Polynomial-time isomorphism test of groups that are tame extensions

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- Classical $n^{1/2 \log n}$, quantum $n^{1/3 \log n}$ (Rosenbaum, 2013);
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One motivation:

- Very recently L. Babai announced that graph isomorphism can be solved in time $n^{(\log n)^c}$ for $c \geq 2$;
- In one of the talks he suggested that GROUPI is a bottleneck to put GRAPHI in P.

Some recent results

Polynomial-time algorithms for:

Abelian groups $O(n)$-time (Kavitha, 2007);

Groups with no abelian normal subgroups
Babai et al. (2011) and Babai et al. (2012);

Groups with abelian Sylow towers
Le Gall (2009), Qiao et al. (2011), and Babai and Qiao (2012);

p-groups of genus 2; quotients of generalized Heisenberg groups
Lewis and Wilson (2012) and Brooksbank et al. (2015).

And a group class with $n^{O(\log \log n)}$-time algorithm:

Central-radical groups Grochow and Qiao (2014).

Why these group classes?

- Groups with no abelian normal subgroups;
- Groups with abelian Sylow towers;
- p-groups of genus 2 and quotients of generalized Heisenberg groups;
- Central-radical groups.

A possible explanation for successes over these group classes?

In Grochow and Qiao (2014) we provide some explanation from the perspective of *extension theory of groups*.

A strategy for group isomorphism...

A divide and conquer strategy

Given two groups G_1 and G_2, consider the following recipe...

1. Agree on some characteristic (normal) subgroup $\mathcal{S}$.
   - e.g. center, commutator subgroup, etc.
2. Slice into the normal parts and the quotient parts.
   - To get $\mathcal{S}(G_i)$ and $G_i/\mathcal{S}(G_i)$.

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3. (Divide) Test isomorphism of the two parts respectively.
   - If both parts are isomorphic respectively, identify the normal part by A and quotient part by Q, continue.
   - Otherwise `not isomorphic`.
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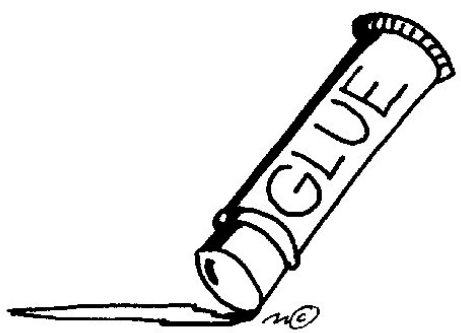
4. (Conquer) ...?

After step 3, we call G_1 and G_2 *extensions* of A by Q.
Q: How do the normal part A, and the quotient part Q glue together?



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By extension theory, two functions arise as the "glue."

Action The conjugation action of Q on A; a homom. $Q \to \mathrm{Aut}(A)$;

2-cocycle How different is from semidirect product; a function $Q \times Q \to A$ satisfying the 2-cocycle identity.

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Lemma (Folklore, cf. Grochow and Qiao (2014))

$G_1 \cong G_2$ if and only if actions and 2-cocycles are the same up to the action of $\mathrm{Aut}(A) \times \mathrm{Aut}(Q)$.

An algorithmic problem about extensions

If the normal subgroup is elementary abelian ($\cong \mathbb{Z}_p^d$)...

Problem (Extension pseudo-congruence problem)

Given two groups that are extensions of $\mathbb{Z}_p^d$ by Q, and $\mathrm{Aut}(Q)$ by a set of generators, decide whether the two extensions are the same under $\mathrm{Aut}(\mathbb{Z}_p^d) \times \mathrm{Aut}(Q)$ in time $\mathrm{poly}(|Q|, p^d)$.

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- Solving this problem will solve group isomorphism in general (Cannon and Holt, 2003);
- For $Q = \mathbb{Z}_p^e$ and central extensions, this is p-group isomorphism and considered difficult.

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Space The set of $n \times n$ matrices, $M(n, \mathbb{C})$;

Group action $A \in \mathrm{GL}(n, \mathbb{C})$ sends $B \in M(n, \mathbb{C})$ to ABA^{-1};

Canonical form (1) B is a direct sum of Jordan blocks; (2) Each Jordan block is determined by the size and the eigenvalue.

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On the other hand, consider a similar problem:

Space The set of pairs of $n \times n$ matrices, $M(n, \mathbb{C}) \oplus M(n, \mathbb{C})$;

Group action $A \in \mathrm{GL}(n, \mathbb{C})$ sends $(B, C) \in M(n, \mathbb{C}) \oplus M(n, \mathbb{C})$ to (ABA^{-1}, ACA^{-1});

Canonical form A long-standing open problem; believed to be intractable.

The tame-wild dichotomy

Definition

A classification problem is *tame*, if the indecomposables of dimension d come from a finite number of 1-parameter families. It is *wild* if it "contains" the problem of classifying pairs of matrices under simultaneous conjugation.

Theorem (Drozd, 1970's)

The classification problem for representations of associative algebras over algebraically-closed fields are either tame or wild.

The tame setting

(We consider extensions of $\mathbb{Z}_p^d$ by Q.)

Theorem (Grochow and Qiao (2015))

If the group algebra $\overline{\mathbb{F}_p}Q$ is tame, then the extension pseudo-congruence problem can be solved.

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$\overline{\mathbb{F}_p}Q$ is tame, iff the Sylow p-subgroup of Q is:

- cyclic. (Finite; Higman (1954).)
- p=2 and dihedral, semi-dihedral, or generalized quaternion. (Tame and not finite; Bondarenko (1975), Ringel (1975), Bondarenko and Drozd (1982) and Crawley-Boevey (1989).)

Othercases are wild (Kruglyak (1963) and Brenner (1970)).

The difference b/w tame and wild

Theorem

Let $n(Q, p, d)$ be the number of indecomposable modules of Q over $\mathbb{F}_p$ of dimension d.

- *If $\overline{\mathbb{F}_p}Q$ is tame, then $n(Q, p, d) \leq \mathrm{poly}(|Q|, p^d)$.*
- *(J. Rickard) If wild, then $n(Q, p, d) = p^{\Omega(d^2)}$.*

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- Does not follow from the definition of tame/wild because of finite fields.
- Rather, this is about determining the number of 1-parameter families and finite cases.
- Finite case is known by Higman (1954).
- Wild case by explicit construction.
- Tame case by examining the explicit classification as in Crawley-Boevey (1989).

The cohomology aspect

Theorem

Let $m(Q, p, d)$ be the order of the 2-cohomology group of Q w.r.t. any fixed $\mathbb{F}_p Q$ module of dimension d. If $\overline{\mathbb{F}_p Q}$ is tame, then $m(Q, p, d) \leq p^{3d}$.

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The algorithm: given two 2-cocycles $f, g : Q \times Q \to \mathbb{Z}_p^d$ w.r.t. $\mathbb{F}_p Q$ module M:

1. Compute $J \leq \mathrm{Aut}(\mathbb{Z}_p^d) \times \mathrm{Aut}(Q)$ that preserves M;
2. View the given two 2-cocycles as two points in $H^2(Q, M)$;
   - The problem reduces to test if some $\alpha \in J$ that sends f to g.
3. Apply the pointwise transporter algorithm.
   - Runs in time $\mathrm{poly}(|H^2(Q, M)|)$.

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(Ingredients from permutation group algorithms (Luks, 1991) and routines about 2-cohomology classes (Grochow and Qiao, 2014)).)

The last slide. . .

In this work, we show:

- A concrete example on how the tame-wild dichotomy affects the efficiency of an algorithm for group isomorphism test.
- The bounds rely critically on the known descriptions of indecomposables for semi-dihedral groups.

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Question for further study:

- Go into the wild!

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For the action aspect: given two $\mathbb{F}_p Q$ modules M and N of dimension d. Let R be the set of indecomposables of $\mathbb{F}_p Q$ of dimension $\leq d$.

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   - $M = L_1^3 \oplus L_2^3 \oplus L_3^2$, and $N = L_1^2 \oplus L_2^3 \oplus L_3^3$.
2. The induced action of $\mathrm{Aut}(Q)$ permutes the indecomposables;
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(Ingredients from computational representation theory (Chistov et al., 1997; Brooksbank and Luks, 2008) and permutation group algorithms (Luks, 1999; Babai and Qiao, 2012).)