$ and let $\bar{j}$ represent the follower most distant from i in structure σ. As an alternative structure, let $\sigma^{'} = \sigma_j^{'} \times \sigma_{-j}$ and $\sigma_{jj}^{'} = 1$ producing $\sigma' = h^-(i,\sigma;g)$

Proposition 1. *For a structure σ consisting of a single leader and a population of $n-1$ followers and for rewards such that $\lambda'(\mu) \geq 0$, $B_{NL} \geq 0$ is a necessary and sufficient condition that all followers prefer following to leading.*

Proof. See Appendix B □

According to Proposition 1, $B_{NL} \geq 0$ is the condition by which each follower $j \in N^S(i;\sigma)$ prefers $\sigma \in H(i;g)$ to the structure produced by a switch by j to lead. The proof establishes $\lambda'(\mu) \geq 0$ as the condition ensuring that the decision to follow by the successor most distant from i identifies the preference to follow for every member of the tree.

The value of $A_{NL}(j;\sigma)$ reflects the differential for player j between following an existing leader i or leading in the presence of i. The proof of Propositions 1 establishes that $A_{NL}(j;\sigma) \geq 0$ ensures that player $j \in N^S(i;\sigma)$ prefers imitating over leading. The A_1 component of $A_{NL}(j;\sigma)$ is strictly positive. The A_2 component is weakly positive and increasing in μ_j^y.



(a) $\lambda'(\mu) > 0$

(b) $\lambda'(\mu) < 0$

Figure 3: $A_3(j;\sigma)$ as shaped by $\lambda(\mu)$. The most distant follower of i has $\mu_j^s = \mu_j^y = 0$ so that $B_{NL} = A_1 + A_3(0)$. $B_{NL} < 0$ indicates that the most distant follower prefers to lead. For $\lambda'(\mu) \geq 0$, the minimum of $A_3(\mu_j^s)$ is at a boundary value for μ_j^s. For $\lambda'(\mu) < 0$ the minimum can occur for an interior value of μ_j^s so that $B_{NL} \geq 0$ does not ensure $A(j;\sigma) \geq 0$ for all j.

The A_3 component, which can also be expressed as

$$A_3(\mu_j^s) = [(m-1)\lambda(n-2) - 1]\psi(n-2) - [(m-1)\lambda(\mu_j^s) - 1]\psi(\mu_j^s),$$

has $A_3(n-2) = 0$. For any increasing $\psi(\mu)$, $\lambda'(\mu) = 0$ results in an $A_3(\mu)$ that is monotonically decreasing in μ for $A_3(0) > 0$, monotonically increasing in μ for $A_3(0) < 0$, and constant at zero if $A_3(0) = 0$. Use $A_3^0(\mu)$ to indicate the $A_3(\mu)$ produced by $\lambda'(\mu) = 0$.

Given $\psi(\mu)$, for $\lambda'(\mu) \geq 0$ then $A_3(\mu) \geq A_3^0(\mu)$ for $0 < \mu < n-2$. As a result, the minimum of $A_3(\mu_j^s)$ is at one of the extremes. Either $A_3(0) \geq 0$ and $A_3(\mu_j^s) \geq A_3(n-2) = 0$ for all possible μ_j^s or $A_3(0) < 0$ is the minimum value of $A_3(\mu_j^s)$. In the former case, $A(j;\sigma) > 0$ for all j. In the latter case, $A(j;\sigma) > A(\bar{j};\sigma)$ for all $j \in N^S(i;\sigma)\backslash\bar{j}$. For the most distant follower, $A(\bar{j};\sigma) \geq 0$ if $A_3(0) \geq -A_1$ or equivalently, $B_{NL} \geq 0$. Figure 3a is illustrative. The example employs $\phi(\mu) = \mu^{1-a}/(1-a)$ and $\phi(\mu) = c\mu^{1-b}/(1-b)$ with $a > b$ so that $\lambda'(\mu) > 0$.

Reflecting the fact that the payoffs at both the top and the bottom of the tree depend only on the size of the tree and not its organization, the condition $B_{NL} \geq 0$ underpinning Proposition 1 is determined by universal parameters independent of the particular i, the characteristics of $\sigma \in H(i;g)$, or the nature of $g \in G(n)$. Curvature in the increasing $\phi(\mu)$ and $\psi(\mu)$ functions also plays no role so long as $\lambda'(\mu) \geq 0$.

Corollary 1. *For a structure σ consisting of a single leader and a population of $n-1$ followers and the condition $B_{NL} \geq 0$, $\lambda'(\mu) < 0$ can produce a preference among middle distance followers to lead.*

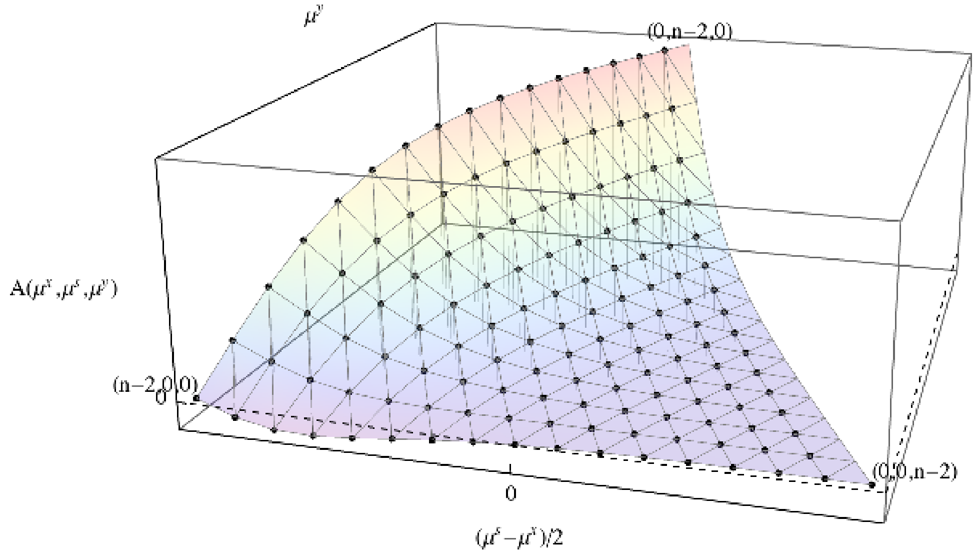


Figure 4: $A_{NL}(j;\sigma)$ surface produced for $n = 18$ with $\lambda'(\mu) < 0$ and $B_{NL} > 0$. Each point on the surface represents a unique feasible triplet $(\mu_j^x, \mu_j^y, \mu_j^s)$. The height of the point is $A_{NL}(j;\sigma)$. For $\sigma \in H(i;g)$, $\mu_j^x + \mu_j^y + \mu_j^s = n - 2$. Each follower occupies a point on the surface. Each point can be occupied by zero, one, or multiple players with the condition that at least one player occupy the lower left corner. The far corner is always the highest point. For $B_{NL} \geq 0$, all three corners are positive with only the near left corner at zero for $B_{NL} = 0$. For $B_{NL} < 0$ the near left corner is negative. For $\lambda'(\mu) \geq 0$, one of the near corners is the lowest point on the surface. For $\lambda'(\mu) < 0$, local convexity in the $\mu^y = 0$ plane allows a low point along the near edge. Any σ with a player located at a point with $A(j;\sigma) < 0$ cannot be an equilibrium.

Proposition 1 relies on the weak concavity of $A_3(\mu_j^s)$ to ensures $A(j;\sigma) \geq 0$ for all $j \in N^S(i;\sigma)$ when $B_{NL} \geq 0$. Given $\psi(\mu)$, for $\lambda'(\mu) < 0$ then $A_3(\mu) \leq A_3^0(\mu)$ for $0 < \mu < n-2$, introducing the possibility that $A_3(\mu_j^s)$ is at its minimum at an interior $0 < \mu < n-2$. This introduces the possibility that $A(j;\sigma) \leq 0$ for some middle distance $j \in N^S(i;\sigma)$ of even when $A_3(0) \geq -A_1$ (equivalent to $B_{NL} \geq 0$). The example included in Figure 3b illustrates the consequence of $\lambda'(\mu) < 0$. The dip in $A_3(\mu_j^s)$ below $-A_1$ indicates that some middle-distance tree nodes are inferior to leading. The surface in Figure 4 presents the $A(j;\sigma)$ for all possible nodes for any $\sigma \in H(i;g)$ from $g \in G(n)$, $n = 18$ in this example. A structure with followers occupying one or more of the nodes with $A(j;\sigma) < 0$, found on the leading edge and in the second row from the leading edge, is not an equilibrium. For $\lambda'(\mu) < 0$, the decision to lead or follow for all $j \in N^S(i;\sigma)$ cannot be identified from the preference of i's most distant follower.

To have $\mu_j^s = n-2$ requires $\mu_j^x = \mu_j^y = 0$. This describes a follower who is the sole direct successor of i so that the remainder of the population links to i though j. A follower in such a position always prefers following i to leading because she retains her distance advantage over the remaining population while gaining the conformity reward for i.

Examination of the social impact of $\lambda'(\mu) < 0$ resumes in Section 5. Until then, analysis will be dedicated to identifying the equilibrium structures supported by $\lambda'(\mu) \geq 0$. Taking advantage of the inconsequentiality of the particular curvature of ϕ and ψ, analysis will continue with the linear payoff of (2), for which $\lambda'(\mu) = 0$.[12]

For $\pi(i;\sigma) = r_c\mu_i^c + r_e\mu_i^e$, A_{NL} and B_{NL} become, respectively,

$$A(j;\sigma) = (m-1)r_c + mr_e\mu_j^y + ((m-1)r_c - r_e)(n-2-\mu_j^s) \tag{7}$$

and

$$B(n,m) = (m-1)\frac{r_c(n-1)}{r_e(n-2)} - 1$$

for $r_e \neq 0$. Proposition 1 applies so that $B \geq 0$ is a necessary and sufficient condition for $\pi(j,\sigma) \geq \pi(j,\sigma')$ for all $j \in N^S(i;\sigma)$ for $\sigma \in H(i;g)$ and $\sigma' = \{h^-(i,\sigma;g)|\sigma'_{jj} = 1\}$. An evaluation of Proposition 1 based on the linear payoff function can be found in Appendix B.

A useful re-expression of the condition $B \geq 0$ defines

$$B(n,\theta) := \theta - \left(1 - \frac{1}{1-n}\right) \tag{8}$$

where

$$\theta = \frac{(m-1)r_c}{r_e}.$$

Throughout the paper, how θ compares to some threshold value determines whether all players prefer to follow an existing leader or whether there exists some player who prefers to lead in the presence of another leader. The condition $B \geq 0$ is just one expression of this threshold. Other threshold values for θ arise when analysis turns to more complicated social structures involving multiple leaders, considered in Section 4. It is worth understanding environmental influencers on the individual's decision in support of Proposition 1 to better understand the role of the parameter θ and the condition $B \geq 0$ here and in future settings.

[12] Linearity allows for aggregation in expectation over possible states. Curvature in the reward components means accounting for each possible state separately, adding complexity to the equations without additional insight.

Population	j leads	j follows
$\{i\}$	$\frac{1}{m}r_c$	r_c
$N^x(j;\sigma)$	$\frac{1}{m}(r_c+r_e)\mu^x$	$r_c\mu^x$
$N^y(j;\sigma)$	$\frac{1}{m}(r_c+r_e)\mu^y$	$(r_c+r_e)\mu^y$
$N^S(j;\sigma)$	$(r_c+r_e)\mu^s$	$(r_c+r_e)\mu^s$

Table 4: Expected contribution to j's payoff according to j's decision

Using (3) to express $A(j;\sigma)$ in term of μ_j^x and μ_j^y yields

$$A(j;\sigma) := (m-1)r_c + (m-1)(r_e+r_c)\mu_j^y + ((m-1)r_c - r_e)\,\mu_j^x. \tag{9}$$

How followers value the $N^x(j;\sigma)$ and $N^y(j;\sigma)$ populations determines how $A(j;\sigma)$ changes as a reflection of j's position in the structure. As reported in Table 4, the $N^S(j;\sigma)$ population contributes equally whether j leads or follows. The first term in (9) reflects the gain of certain conformity with i when following. The certain conformity and ensuing adoption of the $N^y(j;\sigma)$ population in the second term of (9) provides the strongest incentive to follow. The $N^x(j;\sigma)$ population offers both opportunity and sacrifice. When following, player j gains certain conformity with the $N^x(j;\sigma)$ population. By leading, with probability $1/m$, player j retains conformity and gains a timing advantage over the population.

The numerator of θ reflects the marginal contribution of a player in the $N^x(j,\sigma)$ population to j's payoff when j follows i. The denominator of θ captures the marginal payoff contribution of a player in $N^x(j,\sigma)$ when j leads. For $\theta > 1$, the reward to conformity with the $N^x(j,\sigma)$ population exceeds the expected gain from capturing a distance advantage over $N^x(j,\sigma)$ so that the coefficient on μ_j^x is positive. For $\theta < 1$, the $N^x(j,\sigma)$ population rewards j's leading more than following i in expectation.

More simply, the scenarios that contribute to make following attractive are those scenarios in which the reward to conformity is relatively high (r_c/r_e is large) and chance coordination by independent actors is unlikely (m is large). Conversely, the scenarios that undermine following are those in which there is a high premium to preempting and the opportunity to coordinate by chance is high, so that low r_c/r_e and low m yield $B < 0$.

Lemma 2. *Among the set of structures consisting of a single leader and a population of $n-1$ followers, each follower maximizes own payoff as a follower of i by minimizing μ_j^x.*

Lemma 2 identifies the optimal follow action. Lemma 2 emerges from the fact that μ_j^s is independent of j's action, that $\mu^e(j) = \mu_j^y + \mu_j^s$, and that, since for $\sigma \in H(i;g)$ with $\mu_j^y = n - 2 - \mu_j^x - \mu_j^s$, increasing μ_j^x decreases $\pi(j;\sigma)$. Let $H'(i;g)$ be the set of $\sigma \in H(i;g)$ such that each $j \neq i$ employs an imitation strategy that minimizes μ_j^x. By Lemma 2, for $\sigma \in H'(i;g)$, no player can do better for herself as a follower.

Let $H^*(i;g)$ be the set of $\sigma \in H(i;g)$ such that each $j \neq i$ employs an imitation strategy offering a distance-minimizing connection to the leader. Note that if σ' exists such that $\{\sigma, \sigma'\} \in H^*(i;g)$, then $d_j(\sigma) = d_j(\sigma')$ for all $j \in N$. As a result, all $\sigma \in H^*(i;g)$ offer exactly the same payoff profile. Similar to $H^*(i;g)$, let $h^*(i;g)$ be the set of strategies for which each successor of i imitates the player offering the shortest distance from i.

Proposition 2. *Among the set of structures consisting of a single leader and a population of $n-1$ followers, minimizing distance to the leader is a subset of minimizing μ_j^x.*

Proof. See Appendix B □

Following Proposition 2, $H^*(i;g) \subseteq H'(i;g)$. In order to have $H^*(i;g) \subset H'(i;g)$ requires the existence of a $\sigma \in H^*(i;g)$ satisfying three criteria. For $\sigma \in H^{'}(i;g)$ there exists $\sigma' \in H'(i;g)$ if and only if there exists j with the following three properties:

1. There exists some $j' \in N^d(j,g)$ with $d_{j',i} = d_{j,i}$, indicating that j' is equal distance to the leader as is j and that j has the option to imitate j'.

2. $\mu^y(j;\sigma) = 0$, indicating that there are no successors to i of greater distance to i than j without also being a successor to j.

3. All successors to j have no alternative option to link to i but through j.

The criteria make it possible for some j to increase her distance to i (property 1) while preserving μ_j^x minimization for herself (property 2) and for everyone else (property 3). In this case there exists $\sigma' \in H'(i;g) \backslash H^*(i;g)$ where $\sigma^{'}_{jj'} = 1$ and $\sigma^{'}_{-j} = \sigma_{-j}$. Figure 5 includes an illustrative example.

Proposition 3. *If a structure σ consisting of a single leader and a population of $n-1$ followers is an equilibrium then σ is a member of the subset in which all followers minimize μ_j^x.*

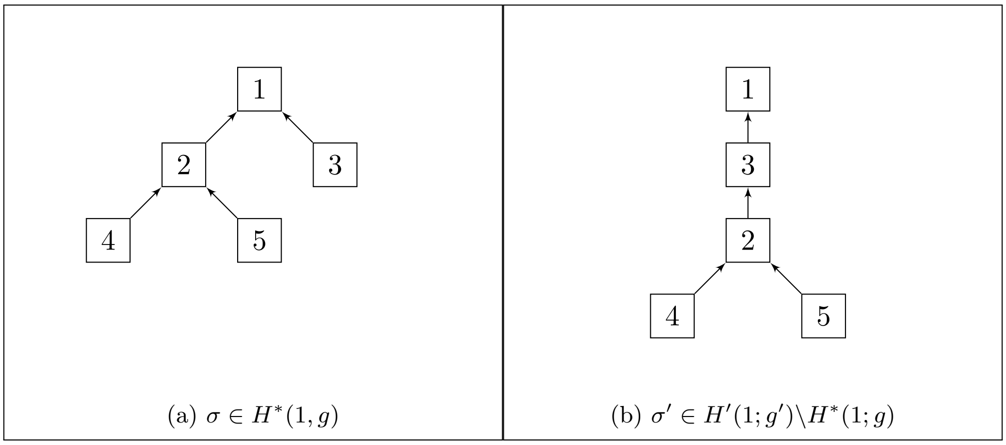


Figure 5: An example of $\sigma' \in H'(i; g)\backslash H^*(i; g)$ based on $N^d(2; g) = \{1, 3\}$, $N^d(4; g) = \{2, 5\}$, and $N^d(5; g) = \{2, 4\}$, $\sigma \in H^*(1; g)$. The conditions for a non-empty $\sigma' \in H'(1; g)\backslash H^*(i; g)$ are satisfied. For players $i = 1, 2, 4, 5$, $\pi(i; \sigma) = \pi(i; \sigma')$ while $\pi(3; \sigma) > \pi(3; \sigma')$.

Proof. See Appendix B □

Proposition 3 excludes $\sigma \in H(i; g)\backslash H'(i; g)$ from the set of Nash equilibria while preserving $\sigma \in H'(i; g)$ as a candidate member of the set of Nash equilibrium. Following Lemma 2, Proposition 3 establishes that for $\sigma \in H(i; g)\backslash H'(i; g)$, some follower $j \in N^S(i; \sigma)$ is able to improve her reward while remaining in $N^S(i; \sigma)$. For $\sigma \in H'(i; g)$, no player (including i) is able to improve her own reward while preserving the structure's membership in $H(i; g)$.

Returning to Figure 5, the action by j to minimize μ_j^x but not d_j benefits some player $j' \in N^S(i; \sigma)\backslash\{j\}$ without cost to any player. In a sense, any structure $\sigma \in H'(i; g)\backslash H^*(i; g)$ is socially preferred to a structure $\sigma \in H^*(i; g)$.

Proposition 4. *For the set of structures consisting of a single leader and a population of $n - 1$ followers, $B \geq 0$ is a necessary and sufficient condition to have the subset in which all followers minimize μ_j^x as Nash equilibria.*

Proof. See Appendix B □

By Proposition 4 for every $i \in N$ there is an equilibrium structure for which i is a leader to the remaining population. The only condition needed to produce this set of structures as equilibria is $B \geq 0$. As observed with Proposition 1, $B \geq 0$ is the product of the environment and preferences, independent of the particular i or the characteristics of $\sigma \in H'(i; g)$ or $g \in G(n)$. Proposition 4 follows naturally from Propositions 1 and 3.

Corollary 2 follows from $H^*(i; g)$ being a non-empty subset of $H'(i; g)$.

Corollary 2. *The structures consisting of a single leader and a population of $n-1$ followers, all of whom minimize their distance to the leader, are Nash equilibrium if $B \geq 0$.*

For the more complicated social structures developed in the sections that follow, analysis will proceed with optimizing followers minimizing d_{ji}. Allowing for the larger set of actions available when minimizing μ_j^x adds to the complexity of derivations without substantively altering the outcomes.[13]

Let g^c represent the special case of a complete graph. The $\{\sigma^c\} = H^*(i; g^c)$ is a star network. Since $g^c \in G(n)$, by Proposition 4 and Corollary 2, the star network is an equilibrium structure when $B \geq 0$. The set $H'(i; g^c)$ consists of additional equilibrium structures in which i leads, $n-2$ players link directly to i and one player links indirectly to i through one of the $n-2$ direct successors.

3.2 Leading without followers

This subsection identifies candidate structures for equilibrium for when $B < 0$, a setting that, according to Proposition 1, excludes the set of structures $H(i; g)$.

Defined formally in the appendix, let $h_L(i, \mu_i^s; g)$ represent the set of structures in which i's successor population is of size $\mu_i^s < n-1$ and the $n - \mu_i^s - 1$ most distant players on g from i lead rather than follow. For $\sigma \in h_L^*(i, \mu_i^s; g)$, the $N^S(i; \sigma)$ population of successors to i is the μ_i^s players closest to i according to g with each member of $N^S(i; \sigma)$ minimizing her distance to i.

Observe that for $\sigma \in h_L(i, \mu_i^s; g)$ and $j \in N^S(i; \sigma)$,

$$\underbrace{1 + \mu_j^x + \mu_j^y + \mu_j^s}_{=\mu_i^s} + \mu^l = n$$

where $\mu^l = |N^L(\sigma)|$. Let

$$C(\mu_i^s; \theta) = \theta - \left(1 - \frac{1}{\mu_i^s}\right). \tag{10}$$

[13]While $\sigma' \in H'(i; g) \backslash H^*(i; g)$ is socially preferred to $\sigma \in H^*(i; g)$, it is reasonable to consider σ' as unlikely to emerge from autonomous play for a number of reasons, including the foregone opportunity to exploit disequilibrium play by other followers and the increased exposure to adverse outcome that may result from the disequilibrium play of others. Though not proven, I have found no counterexample to the notion that $H^*(i; g) \subset H'(i; g)$ only when the structure $\sigma \in H'(i; g) \backslash H^*(i; g)$ is inconsequential to the optimizing decisions of the players.

Allow μ^* to represent the value of μ_i^s that solves $C(\mu_i^s;\theta) = 0$,

$$\mu^* = \frac{1}{1-\theta}, \tag{11}$$

where $B < 0$ ensures $\theta < 1$. Defined below, $\bar{n}$ is an integer near μ^*, $|\bar{n} - \mu^*| < 1$.

Proposition 5. *For $B < 0$ and $\theta \neq 0$, if a structure σ consisting of a single leader with a successor population consisting of the μ_i^s closest followers and in which the remaining $n - \mu_i^s - 1$ players all lead is an equilibrium, then $\mu_i^s = \bar{n}$.*

Proof. See Appendix B $\square$

The proof of Proposition 5 establishes that for $\sigma \in h_L(i, \mu_i^s; g)$, the same $A(j;\sigma) \geq 0$ condition developed in Proposition 1 ensures that player $j \in N^S(i;\sigma)$ prefers her current imitation strategy over leading, implying the $N^l(\sigma)$ population does not influence j's decision. Identify the most distant successor of i given σ as $\bar{j}(\mu_i^s)$ so that $A(\bar{j}(1);\sigma)$ is the value of $A(\bar{j};\sigma)$ for a σ in which $\mu_i^s = 1$ and $A(\bar{j}(n-1);\sigma)$ is the value of $A(\bar{j};\sigma)$ for a $\sigma \in H(i;g)$. The proof establishes that $A(\bar{j}(1);\sigma) > 0$, $A(\bar{j}(\mu_i^s);\sigma)$ is decreasing in μ_i^s and, since $B < 0$, $A(\bar{j}(n-1);\sigma) < 0$. At size $\mu_i^s = \bar{n}$, $\bar{j}(\bar{n}) \in N^S(i;\sigma)$ prefers following to leading while at size $\mu_i^s = \bar{n}+1$, $\bar{j}(\bar{n}+1) \in N^S(i;\sigma')$ prefers to lead. For $\sigma \in h_L^*(i, \bar{n}; g)$, no $j \in N^S(i;\sigma)$ can improve her payoff within the tree nor by leading and no $j \in N^l(\sigma)$ can improve her payoff by joining the i-led tree. Thus, the structure $\sigma \in h_L^*(i, \bar{n}; g)$ is a candidate Nash equilibrium.

A non-trivial set of alternatives and a preference for conformity, meaning $m > 1$ and $r_c > 0$ so that $\theta > 0$, are prerequisite for the existence of a non-trivial tree as a possible equilibrium. For $\theta = 0$, $A(j;\sigma) \leq 0$. When due to $m = 1$, the coordination problem is solved trivially without the leader-follower structure. When due to $r_c = 0$, there is nothing to be gained by delaying adoption.

Similar to the finding in Proposition 4, for $\mu_i^s = \bar{n} < n$, it is the candidate equilibrium size of the tree, according to Proposition 5, that is determined by universal parameters independent of the particular i or the characteristics of $\sigma \in H^*(i;g)$ or $g \in G(n)$.

4 Single and multiple leaders

Proposition 4 establishes that for $B \geq 0$, the set of Nash equilibria includes $H^*(i;g)$ and excludes $\sigma \in H(i;g) \backslash H'(i;g)$. Additionally, Proposition 1 excludes from the set of possible Nash equilibria the set of structures $\sigma \in h^*(i;g)$ for which $\{i,j\} = N^L(\sigma)$ and where either i or j can link to any member of the other leader's tree. There remain other multiple-leader structures, $\sigma \notin H(i;g)$, yet to be identified as included or excluded from the set of Nash equilibria. Likewise, for $B < 0$, Proposition 5 falls short of establishing $\sigma \in h^*_L(i,\bar{n};g)$ as an equilibrium but does establish that the structures of $h_L(i,\tilde{n};g)$ for $\tilde{n} \neq \bar{n}$ cannot be equilibria. This section establishes σ as an equilibrium structure if and only if $\sigma \in h^*_L(i,\bar{n};g)$ when $B < 0$ and identifies conditions for alternatives to $\sigma \in H^*(i;g)$ to be included among the Nash equilibrium structures when $B \geq 0$.

Let $h(i_A,i_B;g)$ be the set of σ given $g \in G(n)$ such that $\{i_A,i_B\} \in N^L(\sigma)$ with successor populations $N^S(i_h;\sigma) \neq \emptyset$ for $h = A,B$. Let $H(i_A,i_B;g)$ represent the subset of $h(i_A,i_B;g)$ such that $\{i_A,i_B\} = N^L(\sigma)$. In $h^*(i_A,i_B;g)$ and $H^*(i_A,i_B;g)$ are structures σ in which each successor employs the shortest path to the chosen leader. Let $\mu^s_h = |N^S(i_h;\sigma)|$ indicate the number of successors in the i_h-led tree. Without loss of generality, assume $\mu^s_A \geq \mu^s_B$.

For $h = A,B$, let j_h represent $j \in N^S(i_h;\sigma)$. With two non-trivial trees, there is a need to identify and label populations in the i_{-h}-led tree based on their position relative to j_h. Let $N^\alpha(j_h;\sigma)$ be the set of successors of i_{-h} who are of distance no greater than d_{j_h,i_h} and let $N^\beta(i_h;\sigma)$ be the set of successors of i_{-h} who are of a distance greater than $d_{j_h i_h}$. Let $\mu^\alpha_j = \mu^\alpha(j;\sigma) = |N^\alpha(j;\sigma)|$ and $\mu^\beta_j = \mu^\beta(j;\sigma) = |N^\beta(j;\sigma)|$. The node labels in Figure 2 identify the agent's position relative to player j with $\mu^\alpha_j = 2$ and $\mu^\beta_j = 4$.

4.1 $B < 0$

Given a leader i_A, consider the set of structures in which some or all of the $n - \overline{n}$ individuals not in $N^S(i_A;\sigma)$ form a second hierarchy.[14]

Recalling that $\theta = (m-1)r_c/r_e$, $B = \theta - (1-(n-1)^{-1})$, and $C = \theta - (1-(\mu^s_i)^{-1})$, let

$$D(j;\sigma) := (m-1)\left(r_c + (r_e + r_c)\mu^y_j\right) + ((m-1)r_c - r_e)\,\mu^x_j - r_e\mu^\alpha_j \tag{12}$$

[14] For any two $\{j_1,j_2\} \notin N^S(i_A;\sigma)$, the ability to form such a hierarchy is not assured by the assumption of strong connectivity since it may require the chain of links to pass through a member of $N^S(i_A;\sigma)$.

and

$$E(i_h;\theta,n,\sigma) := \theta - \left(1 - \frac{1}{\mu_h^s}\right) - \frac{\mu^\alpha(\bar{j}_h;\sigma)}{\mu_h^s}. \tag{13}$$

Proposition 6. $B < 0$ *is a necessary and sufficient condition for the set of* σ *in which the* $\bar{n}$ *players closest to* i *follow* i*, while the* $n - \bar{n} - 1$ *remaining players lead, is the set of equilibrium structures.*

Proof. See Appendix B □

The proof establishes that $D(j;\sigma) \geq 0$ ensures that player $j \in N^S(i_h;\sigma)$ for $h = A, B$ prefers her current follow strategy over leading. As with $A(j;\sigma)$, $D(j;\sigma)$ is not necessarily monotonic in the player's distance from the leader and yet $D(\bar{j}(\mu_i^s);\sigma) \geq 0$ implies $D(j;\sigma) \geq 0$ for all $j \in N^S(i;\sigma)$. For $E(i_h;\sigma) \geq 0$, the most distant successor of i_h prefers her current position in the i_h-led tree to leading.

Recognize that $D(j;\sigma) = A(j;\sigma) - r_e\mu_j^\alpha$ and that $E(i_h;\theta,n,\sigma) = C(\mu_h^s;\theta) - \mu^\alpha(\bar{j}_h)/\mu_h^s$. The minimum threshold value on θ to maintain a second tree in the presence of an existing tree is greater than the threshold necessary to maintain an equal sized tree in which all of the non-members lead. This is because player j gains a distance advantage over the $N^\alpha(j;\sigma)$ population when leading but not when following i.

Since $\mu_j^\alpha \geq 1$ for all $\sigma \in h(i_A, i_B; g)$,

$$\left(1 + \frac{\mu^\alpha(\bar{j}_h) - 1}{\mu_h^s}\right) \geq 1 > \left(1 - \frac{1}{n-1}\right). \tag{14}$$

Thus, $E(i_h;\sigma) \geq 0$ imposes a higher threshold for θ than does the condition $B \geq 0$. A multiple leader structure cannot be an equilibrium when $B < 0$. The set of Nash equilibria are drawn from $h_L(i, \mu_i^s; g)$ only. The set $h_L^*(i, \bar{n}; g)$ constitutes the set of Nash equilibria.

There are two features of this solution worth exploring. First, starting from the structure $\sigma \in h_L^*(i_A, \overline{n}; g)$ and $i_B \in N^L(\sigma)$, the condition $E(i_B;\sigma) < 0$ for all $\mu_B^s > 0$ makes it imprudent for any of the remaining $n - \bar{n} - 2$ leading players to instead follow i_B since following would lower the player's own expected payoff.

Second, consider the impact of a second non-trivial tree on the original i_A-led structure. By the nature of the structure, $\mu^\alpha(\bar{j}_A) = 0$ if and only if $\mu_B^s = 0$. Otherwise, $\mu^\alpha(\bar{j}_A) \geq 1$ for $\mu_B^s > 0$. For $\mu_B^s = 0$, then $E(i_A) = C(\mu_A^s)$, the equation used to solve $\overline{n}$. For $\mu_B^s > 0$, there

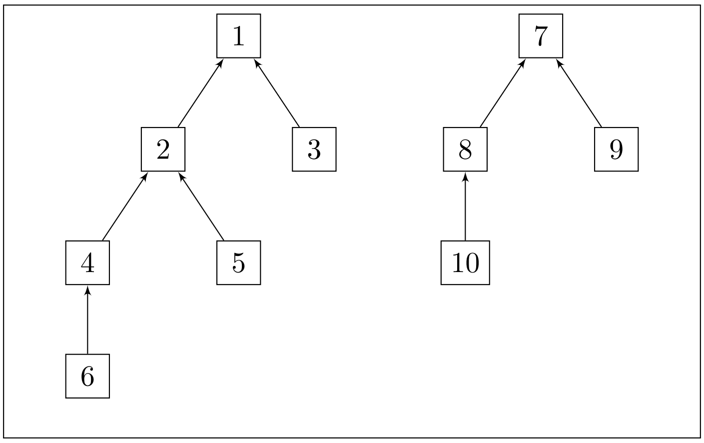


Figure 6: An example of $\sigma \in H(i_A, i_B; g)$ with $i_A =$ player 1, $i_B =$ player 7, and $d\mu = 2$. With $d_{6,1} = 3$ and $d_{10,7} = 2$, $\mu^\beta(10) = 1$. If $N^d(10; g) = \{5, 8\}$, then $\mu^\beta_A(10) = 0$.

is no positive value of μ^s_A for which $E(i_A) \geq 0$ given $B < 0$. The result points to a fragility of the $\sigma \in h^*_L(i, \bar{n}; g)$ equilibrium structure. A single deviation that results in a second non-trivial tree undermines the entire i_A-led structure.

Theorem 1. *Given a non-trivial choice, a preference for conformity, and a strongly connected population, for every $i \in N$ there exists an equilibrium structure with player i as the only leader.*

Proof. Follows directly from Proposition 4 and Proposition 6. □

Propositions 4 and 6 establish that a single-leader structure led by player i for all $i \in N$ is among the equilibrium set. The tree structure is supported by the presence of a meaningful choice between options and a desire to conform, reflected in $\theta > 0$.

4.2 $B \geq 0$

For $\sigma \in H(i_A, i_B; g)$, let $d\mu = \mu^s_A - \mu^s_B$ so that $d\mu$ captures the population size differential between the two trees. Let $N^{AB}(i_A, i_B; \sigma)$ represent the set of followers possessing potential links to predecessors in both trees. A strongly connected g ensures that some member of the i_h-led tree is able to link to some member of the i_{-h}-led tree, $h = A, B$.

Let $\sigma' = \sigma_{-j_h} \times \sigma'_{j_h}$ be the structure produced by j_h switching predecessors in order to become a member of the i_{-h}-led tree. The alternative structure identifies populations $N^\beta_{-h}(j_h) = N^\beta(j_h; \sigma')$ and $N^y_{-h}(j_h) = N^y(j_h; \sigma')$. The former is the population of players in j_h's current tree who are more distant from i_h than is j_h from i_{-h} in σ'. The latter is the population in the i_{-h} led tree more distant from i_{-h} than j_h in σ'. Let $\mu^\beta_{-h}(j_h) = |N^\beta_{-h}(j_h)|$ and $\mu^y_{-h}(j_h) = |N^y_{-h}(j_h)|$.

As illustration, for player 10 in Figure 6, $d_{10,7} = 2$. In the player 1-led tree, only player 6 has a distance $d_{6,1} = 3 > d_{10,7}$ and thus $\mu^{\beta}(10) = 1$. If $N^d(10; g) = \{3, 8\}$, then $\mu^{\beta}_A(10) = 0$ and $\mu^{y}_A(10) = 1$ since, as a successor of player 1, $d_{10,1} = 2$ and no successor of player 7 has $d_{j,7} > 2$.

Let

$$F_A(j_A; \theta, m, \sigma) := \theta - \frac{\mu^{\beta}_B(j_A) - \mu^{\beta}(j_A) - m(\mu^{y}(j_A) - \mu^{y}_B(j_A))}{d\mu - 1 - \mu^{s}(j_A))}, \tag{15}$$

$$F_B(j_B; \theta, m, \sigma) := \frac{\mu^{\beta}(j_B) - \mu^{\beta}_A(j_B) + m(\mu^{y}(j_B) - \mu^{y}_A(j_B))}{d\mu + 1 + \mu^{s}(j_B)} - \theta. \tag{16}$$

For $\sigma \in H(i_A, i_B; g)$, members of $N^{AB}(i_A, i_B; \sigma)$ have the option to switch leaders. All followers have the option to lead. Let $H^{+}(i_A, i_B; g)$ be the subset of $H^{*}(i_A, i_B; g)$ satisfying the three conditions of Proposition 7.

Proposition 7. $B \geq 0$ *allows multiple leader equilibrium structures when these feasible conditions are met:*

1. no leader is capable of linking directly with a member of another tree,
2. the most distant follower in each tree prefers following to leading despite the presence of other trees, and
3. that all followers capable of linking to a member of another tree prefer their current position.

Proof. See Appendix B □

The proof of Proposition 7 confirms that $D(j; \sigma) \geq 0$ ensures that player $j \in N^{S}(i_h; \sigma)$ for $h = A, B$ prefers her current imitation strategy over leading so that for $E(i_h; \sigma) \geq 0$ the most distant successor of i_h prefers her current position in the i_h-led tree to leading. The proof also establishes that $F_h(j_h) \geq 0$ ensures that player $j_h \in N^{AB}(i_A, i_B; \sigma) \cap N^{S}(i_h; \sigma)$ prefers her current imitation strategy over any available alternate imitation strategy that places her in the i_{-h}-led tree.

The higher minimum threshold to satisfy $E(i_h; \sigma) \geq 0$ rather than $B \geq 0$ follows from the weaker reward to following in a multi-leader setting. Consider $\sigma_1 \in H^{*}(i; g)$ and $\sigma_2 \in$

$H^*(i_A, i_B; g)$. Though the reward to following in a multi-leader setting depends on the size of various position-specific relative populations, the structure-independent differential

$$\mathbb{E}(\pi(\bar{j}, \sigma_1) - \pi(\bar{j}_h, \sigma_2)) = \frac{m-1}{m}(n - 1 - \mu^s_h)r_c$$

reveals a reward to following in the multi-leader σ_2 that declines relative to following in the single-tree structure of σ_1 as the size of $\bar{j}_h$'s affiliated tree decreases.

The attraction to lead, on the other hand, depends only on the total size of the follower population and not on how the followers are distributed among leaders. Let $\sigma^{'}_h$, $h = 1, 2$ represent the structure produced when $\bar{j}_h$ switches to lead. The differential

$$\mathbb{E}(\pi(\bar{j}; \sigma^{'}_1) - \pi(\bar{j}_h; \sigma^{'}_2)) = r_e/m$$

is independent of n and μ^s_h. The non-zero value reflects that with two trees, there is one less follower, i_B. Thus, maintaining followers in a multi-leader setting requires a θ that more strongly penalizes leading.

The maximum possible lower bound on θ is generated by $\mu^\alpha(j_h) = n - 3$ and $\mu^s_h = 1$. A sufficient condition to ensure no follower prefers to lead is $E' \geq 0$ where

$$E'(\mu^s_h; \theta, n) := \theta - (n - 3)\,.$$

Though already established by Proposition 1, compute F_h for i_A and i_B and the condition $N^d(i_h; g) \cap \{N^S(i_{-h}; \sigma), i_{-h}\} \neq \emptyset$ violates $F_h \geq 0$. That is, if either leader can link with a member of the other tree, $\sigma \in H(i_A, i_B, ; g)$ cannot be an equilibrium. Given $E(i_h) \geq 0$, $\sigma \in H^*(i_A, i_B; g)$ is an equilibrium only if every $j_h \in N^{AB}(i_A, i_B; \sigma)$ has $F_h(j_h; m, r_c, r_e) \geq 0$.

Generally, $F_A \geq 0$ and $F_B < 0$ so that individuals in the smaller of the two trees prefer joining the larger tree to gain the larger conformity reward. There are two scenarios, described here informally based on the formal findings in the proof of Proposition 7, in which followers prefer their position in a smaller tree to a switch to the larger tree. Such exceptions are the basis that produce $\sigma \in H^+(i_A, i_B; g)$.

The first exception consists of a structure in which (i) the less populous tree has a population concentrated near i_B and (ii) the more populous i_A-led tree has a bulge so that there is a large

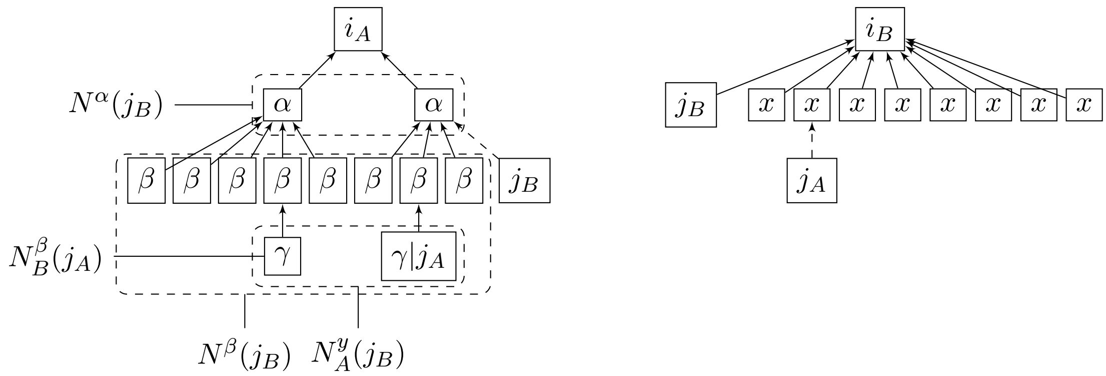


Figure 7: Illustration of Example 4 as a two-leader structure equilibrium. The dashed link is the position available to j_h in the i_{-h} tree. Here, $d\mu = 3$, $\mu_B^\beta(j_A) = 1$, $\mu^\beta(j_B) = 10$, $\mu_A^y(j_B) = 2$, $\mu_A^s = 12$, and $\mu_B^s = 9$. Let $m = 2$, then $F_A \geq 0$ implies $\theta \geq \frac{1}{2}$, $F_B \geq 0$ implies $\theta \leq \frac{3}{2}$, and $E(i_B) \geq 0$ implies $\theta \geq \frac{11}{9}$. There is nontrivial support $\theta \in [\frac{11}{9}, \frac{3}{2}]$ for which $\sigma \in H^+(i_A, i_B; g)$ is a Nash equilibrium.

number of followers at a distance just below the most distant follower of i_B. The structure depicted in Figure 7 conforms to these features. Example 4 illustrates how this structure is advantageous to j_B.

Example 4. With $N^{AB}(i_A, i_B; \sigma) = \{j_A, j_B\}$, the structure depicted in Figure 7 is in $H^+(i_A, i_B; g)$ for conforming values of θ. Player j_B benefits from the possibility of holding a distance advantage over the large population of β-labeled players. She would lose that advantage were she to switch to the i_A-led tree. Player j_A does not gain advantage over the β population with a switch to the i_B-led tree and thus also prefers to stay. The large population of x-labeled players is needed to counter the benefits to j_B of the α- and γ-labeled populations were she to switch.

Formally, with $\mu^y(j_B) = \mu^s(j_B) = 0$, $F_B(j_B; \theta) \geq 0$ reduces to

$$\theta \leq \overline{\theta} \equiv \frac{\mu^\beta(j_B) - m\mu_A^y(j_B)}{d\mu + 1}. \tag{17}$$

In addition, with $\mu^y(j_A) = \mu^s(j_A) = 0$, $F_A(j_A; \theta) \geq 0$ reduces to

$$\theta \geq \underline{\theta} \equiv \frac{\mu_B^\beta(j_A)}{d\mu - 1}. \tag{18}$$

The two conditions define upper and lower bounds on permissible θ to have $\sigma \in H^+(i_A, i_B; g)$. In general, each member of $N^{AB}(i_A, i_B; \sigma)$ imposes her own threshold on θ. For a particular σ and g, the support producing $\sigma \in H^+(i_A i_B; g)$ may be empty, with $\overline{\theta} \leq \underline{\theta}$, or may have θ fall outside of the support. The lower bound on θ established by the condition $E(i_B) \geq 0$ can be greater than that produced by $F_A \geq 0$, in which case the most distant follower of i_B will lead before j_A considers switching to the i_B-led tree.

The second exception, illustrated in Example 5, consists of structures in which a follower in the less populous i_B-led tree has a sufficiently large $\mu^y(j_B)$ such that the larger conformity reward offered by the i_A-led tree does not compensate for the loss of a distance advantage over the $N^y(j_B;\sigma)$ population.

Example 5. With $N^{AB}(i_A, i_B; \sigma) = \{6, 9\}$ and $N^d(9; g) = \{5, 7\}$, the structure σ depicted in Figure 6 satisfies $F_B \geq 0$ with $\theta \leq 1 + \frac{m}{3}$. The $E(i_B) \geq 0$ condition is satisfied with $\theta \geq 2$. The condition $E(i_A) \geq 0$ is less stringent, requiring only that $\theta \geq 7/5$. To support $\sigma \in H^+(i_A, i_B; g)$ as an equilibrium requires $m \geq 3$. For, say, $m = 4$, then $r_c/r_e \in [6/9, 7/9]$ produces a non-empty $H^+(i_A, i_B; g)$. In this range the certainty of the conformity reward discourages player 10 from leading while the relatively high premium for leading pays enough to keep 9 from switching to the greater conformity reward offered by the larger player 1-led tree. In this example, $\sigma \in H^+(i_A, i_B; g)$ is preserved as the number of alternatives increases by a conforming r_c/r_e where $\lim_{m\to\infty} r_c/r_e \in (0, 1/3]$.

The structure in Figure 6 also illustrates how $\sigma \in H^+(i_A, i_B; g)$ depends on the possible links between players. Multi-leader structures are feasible as equilibrium (though not assured) due to g-identified impediments to association. Consider the same σ but a g' such that $N^{AB}(i_A, i_B; \sigma) = \{2, 9\}$ instead of $N^{AB}(i_A, i_B; \sigma) = \{6, 9\}$. Because $F_A(2; \theta) < 0$, then $\sigma \notin H^+(i_A, i_B; g)$. By switching trees, player 2 retains her distance advantage over players $\{4, 5, 6\}$ while forming a larger tree under player 7 than under player 1. Though player 2 is not a leader, the feature that attracts her to join the player 7-led tree is the same as would attract player 1: to gain larger conformity reward while retaining a distance advantage over her population of successors.

5 Non-conforming environments

5.1 Decreasing $\lambda(\mu)$

Consider, again, the nonlinear payoff function of (1),

$$\pi_{NL}(i;\sigma) = \phi(\mu_i^c) + \psi(\mu_i^e)$$

with $\lambda(\mu) = \phi(\mu)/\psi(\mu)$, $\phi(0) = \psi(0) = 0$, and $\phi'(\mu), \psi'(\mu) > 0$. Recall that $B_{NL} \geq 0$ ensures $\sigma \in H^*(i;g)$ is an equilibrium when $\lambda'(\mu) \geq 0$ but not when $\lambda'(\mu) < 0$, according to Proposition 4 and Corollary 1, respectively. With $\lambda'(\mu) \geq 0$, dissatisfaction with following, if present, originates with the most distant follower. In contrast, as seen if Figure 4, $\lambda'(\mu) < 0$ produces local convexity in $A(j;\sigma)$ with respect to μ_j^s that causes middle distance followers with $\mu_j^s > 0$ and low μ_j^y to prefer leading while the most distant follower, with $\mu_j^s = \mu_j^y = 0$, prefers following.

Recall from Section 3, given $\sigma \in H^*(i;g)$ player j's $N^S(j;\sigma)$ population provides a foundational reward $\phi(\mu_j^s) + \psi(\mu_j^s)$ independent of j's decision to lead or follow. Players with a substantial $N^y(j,\sigma)$ population have a strong incentive to follow as it produces a large positive A_2 component in (5). For those players with μ_j^y at or near zero, the decision between following and leading comes down to how to best position the $N^x(j;\sigma)$ population to supplement the $N^S(j;\sigma)$-assured reward. The $N^x(j;\sigma)$ population contributes towards a certain conformity reward when j follows and towards an uncertain preemptive reward when j leads.

The sign of $\lambda'(\mu)$ indicates the change in relative marginal contribution between conformity and preemption rewards as μ increases. With $\lambda'(\mu) = 0$ the relative contribution is constant. For $\lambda'(\mu) > 0$, the relative importance of conformity relative to preemption increases as μ_j^s increases so that the contribution of conformity is relatively weak when μ_j^s is small and strongest when μ_j^s is large. If the most distant follower, despite being in the position that most favors preemption over conformity, still prefers following to leading, then so, too, do all other followers.

For $\lambda'(\mu) < 0$, the relative importance of conformity relative to preemption decreases as μ_j^s increases so that the relative contribution of conformity is at its strongest when μ_j^s is small and becomes weaker as μ_j^s increases. A follower with $\mu_j^s > 0$ extracts the substantial component of the conformity from her successors. The marginal contribution of positioning the $N^x(j;\sigma)$ population to contribute to the conformity reward declines relative to the preemption reward as

μ_j^s increases. Taking advantage of the certainty of her successors in establishing her own smaller hierarchy, the follower may find it beneficial to concede conformity with the larger population. If lucky, player j as a leader of her own hierarchy chooses the same as i, adding the $N^x(j;\sigma)$ population to her own $N^s(j;\sigma)$ successors for the large preemption reward.

Given $B_{NL} \geq 0$ settings with $\lambda'(\mu) < 0$ might, in contrast to $\lambda'(\mu) \geq 0$, instead tend to organize into multiple small-conforming groups, each headed in its adoptions by an autonomous leader. Also in contrast to $\lambda'(\mu) \geq 0$, among the features that make the multiple leader structures prevalent when $\lambda'(\mu) < 0$ is the connectivity of $g \in G(n)$. A single leader can be sustained despite $\lambda'(\mu) < 0$ when either all followers have zero or near zero values for μ_j^s or when those followers with $\mu_j^s > 0$ all also have $\mu_j^y > 0$. A g with few links, forcing followers into a small number of long imitation branches, cannot sustain a single leader structure as equilibrium.

5.2 Excessive popularity

Consider a penalty for excessive popularity in the nature of Arthur (1994).

Example 6. For $g \in G(n)$, consider a linear increasing conformity reward for a population not in excess of $n^\dagger$,

$$\phi(\mu^c) = \begin{cases} r_c\mu^c & \text{for } \mu^c + 1 \leq n^\dagger \\ 0 & \text{otherwise} \end{cases} \tag{19}$$

with $n/2 \leq n^\dagger < n - 2$. The result is an environment that supports a tree of size $\mu_A^s \leq n^\dagger - 1$. The environment imposes a penalty for excessive conformity should $f_{i_A}(a_{i_A}) = f_{i_B}(a_{i_B})$, though the reward to early preemptive adoption remains. Those not in the primary tree may lead or join to form a second tree. For a potential second tree, for illustrative purposes assume an equal number of successors at each distance through the first $n_B - 1$ successors so that for $n_A > n_B$, the additional players are of greater distance from i_A than is the most distant follower of i_B. In this case,

$$\theta \geq \frac{m}{m-1}\left(1 - \frac{1}{n - n^\dagger - 1} + \frac{\mu^\alpha(j_B;\sigma)}{n - n^\dagger - 1}\right) \tag{20}$$

ensures $\sigma \in \{H^*(i_A, i_B; g) | \mu_A^s = n^\dagger - 1\}$ is an equilibrium. The condition in (20) supports the formation of a population of size $\mu_A^s = n^\dagger - 1$ as successors of some i_A leader and is necessary to have $\mu_B^s = n - n^\dagger - 1$ succeed some i_B leader. The threshold on θ is higher than that produced

Players	Contacts	
	1	2
i	x	s_1
j	i	s_2
x	i	j
s_1	j	s_2
s_2	j	s_1

Table 5: Links of g in an example for which not all $\sigma \in \{H^*(i;g)\}_{i \in N}$ are subgame perfect equilibrium.

by the condition $E(i_B) \geq 0$. When this condition does not hold, there is no interior value to $0 < \mu_B^s < n - n^\dagger - 1$ that is an equilibrium.[15]

5.3 Sequential play

5.3.1 Subgame perfect hierarchies

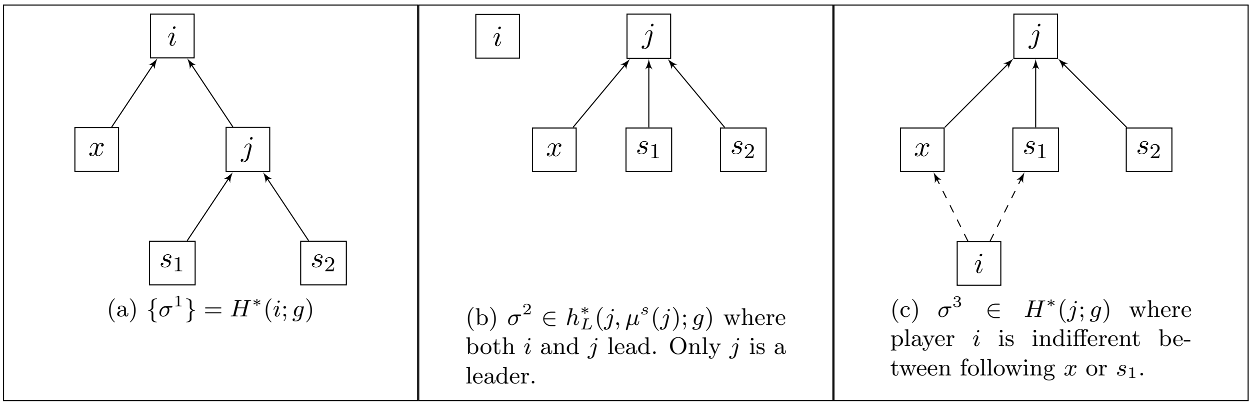
The single leader structure can also be supported as a subgame perfect equilibrium (SPE) in a game with σ established through sequential moves. Typically, the first mover can establish herself as the leader and the remaining population adopts the following strategy to best accommodate this reality generating the same single leader Nash equilibrium structures produced by simultaneous play. There are instances, though, in which the first mover does not end up as the leader in the equilibrium structure, as illustrated in Example 7 below. The inability of player i to lead in a SPE indicates a susceptibility of the Nash equilibrium $\sigma \in H^*(i;g)$ to disruption by a player whose deviation from Nash equilibrium play, in a cascade of best responses, would lead to a different Nash equilibrium $\sigma \notin H^*(i;g)$ favored by the original deviant player.

Example 7. Table 5 lists the potential links of network g. Each member of the population has two contacts. Figure 8 depicts $\{\sigma^1\} = H^*(i;g)$ and $\sigma^3 \in H^*(j;g)$, both Nash equilibria. Given i, j, x, s_1, and s_2 as the order of play, σ^3, rather than σ^1, is the SPE despite i's first mover option to establish herself as a leader. The full extensive-form game is included in Appendix D.1. Best response to player i leading produces $\sigma^2 \in h_L^*(j, \mu^s(j); g)$ in which both i and j lead but with j attracting all of the followers. Two features present in σ^1 and g are essential to exclude it

[15]For $\mu_B^s < n - n^\dagger - 1$, the necessary condition to have the most distant successor of i_B remain a follower is

$$\theta \geq \frac{m}{m-1}\left(1 + \frac{\mu^\alpha(j_B;\sigma) - 1}{\mu_B^s}\right) + \frac{(m-2)r_c}{(m-1)r_e}\left(\frac{n - n^\dagger - 1}{\mu_B^s} - 1\right)$$

for which, with $\mu^\alpha(j_B;\sigma) \geq 1$ and $\mu_B^s \leq n - n^\dagger - 1$, the threshold on θ declines as μ_B^s is increased.



(a) $\{\sigma^1\} = H^*(i;g)$

(b) $\sigma^2 \in h^*_L(j, \mu^s(j); g)$ where both i and j lead. Only j is a leader.

(c) $\sigma^3 \in H^*(j;g)$ where player i is indifferent between following x or s_1.

Figure 8: The tree structures of σ^1 and σ^3 are both Nash equilibria structures based on g as defined in Table 5. For for moves in the order i, j, x, s_1, then s_2, only σ^3 is a SPE . The structure σ^2 reflects the best response by players x, s_1, and s_2 if faced with both i and j leading. The fact that j prefers σ^2 to σ^1 is what undermines player i's leadership when considering a cascade of best responses.

from the set of SPE. First, player j's has an advantage in attracting followers despite i moving first. Because s_1 and s_2 have no choice but to follow j, j has the larger population of followers regardless of x's decision regarding whom to follow. This compels x to follow j. Second, the potential defector from the actions producing σ^1 must be motivated to defect despite i's lead as is true here with $\pi(j;\sigma^2) > \pi(j;\sigma^1)$. The motivation in this example comes from player x. Player x best responds by following j when both i and j lead.

Similar to the best response cascade discussed in Heal and Kunreuther (2010), were the population to start from σ^1, the cascade of best responses to the single deviation by player j transitions the population from σ^1 to σ^3. As is trivially exemplified in the $n = 2$ example, a Nash equilibrium can be susceptible to a best response cascade that just as easily reverses in direction. This does not contribute to identifying the more "fragile" Nash equilibria prone to transition to a more stable alternative Nash equilibrium. SPE offers such a refinement as to avoid cascades that would reverse from the destination structure.

5.3.2 Multiple hierarchies as subgame perfect

Appendix D.2 offers an analysis of the $\sigma \in H^*(i_A, i_B; g)$ structure with $F_B \geq 0$ in support of the two leader equilibrium illustrated in Example 5. Recall that $N^{AB} \cap N^S(7) = \{9\}$ and $\mu^y(9) > 0$ supports a $H^*(i_A, i_B; g)$ structure as a Nash equilibrium. Such a σ is not a SPE regardless of the order of play within the players of the player 7-led tree. The reason is that as the only conduit through which player 7 and the $N^S(7, \sigma)$ population can join $N^S(1, \sigma)$, player 9's switch to join the player 1-led tree enables the remainder of $N^S(7, \sigma)$ to also join in following $N^S(1, \sigma)$ in a cascade of best responses. The end result is to player 9's advantage as she now precedes player 7 and the entire former $N^S(7, \sigma)$ population.

6 Conclusion

The model offers a structure by which to consider the social structures that may arise to support trends among subjective products. Driven by a desire for conformity, personal contacts provide scaffolding upon which the population establishes information pathways facilitating both informed decisions and channels with which to exert influence. With the tacit support of the entire population, a leader identifies the choice for adoption. The choice disseminates to and through followers via a network of imitations.

A desire to preempt a trend means that the number of followers and whether multiple leaders can be present in equilibrium depends on the tradeoff between following an existing leader or acting autonomously in the presence of that leader. The parameter θ captures this tradeoff in the linear rewards model. Interestingly, the term is relevant to the decisions of the population's followers, not its leader(s). Everyone wants to be the leader. It is the willing participation of the followers that makes the structure an equilibrium.

Left unresolved in the current analysis is the process by which the coordinating Nash equilibrium structure can emerge. The substantial coordination involved, confounded by the asymmetry of the equilibrium payoff, makes the realization of an equilibrium structure in a single round of play highly unlikely. The analysis developed here rests on the possibility that coordination can emerge as the consequence of building consistency in player relationships. Computational analysis points processes by which the coordinating structure of the static Nash equilibrium solution can emerge as the consequence of reactive path dependent repeated play. Whether the identified Nash equilibria emerge as the final product of dynamic equilibria play in a repeated game is another question altogether.

A Appendix: Foundations

Formally, define

- $h(i;g) = \{\sigma | i \in N^L, N^S(i;\sigma) \neq \emptyset\}$ as the set of structures in which i leads,
- $H(i;g) = \{\sigma | N^L(\sigma) = \{i\}, N^S(i;\sigma) = N \backslash \{i\}\}$ as the set of structures in which i uniquely leads,
- $h_L(i, \mu_i^s; g) = \{\sigma \in h(i;g) | N^L(\sigma) = N \backslash N^S(i;\sigma)\}$ as the set of structures in which i has μ_i^s followers and is the unique leader,
- $h(i_A, i_B; g) = \{\sigma \in h(i_A; g) \cap h(i_B; g)\}$ as the set of structures in which $\{i_A, i_B\}$ are leaders,
- $H(i_A, i_B; g) = \{\sigma \in h(i_A, i_B; g) | N^L(\sigma) = \{i_A, i_B\}\}$ as the set of structures in which only $\{i_A, i_B\}$ lead and are leaders,
- $N^c(i;a) = \{j \in N \backslash \{i\} | O_i = O_j\}$ as, for action profile a, the set of conforming adopters,
- $N^e(i;a) = \{j \in N^c(i;a) | d_j > d_i\}$ as, for action profile a, the set of ensuing adopters,
- $N^S(i;\sigma) = \{j \in N | \sigma_{ji} = 1 \text{ or } \sigma_{jj_1} = \ldots = \sigma_{j_\tau i} = 1\}$ as, for structure σ, the set of players who are successors to i,
- $N^x(j;\sigma) = \{j_x \in N^S(i;\sigma) | d_{xi} \leq d_{ji}\}$ as, for structure σ, the set of players who are as close or closer to leader i as is j,
- $N^y(j;\sigma) = \{j_y \in N^S(i;\sigma) \backslash N^S(j,\sigma) | d_{yi} > d_{ji}\}$ as, for structure σ, the set of players who are farther from leader i than is j but not successor to j,
- $N^\alpha(j_h;\sigma) = \{j_\alpha \in N^S(i_{-h};\sigma) | d_{j_\alpha i_{-h}} \leq d_{j_h i_h}\}$ as, for structure σ, the set of players who are as close or closer to leader i_{-h} as is j to i_h,
- $N^\beta(j_h;\sigma) = \{j_\beta \in N^S(i_{-h};\sigma) \backslash N^S(j,\sigma) | d_{j_\beta i_{-h}} > d_{j_h i_h}\}$ as, for structure σ, the set of players who are farther from leader i_{-h} than is j to i_h,
- $N^{AB}(i_A, i_B;\sigma) = \{j | N^d(j;g) \cap \{i_A, N^S(i_A;\sigma)\} \neq \emptyset, N^d(j;g) \cap (i_B, N^S(i_B;\sigma)\} \neq \emptyset\}$ as, for structure σ, the set of players with potential links to members of both of the i_A-led tree and the i_B-led tree,

and recognize that for $g \in G(n)$

- $N^d(i_h; g) \cap \{N^S(i_{-h}; \sigma), i_{-h}\} = \emptyset$ implies $N^{AB}(i_A, i_B; \sigma) \cap N^S(i_h; \sigma) \neq \emptyset$.

An * on the set of structures indicate that all followers imitate the contact offering the shortest distance to the leader, that is $a_j = \underset{N^d(j;g)}{\arg\min}\, d_{ji}\ \forall j \in N^s(i; \sigma)$. The sets $h^*_L(i; g)$ and $h^*(i_A, i_B; g)$ have the additional condition that the $N^l(\sigma)$ population are at least as distant from the leader as is the most distant follower measured in the potential links of g, $d_{ij} \geq d_{i\bar{j}(\mu^s_{i_h})}$ for $j \in N^l(\sigma)$, $h = \emptyset, A, B$.

A.1 Utility of interactions

Individuals face a discrete choice in which they receive utility from the interaction between their own choice and the choices of other members in the population. Let the $m \times \bar{d}$ matrix ω_i denote the adoption of an option with element $w_{i,o,d} = 1$ if player i adopts option $o_i \in O$ at distance $d_i = d$. Otherwise, $\omega_{i,o,d} = 0$. Let $\omega_{-i} = (\omega_1, \ldots, \omega_{i-1}, \omega_{i+1}, \ldots, \omega_n)$ represent the choices of all agents other than i. Individual utility can be defined broadly as the sum of three elements,

$$V(\omega_i) = u(\omega_i) + S(\omega_i, \omega_{-i}) + \epsilon(\omega_i).$$

The current analysis considers only the social utility associated with a choice, $S(\omega_i, \omega_{-i})$, setting the innate preferences over the different options, $u(\omega_i)$, and the idiosyncratic random element of utility, $\epsilon(\omega_i)$, each to zero.[16]

Let the $n \times \bar{d}$ matrix Ω_i denote the possession of an option with element $\Omega_{i,o,d} = 1$ when player i adopts option $o_i \in O$ at distance $d_i \leq d$. Otherwise, $\Omega_{i,o,d} = 0$. Let

$$\mu_i = \sum_{j \neq i} \omega_j$$

and

$$\nu_i = \sum_{j \neq i} \Omega_j$$

so that μ_i denotes the aggregate choice for each option at each distance and ν_i denotes the cumulative aggregate choice at each distance.

[16]Also excluded from the utility function is a direct reward from early adoption. The coordination problem of interest is distinct from the utility some people might receive simply by being the first to try new products.

The complementarities of the social choice depends only on the two measures of popularity,

$$\mu_i^c = \mathbf{1}'\mu_i'\omega_i\mathbf{1}$$

and

$$\mu_i^e = \mu_i^c - \mathbf{1}'\omega_i\nu_i'\omega_i\mathbf{1}.$$

Let

$$S(\omega_i, \mu_i^c, \mu_i^e) = \phi(\mu_i^c) + \psi(\mu_i^e),$$

then linearity with $\phi(x) = r_c x$ and $\psi(x) = r_e x$ produces constant cross partials

$$\frac{\partial^2 S(\omega_i, u_i^c, \mu_i^e)}{\partial\omega_{i,o,d}\partial\mu_{i,o,d}} = r_c \text{ and } \frac{\partial^2 S(\omega_i, u_i^c, \mu_i^e)}{\partial\omega_{i,o,d}\partial\nu_{i,o,d}} = r_e, \forall i, o, d$$

so that dependence across players is captured by the two constant coefficients.

A.2 Formal Statement of Lemmas

Lemma 1. *For every $g \in G(n)$ there exists $H(i;g) \neq \emptyset$ for all i.*

Lemma 2. *For $\{\sigma, \sigma'\} \in H(i;g)$ with $\sigma_{-j} = \sigma'_{-j}$ and $\{a_j, a'_j\} \in N^d(j;g)$, then for $\mu^x(j;\sigma) \leq \mu^x(j;\sigma')$,*

$$\pi(j;\sigma) \begin{cases} = \pi(j;\sigma') & \textit{if } \mu^x(j;\sigma) = \mu^x(j;\sigma'), \\ > \pi(j;\sigma') & \textit{if } \mu^x(j;\sigma) < \mu^x(j;\sigma'). \end{cases}$$

B Appendix: Propositions and proofs

B.1 Formal statement and proof of Proposition 1 and Corollary 1

Proposition 1. *For $\sigma \in H(i;g)$, $\sigma' \in h^-(i,\sigma;g)$, and $\lambda'(\mu) \geq 0$, the condition $B_{NL} \geq 0$ is a necessary and sufficient condition for $\pi_{NL}(j,\sigma) \geq \pi_{NL}(j,\sigma')$ for all $j \in N^S(i;\sigma)$.*

Proof. Let σ_{-j} indicate the strategies of all players in $N\backslash\{j\}$. For $\sigma \in H(i;g)$, let $\sigma' = \sigma'_j \times \sigma_{-j}$ and $\sigma'_{jj} = 1$ producing $\sigma' \in h^-(i,\sigma;g)$. Let $\mu_j^h = \mu^h(j;\sigma) = |N^h(j;\sigma)|$ for $h = x, y, s$ so that relational populations are identified according to the structure σ. Recall $\phi'(\mu) > 0$ and

$\psi'(\mu) > 0$. For player $j \in N^S(i;\sigma)$,

$$\pi(j;\sigma) = \phi(n-1) + \psi(\mu_j^y + \mu_j^s).$$

The payoff to j when leading is uncertain due to the uncertainty in the outcome of whether $f_i(a_i) = f_j(a_j)$. Expectations are taken over the possible maps f with

$$\mathbb{E}(\pi(j;\sigma^{'})) \quad = \quad \frac{1}{m}(\phi(n-1) + \psi(\mu_j^x + \mu_j^y + \mu_j^s)) + \frac{m-1}{m}(\phi(\mu_j^s) + \psi(\mu_j^s)) \tag{21}$$

The condition $A_{NL}(j;\sigma) \geq 0$, derived from $\mathbb{E}(\pi(j;\sigma) - \pi(j;\sigma')) \geq 0$, ensures that player $j \in N^S(i;\sigma)$ prefers her position as a follower of i over leading.

The condition $B_{NL} \geq 0$ is equivalent to $A(\bar{j};\sigma) \geq 0$ for $\bar{j} = \underset{j \in N^S(i;\sigma)}{\operatorname{argmax}} d_{ji}$. For $\bar{j}$, $\mu^y(\bar{j}) = \mu^s(\bar{j}) = 0$, leaving

$$A_{NL}(\bar{j};\sigma) \quad = \quad (m-1)[\phi(n-1) - \phi(n-2)] + (m-1)\phi(n-2) - \psi(n-2) \geq 0.$$

The first term is strictly positive. $B_{NL} \geq 0$ implies

$$(m-1)\phi(n-2) - \psi(n-2) \geq -(m-1)[\phi(n-1) - \phi(n-2)].$$

For follower j,

$$A_{NL}(j;\sigma) \quad = \quad A_1 + A_2(\mu_j^y, \mu_j^s) + A_3(\mu_j^s) \tag{22}$$

where

$$A_1 = (m-1)[\phi(n-1) - \phi(n-2)]$$
$$A_2(\mu_j^y, \mu_j^s) = [\psi(\mu_j^y + \mu_j^s) - \psi(\mu_j^s)]$$
$$A_3(\mu_j^s) = [(m-1)\lambda(n-2) - 1]\psi(n-2) - [(m-1)\lambda(\mu_j^s) - 1]\psi(\mu_j^s).$$

Recall $\lambda(\mu) = \phi(\mu)/\psi(\mu)$ and $n-2 = \mu_j^x + \mu_j^y + \mu_j^s$. Observe that for $\mu_j^s = n-2$, $A_2(0, n-2) = 0$ and $A_3(n-2) = 0$.

The derivative of $A_3(\mu)$ yields

$$A_3'(\mu) = -(m-1)\psi(\mu)\lambda'(\mu) - [(m-1)\lambda(\mu) - 1]\psi'(\mu).$$

Recall $\psi(0) = 0$ and $\psi'(\mu) > 0$. The sign of the first term is determined by $\lambda'(\mu)$. The sign of the second term is determined by the sign of $(m-1)\lambda(\mu) - 1$. For $\lambda'(\mu) = 0$ only the second term remains. An $A_3(0) > 0$ implies $(m-1)\lambda(n-2) - 1 > 0$ and thus for $\lambda'(\mu) = 0$, $(m-1)\lambda(\mu) - 1 > 0$ so that $A_3'(\mu) < 0$ for all μ. Likewise, $A_3(0) < 0$ implies $A_3'(\mu) > 0$ and $A_3(0) = 0$ implies $A_3'(\mu) = 0$ for all μ. Thus, $\lambda'(\mu) = 0$ produces an $A_3(\mu)$ function that is monotonically converging on $A_3(n-2) = 0$, either from above if $A_3(0) > 0$ or from below if $A_3(0) < 0$. In the former case, $\arg\min A_3(\mu_j^s) = n-2$ so that $A_3(\mu_j^s) \geq 0$ for $\mu_j^s \in [0, n-2]$ indicating that every follower is content to follow rather than lead in the presence of leader i. For the latter, $A_3(\mu_j^s) < 0$ for $\mu_j^s \in [0, n-2)$ with $\arg\min A_3(\mu_j^s) = 0$ so that the most distant follower, with $\mu_j^s = \mu_j^y = 0$ is the most dissatisfied follower. The most distant follower prefers to lead if $A_3(0) < -A_1$.

For $\lambda'(\mu) > 0$, the first term is zero at the origin and otherwise negative. An $A_3(0) > 0$ still ensures $(m-1)\lambda(n-2) - 1 > 0$ but with $\lambda(0) < \lambda(n-2)$, $(m-1)\lambda(0) - 1$ could be less than zero. The $\psi(0) = 0$ in the first term leaves the sign of $A_3'(0)$ to be determined by whether $(m-1)\lambda(0) - 1 > 0$ or $(m-1)\lambda(0) - 1 < 0$. The sign $A_3'(\mu)$ for interior values of μ is subject to the combined effects of $\lambda(\mu)$, $\psi(\mu)$, $\lambda'(\mu)$, and $\psi'(\mu)$. Let $A_3^0(\mu)$ indicate the $A_3(\mu)$ produced by a given $\psi(\mu)$ and where $\lambda'(\mu) = 0$. Despite the freedom in characterizing $A_3(\mu)$, given $\psi(\mu)$, $A_3(\mu) = A_3^0(\mu)$ at $\mu = 0, n-2$ and for $\lambda'(\mu) > 0$, $A_3(\mu) - A_3^0(\mu) \geq 0$ for $0 < \mu < n-2$ so that $A_3^0(\mu)$ is the lower bound for $A_3(\mu)$. Thus, the conditions $\arg\min A_3(\mu_j^s) = n-2$ for $A_3(0) > 0$ and $\arg\min A_3(\mu_j^s) = 0$ for $A_3(0) \leq 0$ for $\mu_j^s \in [0, n-2]$, identified for $\lambda'(\mu) = 0$, remain true for $\lambda'(\mu) > 0$. $\square$

Corollary 1. *For $\sigma \in H(i; g)$, $\sigma' = \{h^-(i, \sigma; g) | \sigma_{jj}' = 1\}$, and $B_{NL} \geq 0$, if $\lambda'(\mu) < 0$ then $\pi_{NL}(j, \sigma) < \pi_{NL}(j, \sigma')$ is possible for some $j \in N^S(i; \sigma) \backslash \{\bar{j}\}$.*

Proof. For $\lambda'(\mu) < 0$, the first term is zero at the origin and otherwise positive. An $A_3(0) > 0$ ensures $(m-1)\lambda(n-2) - 1 > 0$ and thus $(m-1)\lambda(\mu) - 1 > 0$, $\mu \in [0, n-2]$. For $\lambda'(\mu) > 0$ and a given $\psi(\mu)$, $A_3(\mu) - A_3^0(\mu) \leq 0$ for $0 \leq \mu \leq n-2$ so that $A_3^0(\mu)$ is the upper bound for

$A_3(\mu)$. Thus, the condition $\arg\min A_3(\mu_j^s) = n-2$ for $A_3(0) > 0$ and $\arg\min A_3(\mu_j^s) = 0$ for $A_3(0) \leq 0$ is not assured when $\lambda'(\mu) < 0$. Sufficient, but by no means necessary, evidence that $A_3(\mu)$ has a minimum for some interior value of μ would be a positive slope in $A_3(\mu)|_{\mu=n-2}$. Realizing $(m-1)\lambda(n-2) - 1 = A_3(0)$, $A_3'(n-2) > 0$ if

$$(m-1)\psi(n-2)\lambda'(n-2) + A_3(0)\psi'(n-2) < 0$$

which, with $\lambda'(n-2) < 0$, can be true even for $A_3(0) > 0$. □

B.2 Evaluation of Proposition 1 with linear payoff

Proof. Let σ_{-j} indicate the strategies of all players in $N\backslash\{j\}$. For $\sigma \in H(i;g)$, let $\sigma^{'} = \sigma_j^{'} \times \sigma_{-j}$ and $\sigma_{jj}^{'} = 1$ producing $\sigma^{'} \in h^{-}(i,\sigma;g)$. Let $\mu_j^h = \mu^h(j;\sigma) = |N^h(j;\sigma)|$ for $h = x, y, s$. For player $j \in N\backslash\{i\}$,

$$\pi(j;\sigma) = r_c(\mu_j^x + \mu_j^y + \mu_j^s + 1) + r_e(\mu_j^y + \mu_j^s). \tag{23}$$

The payoff to j when leading is uncertain due to the uncertainty in the outcome of whether $f_i(k_i) = f_j(k_j)$. Expectations are taken over the possible maps f with

$$\begin{aligned}
\mathbb{E}(\pi(j;\sigma^{'})) &= \frac{1}{m}((r_c + r_e)(\mu_j^x + \mu_j^y + \mu_j^s) + r_c) + \frac{m-1}{m}(r_c + r_e)\mu_j^s \\
&= (r_c + r_e)\mu_j^s + \frac{1}{m}((r_c + r_e)(\mu_j^x + \mu_j^y) + r_c).
\end{aligned} \tag{24}$$

The condition $A(j;\sigma) \geq 0$, derived from $\mathbb{E}(\pi(j;\sigma) - \pi(j;\sigma')) \geq 0$, ensures that player $j \in N^S(i;\sigma)$ prefers her position as a follower of i over leading.

With $\mu_j^y \geq 0$ the first term of $A(j;\sigma)$ as expressed in (9) is strictly positive. For $\theta = (m-1)r_c/r_e > 1$ the second term is also positive so that $A(j;\sigma) > 0$ for all $j \in N\backslash\{i\}$. For $\theta < 1$, the coefficient on μ_j^x is negative. For $\bar{j} = \underset{j \in N^S(i;\sigma)}{\operatorname{argmax}}\, d_{ji}$, $\mu_j^y = \mu_j^s = 0$ and $\mu_j^x = n-2$. With $n-2$ as the maximum possible value for μ_j^x, $A(\bar{j};\sigma) \leq A(j;\sigma)$ for $\theta < 1$. With $B = A(\bar{j};\sigma)/(r_e(n-1))$, $B \geq 0$ is necessary and sufficient to ensure $A(\bar{j};\sigma) \geq 0$ implies $A(j;\sigma) \geq 0$ $\forall j \in N\backslash\{i\}$, for all $\sigma \in H(i;g)$, and for all $i \in N$. □

B.3 Formal statement and proof of Proposition 2

Proposition 2. *For $\sigma \in H(i;g)$ and $a_j \in N^d(j;g)$, $a_j = \underset{N^d(j;g)}{\arg\min} \mu_j^x$ if $a_j = \underset{N^d(j;g)}{\arg\min} d_{ji}$. Structure $\sigma' \in H'(i;g)$ if $\sigma' \in H^*(i;g)$ or if $\sigma' = \sigma_{-j} \times \sigma_j^{'}$ with $a_j^{'} = j'$ where $\sigma \in H'(i;g)$ and where $j \in N^S(i;\sigma)$ satisfies the following three properties*

1. *$j' \in N^d(j,g)$ with $d_{j'i} = d_{ji}$,*

2. *$\mu^y(j;\sigma) = 0$, and*

3. *either $\mu^s(j;\sigma) = 0$ or $\mu^s(j;\sigma) > 0$ with successors $N^S(j;\sigma)$ having no option to link to i but through j.*

Proof. For $\{j_1, j_2\} \in N^d(j;g)$ with $d_{j_1 i} < d_{j_2 i}$, let $\sigma^h = \sigma | \sigma_{jj_h} = 1$, $h = 1, 2$, so that $\mu^x(j, \sigma^1) \leq \mu^x(j, \sigma^2)$. The condition that allows $\mu^x(j, \sigma^1) = \mu^x(j, \sigma^2)$ is $\mu_j^y = 0$. With $j_2 \notin N^S(j;\sigma)$, $\mu_j^y = 0$ implies $j_2 \in N^x(j;\sigma)$ and $d_{j_2 i} = d_{ji} = d_{j_1 i} + 1$. For $\sigma^1 \in H^{'}(i;g)$, a necessary and sufficient condition to have $\sigma^2 \in H^{'}(i;g)$ is that for all $j_s \in N^S(j;\sigma^1)$, $N^d(j_s;g) \subset \{N^S(j;\sigma) \cup \{j\}\}$. The condition establishes that no successor of j has the option to link to i without having the chain of links pass through j, a condition necessary to ensure that $\mu^x(j_s;\sigma^2)$ is minimized for all j_s. □

B.4 Formal statement and proof of Proposition 3

Proposition 3. *If $\sigma \in H(i;g)$ is a Nash equilibrium, then $\sigma \in H'(i;g)$.*

Proof. For player i, leading dominates following since to choose one's own successor as a predecessor pays zero. From $\mu^e(j) = \mu_j^y + \mu_j^s$ and $\mu_j^y = n - 2 - \mu_j^s - \mu_j^x$, increasing μ_j^x decreases $\pi(j;\sigma)$. Among the following options, a player can do no better than to minimize μ_j^x. A player who is not minimizing μ_j^x is not optimizing against her available following options. Thus, any structure $\sigma \in H(i;g) \backslash H'(i;g)$ cannot be a Nash equilibrium. For $\sigma \in H'(i;g)$ each player is optimizing from the set of strategies that preserve $\sigma \in H(i;g)$. □

B.5 Formal statement and proof of Proposition 4

Proposition 4. *$B \geq 0$ is a necessary and sufficient condition for $\{H'(i;g)\}_{i \in N}$ to be a set of equilibrium strategies to the exclusion of all $\{H(i;g) \backslash H'(i;g)\}_{i \in N}$.*

Proof. From Proposition 1, given $B \geq 0$, every player $j \in N\backslash\{i\}$ prefers any structure $\sigma \in H(i;g)$ over the structure produced by player j's deviation to lead. In combination with Proposition 3, $B \geq 0$ implies that no follower in the population can do better for herself than to minimize her μ_j^x. $\square$

Corollary 2. *$B \geq 0$ is a necessary and sufficient condition for $\{H^*(i;g)\}_{i\in N}$ to be a set of equilibrium strategies.*

B.6 Formal statement and proof of Proposition 5

Proposition 5. *For $B < 0$ and $\theta \neq 0$, if $\sigma \in h_L(i,\mu_i^s;g)$ is a Nash equilibrium, then $\sigma \in h_L^*(i,\bar{n};g)$ where $\bar{n}$ is an integer value with $|\bar{n}-\mu^*| < 1$.*

Proof. Let $\mu_j^h = \mu^h(j;\sigma) = |N^h(j;\sigma)|$ for $h = x,y,s$ and $\mu^l = \mu^l(\sigma) = |N^L(\sigma)|$. For $\sigma \in h_L(i,\mu_i^s;g)$, let $\sigma^{'} = \sigma_j^{'} \times \sigma_{-j}$ and $\sigma_{jj}^{'} = 1$, $j \in N^S(i;\sigma)$. The payoff to j when following is uncertain due to the uncertainty in the outcome of whether $f_i(k_i) = f_l(k_l)$ for each $l \in N^L(\sigma)\backslash\{i\}$ with,

$$\mathbb{E}(\pi(j;\sigma)) = r_c(\mu_j^x + \mu_j^y + \mu_j^s + 1) + r_e(\mu_j^y + \mu_j^s) + \frac{1}{m}r_c(\mu^l - 1). \tag{25}$$

The payoff to j when leading is uncertain due to the uncertainty in the outcome of whether $f_j(k_j) = f_l(k_l)$ for each $l \in N^L(\sigma)$ with,

$$\mathbb{E}(\pi(j;\sigma')) = (r_c + r_e)\mu_j^s + \frac{1}{m}(r_c + r_e)(\mu_j^x + \mu_j^y) + \frac{1}{m}r_c(\mu^l(\sigma)). \tag{26}$$

For agent $j \in N^S(i;\sigma)$, the same condition $A(j;\sigma) \geq 0$ as developed for Proposition 1, derived from $\mathbb{E}(\pi(j;\sigma) - \pi(j;\sigma')) \geq 0$ using (25) and (26), ensures that player $j \in N^S(i;\sigma)$ prefers her position in the hierarchy over leading. Let $\bar{j}(\mu_i^s) = \underset{j\in N^S(i;\sigma)}{\arg\max}\, d_{ji}$, then $\mu^y(\bar{j}(\mu_i^s)) = \mu^s(\bar{j}(\mu_i^s)) = 0$ and $\mu^x(\bar{j}(\mu_i^s)) = \mu_i^s - 1$ so that

$$A(\bar{j}(\mu_i^s);\sigma) = (m-1)r_c + ((m-1)r_c - r_e)(\mu_i^s - 1) \tag{27}$$

and $C(\mu_i^s;\theta) = A(\bar{j}(\mu_i^s);\sigma)/r_e\mu_i^s$. With $B < 0$, $((m-1)r_c - r_e) < 0$ so that $A(\bar{j}(\mu_i^s);\sigma)$ decreases as the size of the tree increases. For $\mu_i^s = 1$, $A(\bar{j}(1),\sigma) = (m-1)r_c > 0$ while $B < 0$ means that for $\mu_i^s = n-1$, $A(\bar{j}(n-1);\sigma) < 0$.

For $m = 1$, $A(j;\sigma) = -r_e\mu_j^x < 0$. For $r_c = 0$, $A(j;\sigma) = r_e((m-1)\mu_j^y - \mu_j^x)$ so that the most distant follower, with $A(\bar{j}(\mu_i^s);\sigma) = -r_e(\mu_i^s - 1) \leq 0$, prefers to lead in the presence of other followers. Player $\bar{j}$ is indifferent to leading only when she is the only follower, $m = 1$, and $r_c = 0$. With a non-trivial choice ($m > 1$) and a preference for conformity ($r_c > 0$), the equilibrium structure requires $\mu_i^s \geq 1$.

The value of μ_i^s that sets $C(\mu_i^s;\theta) = 0$ need not be an integer. There exists $\bar{n} \in \{\text{floor}(\mu^*), \text{ceil}(\mu^*)\}$ such that $A(j(\bar{n});\sigma) \geq 0$ and $A(j(\bar{n}+1);\sigma) < 0$. A structure $\sigma \in h_L(i,\bar{n};g)\backslash h_L^*(i,\bar{n};g)$ cannot be an equilibrium because either there are members of $N^S(i;\sigma)$ able to improve their payoff by choosing a different predecessor offering a shorter distance to i or there is a member of $N^L(\sigma)$ able to improve her payoff by choosing to follow a predecessor offering a shorter distance to i than the current $\bar{j}(\mu_i^s)$ player. For $\sigma \in h_L^*(i,\bar{n};g)$, no player is able to improve her payoff through unilateral deviation while preserving a single-leader structure. □

B.7 Formal statement and proof of Proposition 6

Proposition 6. *$B < 0$ is a necessary and sufficient condition for $\{h_L^*(i,\overline{n};g)\}_{i\in N}$ to be the set of equilibrium strategies.*

Proof. Let $\mu_h^s = \mu^s(i_h)$. For $\sigma \in h^*(i_A, i_B; g)$, let $\sigma' = \sigma_{-j} \times \sigma_j^{'}$, with $\sigma_{jj}^{'} = 1$, $j \in N^S(i;\sigma)$. For j, the expected payoff for following and leading are, respectively,

$$\begin{aligned}
\mathbb{E}(\pi(j;\sigma)) &= r_c(1+\mu_j^x+\mu_j^y+\mu_j^s) + r_e(\mu_j^y+\mu_j^s) \\
&+ \frac{1}{m}(r_c(\mu^l(\sigma)-1+\mu_j^\alpha+\mu_j^\beta) + r_e(\mu_j^\beta)), \qquad (28)\\
\mathbb{E}(\pi(j;\sigma')) &= (r_c+r_e)\mu_j^s \\
&+ \frac{1}{m}((r_c+r_e)(\mu_j^x+\mu_j^y+\mu_j^\alpha+\mu_j^\beta) + r_c\mu^l(\sigma)) \qquad (29)
\end{aligned}$$

where $\mu_j^h = \mu^h(j;\sigma) = |N^h(j;\sigma)|$ for $h = x,y,s,\alpha,\beta$ and $\mu^l = \mu^l(\sigma) = |N^L(\sigma)|$. Observe, for $h = A, B$,

$$\underbrace{1+\mu_j^x+\mu_j^y+\mu_j^s}_{=\mu_h^s} + \underbrace{\mu_j^\alpha+\mu_j^\beta}_{=\mu_{-h}^s} + \mu^l = n.$$

The condition $\mathbb{E}(\pi(j;\sigma) - \pi(j;\sigma')) \geq 0$ implies $D(j_h;\sigma) \geq 0$ as reported in (12). For the most distant player(s) from i_h according to σ, $E(i_h;\sigma) = D(\bar{j}(\mu_h^s);\sigma)/r_e\mu_h^s$. With $\mu^y(\bar{j}(\mu_h^s)) = \mu^s(\bar{j}(\mu_h^s)) = 0$, $\mu^x(\bar{j}(\mu_h^s)) = \mu_h^s - 1$, and $\mu^\alpha(\bar{j}(\mu_h^s)) \geq \mu^\alpha(j_h)$ for all $j_h \in N^S(i_h;\sigma)$, $D(\bar{j}(\mu_h^s);\sigma) \geq 0$ implies $D(j_h;\sigma) \geq 0$ for all $j \in N^S(i_h;\sigma)$, so that $E(i_h;\sigma) \geq 0$ is necessary and sufficient to ensure $D(j_h;\sigma) \geq 0$ holds for all $j_h \in N^S(i_h;\sigma)$. Since

$$\left(1 + \frac{\mu_h^\alpha(j_{\mu_h^s}) - 1}{\mu_h^s}\right) \geq 1 > \left(1 - \frac{1}{n-1}\right),$$

the condition $E(i_h;\sigma) \geq 0$ violates $B < 0$. For $\sigma \in h_L^*(i,\bar{n};g)$, no player is able to improve her payoff through unilateral deviation. $\square$

B.8 Formal statement and proof of Proposition 7

Proposition 7. *For*

$$\begin{aligned} H^+(i_A,i_B;g) \quad &= \quad \{\sigma \in H^*(i_A,i_B;g)\} \\ \textit{such that} \quad & \quad N^d(i_h;g) \cap \{N^S(i_{-h};\sigma), i_{-h}\} = \emptyset, \\ & \quad E(i_h,\mu_h^s,\theta,\sigma) \geq 0, \\ & \quad F_h(j_h;\theta,m,\sigma) \geq 0 \textit{ for all } j \in N^{AB}(i_A,i_B;\sigma), \end{aligned}$$

a structure $\sigma \in H(i_A,i_B;g)$ is a Nash equilibrium if and only if $\sigma \in H^+(i_A,i_B;g)$. The set $H^+(i_A,i_B;g)$ is feasibly non-empty.

Proof. For $\sigma \in H(i_A,i_B;g)$, without loss of generality, let $\mu_A^s \geq \mu_B^s$. With $g \in G(n)$, $\{i_h \cup N^{AB}(i_A,i_B;\sigma)\} \cap \{i_{-h} \cup N^S(i_{-h};\sigma)\}$, $h = A,B$, are both nonempty sets. The compliments $\{i_h, N^S(i_h;\sigma)\} \backslash N^{AB}(i_A,i_B,\sigma)$ $h = A,B$ can be nonempty, indicating that possibly i_h and some $j \in N^S(i_h;\sigma)$ have no direct potential link to $\{i_{-h}, N^S(i_{-h};\sigma)\}$ with the current σ.

For player $j_h \in N^S(i_h;\sigma)$, expected payoff for remaining a follower in the i_h-led tree is $\mathbb{E}(\pi_h(j;\sigma))$ as expressed in (28). Let $\sigma'_{h\to -h} = \sigma_{-jh} \times \sigma'_{jh}$, with $j_h \in N^S(i_{-h};\sigma'_{h\to -h})$. That is, $\sigma'_{h\to -h} \in H^*(i_A,i_B;g)$ represents the alternative to $\sigma \in H^*(i_A,i_B;g)$ based on a switch by player $j_h \in N^{AB}(i_A,i_B;\sigma) \cap N^S(i_h;\sigma)$ from the i_h-led tree to the i_{-h}-led tree. Compute

$$\begin{aligned} \mathbb{E}(\pi(j_h;\sigma) - \pi(j_h;\sigma'_{h\to -h})) \quad &= \quad \frac{1}{m}((m-1)r_c(\mu_h^s - \mu_{-h}^s - 1 - \mu^s(j_h)) \\ & \quad + r_e(\mu^\beta(j_h) - \mu_{-h}^\beta(j_h) + m(\mu^y(j_h) - \mu_{-h}^y(j_h)))). \end{aligned}$$

Table 6: Example 2 payoff matrix and game for $n = 2$, $m = 2$, $r_c = 1$, $r_e > 0$

(a) State-Dependent Payoffs

			Player 2		
			lead (options)		follow
			O_1	O_2	
Player 1	lead (options)	O_1	1,1	0,0	$r_e + 1, 1$
		O_2	0,0	1,1	$r_e + 1, 1$
	follow		$1, r_e + 1$	$1, r_e + 1$	0,0

(b) Action-dependent expected payoffs

			Player 2		
			lead (labels)		follow
			A	B	
Player 1	lead (labels)	A	$\frac{1}{2}, \frac{1}{2}$	$\frac{1}{2}, \frac{1}{2}$	$r_e + 1, 1$
		B	$\frac{1}{2}, \frac{1}{2}$	$\frac{1}{2}, \frac{1}{2}$	$r_e + 1, 1$
	follow		$1, r_e + 1$	$1, r_e + 1$	0,0

The condition $F_A \geq 0$ of (15) corresponds to $\mathbb{E}(\pi(j_A;\sigma) - \pi(j_A;\sigma^{'}_{A\to B})) \geq 0$ and the condition $F_B \geq 0$ of (16) corresponds to $\mathbb{E}(\pi(j_B;\sigma) - \pi(j_B;\sigma^{'}_{B\to A})) \geq 0$.

The condition $F_h(i_h) \geq 0$ reduces to

$$-\left(\theta - \left(1 - \frac{1}{\mu^s_{-h} + 1}\right)\right) \geq 0.$$

Since $\mu^s_{-h} \leq (n-2)$, $B \geq 0$ ensures that $F_h(i_h) \leq 0$ for both leaders. The condition holds at equality only if $B = 0$ and $\mu^s_{-h} = (n-2)$, a condition that cannot hold for both leaders simultaneously.

$F_h(j) > 0$ for all $j \in N^{AB}(i_A, i_B; \sigma)$ is feasible. $\square$

C Appendix: Examples

C.1 Example 2

The full payoff matrix associated with each possible outcome is listed in Table 6a. Table 6b is the proper normal-form game based on the options labels $K = \{A, B\}$. The payoff table includes all possible actions by each player and the expected payoff associated with the uncertain outcome produced when both players lead.

C.2 Multiple-leader structures

Two scenarios allow for a multiple leader structure in equilibrium with linear payoff functions. Both feature a σ given g such that a particular follower finds it advantageous and feasible to preserve the multiple leader structure.

C.2.1 Example 4

Let $\mu^h(j) = \mu^h(j;\sigma) = |N^h(j;\sigma)|$ for $h = x, y, s, \alpha, \beta$. For $h = A, B$, let $\sigma' = \sigma_{-j_h} \times \sigma'_{j_h}$ be the structure produced by j_h switching predecessors in order to become a member of the i_{-h}-led tree. The alternative structure identifies populations $N^\beta_{-h}(j_h) = N^\beta(j_h;\sigma')$ and $N^y_{-h}(j_h) = N^y(j_h;\sigma')$. Let $\mu^\beta_{-h}(j_h) = |N^\beta_{-h}(j_h)|$ and $\mu^y_{-h}(j_h) = |N^y_{-h}(j_h)|$.

The structure σ is as depicted in Figure 7. With $\mu^y(j_A) = \mu^s(j_A) = \mu^\beta(j_A) = \mu^y(j_B) = \mu^s(j_B) = \mu^\beta_A(j_B) = 0$, $F_A \geq 0$ and $F_B > 0$ of (17) and (18) jointly imply

$$\frac{\mu^\beta_B(j_A)}{\theta} + 1 \leq d\mu < \frac{\mu^\beta(j_B) - m\mu^y_A(j_B)}{\theta} - 1. \tag{30}$$

The key features needed of σ to satisfy (30) are

1. $\mu^s_B \geq 1 + \mu^\alpha(j_B) + (m\mu^y_A(j_B) - (\theta - 1)\mu^\beta(j_B))/\theta$ so that μ^s_B is larger than μ^s_A less the $N^S(i_A;\sigma)$ followers at distance $d_{j_B,i} + 1$. Each member of the $N^y_A(j_B)$ population requires m members of $N^S(i_B;\sigma)$ to keep j_B in $N^S(i_B;\sigma)$. $\theta = 1$ is the minimum possible threshold on θ derived from $E_h \geq 0$. The stronger condition $\mu^s_B \geq 1 + \mu^\alpha(j_B) + m\mu^y_A(j_B)$ ensures $F_B \geq 0$ over the entire feasible support for θ;

2. a concentration of the i_A-led population at the distance $d_{j_B,i_B} + 1$ sufficiently large to have $\mu^s_A \geq \mu^s_B$ despite feature 1;

3. $d_{j_B,i_A} \geq d_{j_B,i_B} + 1$; and

4. $d_{j_A,i_B} = d_{j_B,i_B} + 1$.

Figure 7 is an equilibrium structure satisfying (30). Feature 1 requires a large x population based on the sizes of the α and γ populations. The β population is sufficiently large to produce $\mu^s_A \geq \mu^s_B$ in accordance with Feature 2. So that j_B prefers the i_B-led tree, she cannot benefit from the β population were she to switch, which is captured by Feature 3. Feature 4 puts j_A

in a position where she fails to share in j_B's distance advantage over the β population from the i_B-led tree, thereby keeping $\mu_B^\beta(j_A)$ small. By Feature 3, the β population exists within the distance range $d_{j_B,i_B}+1$ and d_{j_B,i_A} (inclusive) but Feature 4 constrains the population to have a distance of $d_{j_B,i_B}+1$.

C.2.2 Example 5

The inequality $F_B(j_B) > 0$ supports follower $j_B \in \{N^S(i_B,\sigma) \cap N^{AB}(i_A,i_B;\sigma) | \mu_j^s = 0, \mu_j^y > 0\}$ in her current position, as illustrated in Example 5. The additional imposition of $\mu_A^y(j_B) = 0$ minimizes the attraction of the i_A-led tree to j_B as it implies player j_B has to join the i_A-led tree at the maximum distance.

D Appendix: Sequential play games

D.1 Unsupported leaders

The g network is as listed in Table 5. The extensive-form game depicted in Table 7 only includes the decisions of i, j, and x in that order of play since $s_1, s_2 \in N^S(j; \{\sigma^1, \sigma^2, \sigma^3\})$. Each player has the option to lead, labeled "L", or to imitate the first or second contact. Under "Strategies" are the actions employed to achieve each structure, identified by number in the top row of the table. The payoff to each player in each structure is listed in the "Payoff" section of Table 7. Those payoff areas labeled "Loop" are strategies that produce self-referencing imitation loops with no leader within the loop. Since this generates a zero payoff for those in the loop, a structure that includes a loop is never an equilibrium strategy.

Structure 5 in the decision tree generates $\sigma^1 \in H^*(i;g)$. The two hierarchies that make up $H^*(j;g)$ are produced in structures 12 and 20. The subgame perfect structure-dependent strategy of each player is shaded (if viewed in color, they are colored blue for player n, orange for player j, and yellow for player i).

Both structures 12 and 20 are subgame perfect solutions to this sequential-play game while 5 is excluded from the subgame perfect equilibrium set.

Player	Strategy																										
i	L	L	L	L	L	L	L	L	L	x	x	x	x	x	x	x	x	x	s_1	S	s_1	s_1	s_1	s_1	s_1	s_1	s_1
j	L	L	L	i	i	i	s_2	s_2	s_2	L	L	L	i	i	i	s_2	s_2	s_2	L	L	L	i	i	i	s_2	s_2	s_2
x	L	i	j	L	i	j	L	i	j	L	i	j	L	i	j	L	i	j	L	i	j	L	i	j	L	i	j
	Structure																										
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27
Player	Payoff																										
i	0	1.1	0	3.3	4.4	4.4	Loop			0.1	Loop	0.4	3.4	Loop		Loop			0.3	0.4	1.4	Loop			Loop		
j	2.2	2.2	3.3	2.3	2.4	3.4				2.2		4.4	2.4						3.3	4.4	4.4						
x	0	0.1	0.3	0	2.4	0.4				1.1		1.4	4.4						0	1.4	0.4						

Table 7: The Nash equilibrium supported-structure $5 \in H^*(i, \sigma)$ is not supported by a subgame perfect analysis with i, j, x, s_1, s_2 as the order of play. The preferred choice of the each player given the downstream choice of the other players is highlighted according to player i = yellow, player j = orange, and player x = light blue. The hierarchy produced in structures 12 and 20 are equivalent and are subgame perfect equilibria with player j leading the entire population, including player i.

Player	Strategy																										
7	L	L	L	L	L	L	L	L	L	9	9	9	9	9	9	9	9	9	10	10	10	10	10	10	10	10	10
8	L	L	L	7	7	7	10	10	10	L	L	L	7	7	7	10	10	10	L	L	L	7	7	7	10	10	10
10	L	8	9	L	8	9	L	8	9	L	8	9	L	8	9	L	8	9	L	8	9	L	8	9	L	8	9
	Structure ID																										
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27
Player	Payoff: player 9 imitates 7																										
7	1.1	1.1	2.2	2.2	3.3	3.3	1.1	Loop	3.3	Loop			Loop			Loop			1.2	1.3	Loop	2.3	Loop		1.3	Loop	
8	0	1.1	0	0.2	1.3	1.3	0.1		0.3										0	3.3		0.3			1.3		
10	0	0.3	0.2	0	0.3	0.3	1.1		1.3										0.2	2.3		3.3			3.3		
9	0.1	0.1	1.2	0.2	1.3	1.3	0.1		2.3										2.2	0.3		0.3			0.3		
	Structure ID																										
	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50	51	52	53	54
Player	Payoff: player 9 joins i_A																										
7	0	0	0	1.1	2.2	1.1	0	Loop	0	0.5	0.5	1.6	1.6	2.7	1.7	0.5	Loop	1.7	0.1	0.2	0.6	1.2	Loop	1.7	0.2	Loop	0.7
8	0	1.1	0	0.1	1.2	0.1	0.1		0.6	0	1.1	0	0.6	1.7	0.7	0.1		0.7	0	2.2	0	0.2		0.7	0.2		0.7
10	0	0.1	0.5	0	0.2	0.5	1.1		1.4	0	0.1	0.6	0	0.7	1.7	1.1		1.7	1.1	1.2	1.6	2.2		2.7	2.2		2.7
9	0.4	0.4	1.5	0.4	0.4	1.5	0.4		2.6	1.5	1.5	2.6	2.6	3.7	3.7	1.5		3.7	0.4	0.4	2.6	0.4		3.7	0.4		3.7

Table 8: Subgame perfect analysis of a multiple leader Nash equilibrium structure. The Nash equilibrium-supported structure 5 is not supported in a subgame analysis (in any order). Play here is in order the 7, 8, 10, 9. Player 9 chooses between imitating player 7 and switching to join the i_A-led tree. The preferred choice of each player given the downstream choice of the other players is highlighted according to player 7 = yellow, player 8 = orange, player 10 = light blue, and player 9 = tan. Strategy sets 15 and 18 are equivalent and are subgame perfect equilibria with player 9 following i_A and players 7, 8, and 10 following 9.

D.2 Multiple hierarchies

A structure $\sigma \in H^*(i_A, i_B; g)$ like that depicted in Figure 6 conforms to the scenario of Example 5. For $m \to \infty$, the particular structure depicted is supported as a Nash equilibrium for $r_c/r_e \in (0, 1/3]$. The SPE analysis of a sequential game is based on this environment with the expected payoffs reported in Table 8 computed based on $m \to \infty$, $r_c = 0.1$, and $r_e = 1$.[17]

The two lower sections of Table 8 include the structure-dependent payoffs to each player in the player 1-led tree depending on whether player 9 imitates player 7 or joins $N^S(i_A; \sigma)$, as indicated in the "Strategies" section of the table. Excluded from the table are the payoffs associated player 9's dominated option to lead. This leaves 54 possible structures as show in Table 8. These are spread out over two sets of columns of 27 payoffs each; the first set is based on player 9 imitating player 7 and the second set, in the lower portion, is based on player 9 joining $N^S(i_A; \sigma)$. The table contains all of the information of the remaining extensive-form tree which is large but straight forward to construct. Each structure requires the construction of the resulting hierarchy to determine individual payoff. Structure 5 with player 9 imitating player 7 is the Nash equilibrium reflected in Figure 6.

In a sequential play game, decisions proceed in the order players 1 through 6 and then players 7, 8, 10, 9. Eliminate A from player 9's action set and only strategy profiles of the upper payoff section of Table 8 can be reached. Structure 5 is also the SPE of this limited action set. Consistent with the Nash equilibrium, given structure 5, player 9 prefers to imitate player 7 to switching to the i_A-led tree.

Allow player 9 to freely choose from $a_9 \in \{L, 7, A\}$, then structures 15 and 18 with $a_9 = A$ are both SPE and structure 5 is not a SPE. Despite the early mover advantage to players 7 and 8, players 9 and 10 are both better off in the i_A-led tree. If player 10 imitates 8, then player 9 imitates 7 but this is not in player 10's interest. Player 10 enables player 9's choice of A by following 9 rather than 8. Without the support of players 9 and 10 as followers, players 7 and 8 are forced to follow 9 as well. The final $\sigma' \in H^*(i_A; g)$ benefits player 9 more than preserving σ because she has a distance advantage over 7, 8, and 10 rather than just player 10, as depicted in Figure 9.

[17]This is for convenience of display only. Any combination of parameters such that the σ depicted in Figure 6 is in $H^+(i_A, i_B; g)$ leaves the relative payoffs reported in Table 8 unaffected.

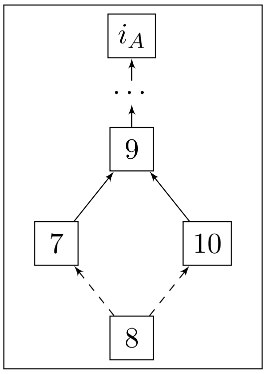


Figure 9: Player 9 with i_A as a predecessor and with player 7 and the former $N^S(7;\sigma)$ population as successors following a best response cascade that transitions from a Nash equilibrium $\sigma \in H^*(i_A, 7; g)$ with two leaders to a Nash equilibrium $\sigma' \in H^*(i_A; g)$ with a single leader.

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