Inequivalent multipartite coherence classes and two operational coherence monotones

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Quantum coherence has received significant attention in recent years, but the structure of multipartite coherent states is unclear. In this paper, we generalize important results in multipartite entanglement theory to their counterparts in quantum coherence theory. First, we give a necessary and sufficient condition for when two pure multipartite states are equivalent under local incoherent operations assisted by classical communications (LICC), i.e., two states can be deterministically transformed to each other under LICC operations. Next, we investigate and give the conditions in which such a transformation succeeds only stochastically. Different from the entanglement case for two-qubit states, we find that the stochastic LICC (sLICC) equivalence classes are infinite. Moreover, it is possible that there are some classes of states in multipartite entanglement that can convert into each other, whereas they cannot convert into each other in multipartite coherence. In order to show the difference among sLICC classes, we introduce two coherence monotones: accessible coherence and source coherence, following the logistics given in [Phys. Rev. Lett. **115**, 150502 (2015)]. These coherence monotones have a straightforward operational interpretation, namely, the accessible coherence characterizes the proficiency of a state to generate other states via quantum incoherent operations, whereas the source coherence characterizes the set of states that can be reached via quantum incoherent operations acting on the given state of interest.

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I. INTRODUCTION

Coherence originates from the "superposition" of quantum states and plays a central role in interference phenomena in quantum physics and quantum information science [1–13]. Coherence is an essential ingredient for multipartite entanglement in many-body systems and a necessary phenomenon of analyzing physical phenomena in quantum optics [14], solid-state physics [15], and nanoscale thermodynamics [16–18] even in biological systems [19–21]. A mathematical framework of quantum coherence as a physical resource has been proposed recently [1]. There are two basic elements in coherence theory: (1) free states and (2) free operations. Free states in the coherence theory are those states which are diagonal in a fixed basis $\{|i\rangle\}$, which we call incoherent states. Free operations (incoherent operations) are some specified classes of physically realizable operations that act invariantly on the set of incoherent states, which is not unique due to practical implications.

Local operations assisted by classical communications (LOCC) is helpful for understanding the structure of entangled states because entanglement cannot be created by LOCC operators. In analogy with entanglement theory, we focus on the protocol that each party performs local incoherent operations assisted by classical communications (LICC) [5,6]. In LICC protocol, the local incoherent operator cannot create global coherence so that multipartite coherence remains a resource. Therefore, LICC can be viewed as a natural setting to explore the structure of a multipartite coherent states.

The first aim in this paper is to understand the structure of multipartite coherent states. For quantum information processing, different structures of states often have different capabilities in state transformation. Just like the resource of entanglement, a LOCC protocol for a multipartite state leads to natural ways of defining equivalent relations in the set of entangled states as well as establishing hierarchies between the resulting classes (structures). States in different classes cannot convert into each other. For example, Greene-Horne-Zeilinger (GHZ) states and W states belong to two different classes via stochastic local operations and class communications (sLOCC), which reveals the existence of two inequivalent kinds of genuine tripartite entanglement [22]. To understand the structure of multipartite coherent states, we also consider the classification of pure coherent states of multipartite quantum systems under LICC even in a stochastic setting. For multipartite coherence, an interesting observation is as follows: If one of the three qubits in the W state is lost, the state of the remaining two-qubit system is still coherent, whereas the GHZ state, which is incoherent after the loss of one qubit. This observation leads to following question: Is it possible that there are some classes of states in multipartite entanglement can convert into each other, whereas they cannot convert into each other in multipartite coherence?

We address the questions above by focusing on the equivalent class of pure multipartite coherent states. First, a necessary and sufficient condition for pure multipartite state transformations via LICC is presented. Second, we investigate and give the conditions in which such a transformation

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succeeds only stochastically, namely, stochastic local incoherent operations and class communications (sLICC). As an application, we investigate the two-qubit sLICC equivalent classes, showing that equivalent classes of bipartite coherence under sLICC are more complex than equivalent classes of bipartite entanglement under sLOCC even in the two-qubit case. This specific incoherent constraint of locality makes us undertand the structure of multipartite coherence because many coherence measures do not have an interpretation in the context of the LICC paradigm [23].

The other aim in this paper is to identify new operational coherence measures for investigating possible LICC transformations among coherent multipartite states. Inspired by similar concepts previously investigated for entanglement [24], we introduce two coherence monotones: accessible coherence and source coherence. Accessible coherence refers to the proficiency of a state to generate other states via free operations, and the source coherence denotes that the set of a state of interest can be obtained via free operations. Both coherence monotones can also be used for single-qubit and multipartite state cases and can be applied for other free operations, such as incoherent operations (ICs), LICC. For single-qubit states achieved via physically incoherent operations (PIO) [12], strictly incoherent operations (SIO) [11] and IC, we obtain explicit formulas of accessible coherence and source coherence. We analyze pure (or mixed) states via IC and derive explicit formulas for the source coherence. These two coherence monotones also have a geometric interpretation. We additionally show how accessible coherence can be computed numerically and provide examples.

This paper is organized as follows. In Sec. II, we first introduce the necessary notation and lemmas we need. In Sec. III, we will explore inequivalent classes of multipartite coherence states. In Sec. IV, we introduce two operational coherence monotones: accessible coherence and source coherence. In Sec. V, we derive the formula for pure states transformation via LICC and its SIO version and then give some examples. In Sec. VI, we derive the formula for pure states transformation via PIO, SIO, and IC and give some examples. We summarize our results in Sec. VII.

II. NOTATION AND PRELIMINARIES

We first introduce the necessary notation. We consider a Hilbert space $\mathcal{H}$ of finite dimension d. The incoherent basis of $\mathcal{H}$ is fixed and is denoted as $\{|i\rangle\}_{i=1}^{d}$ throughout this paper. A unitary operation U is called an incoherent unitary (IU) if $U=\sum_{i=1}^{d} e^{i\theta_i}\,|i\rangle\,\langle\pi(i)|$ with $\pi(i)$ being a permutation. Given a quantum state ρ, its von Neumann entropy is $S(\rho)=-\mathrm{Tr}\,\rho\,\log_2\rho$. For a N-partite state $|\psi\rangle$ defined in $\mathcal{H}_1\otimes\mathcal{H}_2\otimes\cdots\otimes\mathcal{H}_N$, its reduced density operator on a subset $X\subset[N]:=\{1,2,\ldots,N\}$ is denoted as $\rho_X^{\psi}=\mathrm{Tr}_{\overline{X}}\,|\psi\rangle\,\langle\psi|$, where $\overline{X}=[N]\backslash X$.

A general resource theory for a quantum system has two components: free states and free operations. In the resource theory of coherence, a free state σ (incoherent state) can be written as $\sigma=\sum_i\sigma_i|i\rangle\langle i|$ for a fixed basis $\{|i\rangle\}$. Variants of the free operations in the resource theory of coherence have been proposed. A completely positive and trace-preserving (CPTP) map Φ is said to be IC if its Kraus operators K_n are of the form $K_n=\sum_i c(i)\,|j(i)\rangle\,\langle i|$ with $|j(i)\rangle$ being a (possibly many-to-one) function from the index set of the basis onto itself and coefficients $c(i)$ satisfying $\sum_n K_n^{\dagger}K_n=I$ [1]. If every $K_n=\sum_i c(i)\,|i\rangle\,\langle\pi(i)|$, where $\pi(i)$ is a permutation, then the corresponding operation is a SIO [11]. Lastly, the free operation is called a PIO if K_n has the form $K_n=U_nP_n$, where $\{U_n\}$'s are IU operators and $\{P_n\}$'s form an orthogonal and complete set of incoherent projectors [12]. From their definitions, we have following inclusion: PIO $\subset$ SIO $\subset$ IC.

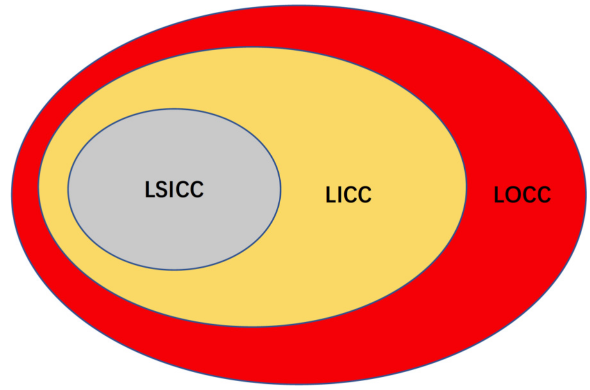

FIG. 1. The hierarchy of LOCC, LICC, and LSICC.

A fundamental class of operations in entanglement theory is LOCC since it allows an operational definition of entanglement [25]. For two bipartite entangled states $|\phi\rangle$ and $|\psi\rangle$, what is the necessary and sufficient condition for transforming $|\phi\rangle$ to $|\psi\rangle$ using LOCC operations? Nielsen showed that this condition of entanglement transformation is related to the algebraic theory of *majorization*:

Lemma 1 ([26]). Given two bipartite pure states $|\phi\rangle$ and $|\psi\rangle$ in the system $\mathcal{H}_A\otimes\mathcal{H}_B$, $|\phi\rangle$ can be converted into $|\psi\rangle$ via LOCC if and only if $\lambda(\phi)\prec\lambda(\psi)$, where $\lambda(\phi)$ denotes the vector of eigenvalues of $\mathrm{Tr}_B(|\phi\rangle\langle\phi|)$. Here, for two d-dimensional vectors $x=(x_1,x_2,\ldots,x_d)$ and $y=(y_1,y_2,\ldots,y_d)$, $x\prec y$ holds if and only if for each k in the range of $1,2,\ldots,d$, $\sum_{i=1}^{k}x_i^{\downarrow}\leqslant\sum_{i=1}^{k}y_i^{\downarrow}$ with equality when $k=d$, where $x_i^{\downarrow}$ means that the elements x_i are arranged in decreasing order.

Coherence-state transformation has also been studied in the literature, motivated by entanglement state transformation. In the single party setting, Du *et al.* [7] obtained the following necessary and sufficient condition of pure state coherence transformation via $\mathcal{O}\in\{\mathrm{SIO},\mathrm{IC}\}$:

Lemma 2 ([7]). For a fixed basis $\{|i\rangle\}$, a pure state $|\phi\rangle$ can be converted into $|\psi\rangle$ via O $\in$ {SIO, IC} if and only if $\lambda[\Delta(\phi)]\prec\lambda[\Delta(\psi)]$, where $\Delta(\rho)=\sum_i\langle i|\,\rho\,|i\rangle\,|i\rangle\langle i|$, and for convenience, we denote $\Delta(|\psi\rangle\langle\psi|)$ as $\Delta(\psi)$.

In the multipartite setting, the class of LICC can be defined accordingly when the local incoherent operations are IC operators [5,6]. If the local operations are SIO operations, we call such protocol LSICC. These protocols have been used to study the relationship between coherence and entanglement [5,6]. It is easy to see the following inclusion: LSICC $\subset$ LICC $\subset$ LOCC. The hierarchy of the sets has been show in Fig. 1.

Chitambar and Hsieh [5] studied pure state transformation under LICC and found the following:

Lemma 3 ([5]). Suppose that two bipartite pure states $|\psi\rangle$ and $|\phi\rangle$ have reduced density matrices that are diagonal in the incoherent bases for both parties and both states. Then, $|\phi\rangle$ can be converted into $|\psi\rangle$ via LICC if and only if the squared Schmidt coefficients of $|\psi\rangle$ majorize those of $|\phi\rangle$, i.e., $|\phi\rangle \prec |\psi\rangle$.

Shi *et al.* [2] and Streltsov *et al.* [3] studied mixed state transformation of single-qubit systems via $\mathcal{O} \in \{\text{SIO}, \text{IC}\}$ and obtained the following result.

Lemma 4 ([2,3]). The state $\rho = \frac{1}{2}\begin{pmatrix} 1+r_z & r_x+ir_y \\ r_x-ir_y & 1-r_z \end{pmatrix}$ can be converted into $\sigma = \frac{1}{2}\begin{pmatrix} 1+s_z & s_x+is_y \\ s_x-is_y & 1-s_z \end{pmatrix}$ via SIO and IC if and only if the following inequalities are satisfied:

$$s_x^2 + s_y^2 \leqslant r_x^2 + r_y^2, \quad \frac{1-r_z^2}{r_x^2+r_y^2}\left(s_x^2+s_y^2\right) + s_z^2 \leqslant 1. \quad (1)$$

Lastly, we also consider LICC protocols that succeed in coherence states transformation only stochastically. Analogous to sLOCC in entanglement theory, we call these operations stochastic LICC operations and use the notation: sLICC.

III. INEQUIVALENT CLASSES OF MULTIPARTITE COHERENCE STATES

In this section, we will explore inequivalent classes of multipartite coherence states. First, we will give a necessary and sufficient condition for when two multipartite coherence states can be interconverted with certainty under LICC. We then study the interconversion of multipartite coherence states which only succeeds with a strictly positive probability. This allows us to define inequivalent classes of multipartite coherence states. The discussion in this section closely follows the inequivalent classes of multipartite entanglement states in Refs. [22,27].

We start with the following lemma showing that the number of product terms of a multipartite coherent pure state is a sLICC monotone. Note that the number of product terms has been shown by other authors to be IC monotone [11,28]. We extend this idea and show that the number of product terms is also a sLICC monotone for multipartite coherence. We say that two pure states $|\psi\rangle$ and $|\phi\rangle$ are $\mathcal{O}$ equivalent if they can be transformed into each other by means of operations in the set $\mathcal{O}$.

Lemma 5 (LICC and sLICC monotone). The number of nonzero product terms in the fixed basis does not increase under LICC (respectively, sLICC).

Proof. Suppose there are many but finite rounds in the LICC (respectively, the sLICC) protocol. At the kth round, Alice performs an IC measurement and obtains a state $|\psi^{(k)}\rangle = \sum_{i=1}^{N}\sum_{t=1}^{M}\psi_{it}\,|i\rangle \otimes |t\rangle$ with the number of nonzero product terms in the fixed basis of $|\psi^{(k)}\rangle$ being NM. Alice tells her result to Bob. Then, Bob performs an IC measurement with outcome s at the $(k+1)$th round operation, the resulting state is $|\psi^{(k+1)}\rangle \propto F_s\,|\psi^{(k)}\rangle = \sum_{i=1}^{N}\sum_{t=1}^{M} c_i\psi_{it}\,|j(i)\rangle \otimes |t\rangle$, where $F_s = \sum_i c_i\,|j(i)\rangle\,\langle i|$ and coefficients $c(i)$ satisfying $\sum_s F_s^\dagger F_s = I$ [1] since $|j(i)\rangle$ being a (possibly many-to-one) function from the index set of the basis onto itself. It is clear that $|\psi^{(k+1)}\rangle$ can be expressed as a sum of product terms with no more than MN terms. Consequently, at every round of LICC (respectively, sLICC) protocol, the number of nonzero product terms will not increase as the LICC (respectively, sLICC) protocol continues. ■

Remark. The number of product terms in the fixed basis is similar to Schmidt rank in the resource theory of entanglement, but they do not play the same roles. Schmidt rank, being an entanglement monotone under sLOCC, can be used to classify the structures of entanglement, and this classification is complete for bipartite settings. In other words, two pure entangled states are sLOCC equivalent iff they have the same Schmidt rank. However, the number of product terms alone, despite being a sLICC monotone, is not sufficient to classify all the structures of multipartite pure coherent states. Example 3 below shows that two pure states can be sLICC inequivalent, even though they have the same number of product terms.

Our first main result is the following, which originates from its entanglement counterpart in Ref. [27, Corollary 1].

Theorem 6. Two multipartite pure states $|\psi\rangle$ and $|\phi\rangle$ are LICC equivalent iff they are local IU (LIU) equivalent.

Proof. It is clear that, if $|\psi\rangle$ and $|\phi\rangle$ are LIU equivalent, they are also LICC equivalent since LIU operations performed by each party constitute a special case of LICC protocols.

Next, suppose that two N-partite states $|\psi\rangle$ and $|\phi\rangle$ are LICC equivalent, i.e., there exists a LICC protocol that converts $|\psi\rangle$ to $|\phi\rangle$, which consists of many rounds of local IC operations and communications between the parties. Suppose, without loss of generality, that Alice (A_1) performs the first local IC operation, yielding the ensemble $\mathcal{E} = \{p_k, |\psi_k\rangle\}$. Since the reduced state of the remaining parties cannot be changed by Alice's operation, we have

$$\rho^{\psi}_{A_2\cdots A_N} = \sum_k p_k\,\mathrm{Tr}_{A_1}(|\psi_k\rangle\langle\psi_k|), \quad (2)$$

where $\rho^{\psi}_{A_2\cdots A_N} = \mathrm{Tr}_{A_1}(|\psi\rangle\,\langle\psi|)$. Note that the average entropy is unchanged [27]: $S(\rho^{\psi}_{A_1}) = S(\rho^{\psi}_{A_2\cdots A_N}) = \sum_k p_k S[\mathrm{Tr}_{A_1}(|\psi_k\rangle\langle\psi_k|)]$. From the strict concavity of the von Neumann entropy [25], it must hold that $S[\mathrm{Tr}_{A_1}(|\psi_k\rangle\langle\psi_k|)] = S[\mathrm{Tr}_{A_1}(|\psi\rangle\langle\psi|)]$ for every k, i.e.,

$$|\psi_k\rangle = U_k^{A_1} \otimes I_{A_2\cdots A_N}\,|\psi\rangle\,, \quad (3)$$

where $U_k^{A_1}$ is a unitary operation acting on Alice's system. Since, by assumption, Alice has to perform incoherent operations, then $U_k^{A_1}$ must be local incoherent unitary for all k.

Alice can choose a label k with probability p_k as her "measurement result" and performs the deterministic LIU operation $U_k^{A_1}$ on state $|\psi\rangle$. Other parties' operations follow similarly so that deterministic LIU operations of the LICC protocol are sufficient to obtain final state $|\phi\rangle$. Hence, we conclude that if $|\phi\rangle$ and $|\psi\rangle$ are LICC equivalent, they are also LIU equivalent. ■

It is also possible that two multipartite pure states $|\psi\rangle$ and $|\phi\rangle$ cannot always succeed with certainty in interconverting through operations in the class $\mathcal{O}$, i.e., such a transformation may only succeed stochastically. This allows the structure of multipartite coherence states to be understood operationally (similar to that of multipartite entangled states): If two multipartite pure states $|\psi\rangle$ and $|\phi\rangle$ cannot be transformed to each other with nonzero probability, they must each belong to different types of multipartite coherence structures.

In the remainder of this section, we will focus on sLICC. Next we give our main result:

Theorem 7. Two multipartite pure states $|\psi\rangle$ and $|\phi\rangle$ are equivalent under sLICC if and only if they are related by local (invertible) SIO operators.

First, we have following lemma:

Lemma 8. If E_0 is an IC measurement, then IC measurements E_i can be constructed such that $\sum_i E_i = I$.

Proof. Let $E_1 = I - E_0$. Since $E_1 \geqslant 0$, we have spectral decomposition for $E_1 = \sum_i \lambda_i |\psi_i\rangle\langle\psi_i|$. Setting $M_i = \sqrt{\lambda_i}|i\rangle\langle\psi_i|$, we find that $E_i = M_i^\dagger M_i$ is an IC positive operator-valued measure for every i. ■

Now, we give the proof of Theorem 7.

Proof. If $|\phi\rangle = A_1 \otimes A_2 \otimes \cdots \otimes A_N |\psi\rangle$ holds with SIO (invertible IC) operators A_k with $k = 1, 2, \ldots, N$, then we can find a sLICC protocol for the parties to transform $|\psi\rangle$ into $|\phi\rangle$ with a positive probability of success. Indeed, each party k can perform an M-outcome IC measurement $\{F_0^{(k)}, F_1^{(k)}, \ldots, F_M^{(k)}\}$, where $F_0^{(k)} = \sqrt{\frac{p_k}{\langle\psi_k|A_k^\dagger A_k|\psi_k\rangle}}A_k$ with $0 < p_k \leqslant 1$. It is easy to check that, after all parties have performed their corresponding measurements, the transformation from $|\psi\rangle$ to $|\phi\rangle$ will succeed with probability $p_1 p_2 \cdots p_N$. The analysis also holds for $|\phi\rangle$ converting into $|\psi\rangle$ by observing that $|\psi\rangle = A_1^{-1} \otimes A_2^{-1} \otimes \cdots \otimes A_N^{-1} |\phi\rangle$.

Conversely, suppose that there is a sLICC protocol, consisting of IC measurements $F^{(k)}$ performed by the kth party such that $|\psi\rangle$ is transformed into $|\phi\rangle$. Then, there must exist one branch of all possible protocol outcomes, say $(x_1, x_2, \ldots, x_N)$ in which $|\phi\rangle$ is obtained. Tracking the performed measurement of each party $F_{x_k}^{(k)}$, the corresponding IC operators A_k are obtained as follows:

$$\begin{aligned}\frac{1}{\sqrt{p_k}} I_{A_1\cdots A_{k-1}A_{k+1}\cdots A_N} \otimes F_{x_k}^{(k)} |\psi^{(k-1)}\rangle \\ = I_{A_1\cdots A_{k-1}A_{k+1}\cdots A_N} \otimes A_k |\psi^{(k-1)}\rangle = |\psi^{(k)}\rangle, \end{aligned} \tag{4}$$

with $p_k = \langle\psi^{(k-1)}| F_{x_k}^{(k)\dagger} F_{x_k}^{(k)} |\psi^{(k-1)}\rangle$, $|\psi^{(0)}\rangle = |\psi\rangle$, and $|\psi^{(N)}\rangle = |\phi\rangle$. To summarize,

$$|\phi\rangle = A_1 \otimes A_2 \otimes \cdots \otimes A_N |\psi\rangle . \tag{5}$$

Lastly, A_k must be full rank (hence revertible) since the number of nonzero product terms in $|\psi\rangle$ and $|\phi\rangle$ must be equal, a consequence of Lemma 5. ■

To conclude this section, we use an example to demonstrate that the N-partite coherence states are already more versatile than entanglement when $N = 2$.

Example [Characterization of two-qubit coherence states]. Consider a two-qubit system with the fixed basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$. Our classification is based on the number of product terms R in a two-qubit pure state since this number will not be altered by invertible SIO operators. The following table lists all inequivalent classes of two-qubit states. ■

Now, we will give a detail analysis about this two-qubit coherence states example.

Proof. From Lemma 5, we know that the classification is restricted by the number of product terms under sLICC, thus we list these potential equivalent classes based on the number of product terms. The classification of the number of product terms equal to 1 is trivial: every fixed basis $|ij\rangle$ with $i, j \in \{0, 1\}$ can be converted to each other via local SIO operators.

TABLE I. All inequivalent classes of two-qubit states.

R	Classification
1	$\|00\rangle$
2	$a\|00\rangle + b\|01\rangle$, $a\|00\rangle + c\|10\rangle$, $a\|00\rangle + d\|11\rangle$
3	$a\|00\rangle + b\|01\rangle + c\|10\rangle$, $a\|00\rangle + b\|01\rangle + d\|11\rangle$
4	Infinitely many (based on different $\mathcal{R}$'s)

When the number of product terms equals 2, we can conclude the following three classes: $a|00\rangle + b|01\rangle$, $a|00\rangle + c|10\rangle$, and $a|00\rangle + d|11\rangle$ after considering local SIO operators allowed by each party. A similar method can be used when the number of product terms equals 3.

When the number of product terms equals 4, let $|\psi\rangle = a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle$. If $|\psi\rangle$ can be converted to $|\phi\rangle = a'|00\rangle + b'|01\rangle + c'|10\rangle + d'|11\rangle$ under sLICC, then there exist local SIO operators A and B such that $|\phi\rangle = A \otimes B|\psi\rangle$. Each of the SIO operators A, B can be expressed in two forms: $\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$ or $\begin{pmatrix} 0 & z \\ w & 0 \end{pmatrix}$. If $A = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$ and $B = \begin{pmatrix} z & 0 \\ 0 & w \end{pmatrix}$ or $A = \begin{pmatrix} 0 & x \\ y & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & z \\ w & 0 \end{pmatrix}$, we find that $\frac{ad}{bc} = \frac{a'd'}{b'c'}$ after the SIO operations. If $A = \begin{pmatrix} 0 & x \\ y & 0 \end{pmatrix}$ and $B = \begin{pmatrix} z & 0 \\ 0 & w \end{pmatrix}$ or $A = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$ and $B = \begin{pmatrix} 0 & z \\ w & 0 \end{pmatrix}$, we find that $\frac{ad}{bc} = \frac{b'c'}{a'd'}$ after the SIO operations. Denoted by $\mathcal{R} = \frac{ad}{bc}$, we conclude that states with the same number $\mathcal{R}$ or $\frac{1}{\mathcal{R}}$ are in the same equivalent class under such a transformation. ■

We can show that it is possible that there are some classes of states in multipartite entanglement that can convert into each other, whereas they cannot convert into each other in multipartite coherence even in the two-qubit case.

From the example above, we can see sLICC equivalent class of $|\psi\rangle = a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle$ can form a one-parameter family of states.

Corollary 9. Any state $|\psi\rangle = a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle$ which the number of product terms equals 4 is sLICC equivalent to a state with form $|\psi'\rangle = \alpha(|00\rangle + |01\rangle + |10\rangle) + \beta|11\rangle$, where β is any complex number with $0 < |\beta| < 1$, and α is the real number determined by normalization. That is, the sLICC equivalent class of $|\psi\rangle = a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle$ can form an one-parameter family of states.

Proof. As discussed above, we find the following four operators:

$$\begin{pmatrix} \frac{\alpha}{b\beta} & 0 \\ 0 & \frac{1}{d} \end{pmatrix} \otimes \begin{pmatrix} \frac{d\alpha}{c} & 0 \\ 0 & \beta \end{pmatrix}, \tag{6}$$

$$\begin{pmatrix} \frac{\alpha}{b\beta} & 0 \\ 0 & \frac{1}{c} \end{pmatrix} \otimes \begin{pmatrix} 0 & \frac{c\alpha}{d} \\ \beta & 0 \end{pmatrix}, \tag{7}$$

$$\begin{pmatrix} 0 & \frac{\alpha}{b\beta} \\ \frac{1}{b} & 0 \end{pmatrix} \otimes \begin{pmatrix} \frac{b\alpha}{a} & 0 \\ 0 & \beta \end{pmatrix}, \tag{8}$$

$$\begin{pmatrix} 0 & \frac{\alpha}{c\beta} \\ \frac{1}{a} & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & \frac{a\alpha}{b} \\ \beta & 0 \end{pmatrix} \tag{9}$$

can transform $|\psi\rangle = a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle$ to $|\psi'\rangle = \alpha(|00\rangle + |01\rangle + |10\rangle) + \beta|11\rangle$, where $0 < |\beta| < 1$ and $\alpha = \frac{\sqrt{1-|\beta|^2}}{\sqrt{3}} > 0$. Thus, state $|\psi\rangle = a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle$ is sLICC equivalent to state $|\psi'\rangle = \alpha(|00\rangle + |01\rangle + |10\rangle) + \beta|11\rangle$ via those operators above. ■

Remark. In entanglement theory, the degree of entanglement can be measured by concurrence C_E [29]. If we write $|\psi\rangle = a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle$, then $C_E(|\psi\rangle) = 2|ad - bc|$ [29]. Suppose that $|\psi\rangle$ and $|\phi\rangle$ are sLOCC equivalent, then there exist local invertible operators A and B such that $|\phi\rangle = A \otimes B|\psi\rangle$ [22]. Consequently, $C_E(|\phi\rangle) = \det(A)\det(B)C_E(|\psi\rangle)$. There are only two sLOCC equivalent classes: $C_E(|\psi\rangle) = 0$ and $C_E(|\psi\rangle) \neq 0$. Equivalently, the classification depends on either $ad = bc$ or $ad \neq bc$. The classification of sLICC is different from the classification of sLOCC in entanglement theory. As shown in the two-qubit case, sLICC classification depends both on the number of product terms and on the number $\mathcal{R} = \frac{ad}{bc}$. Because the sLICC equivalent class of $|\psi\rangle = a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle$ can form a one-parameter family of states, we can further simplify $\mathcal{R}$ as $\mathcal{R} = \frac{\beta}{\alpha} = \frac{3\beta}{\sqrt{1-|\beta|^2}}$.

IV. TWO OPERATIONAL COHERENCE MONOTONES: ACCESSIBLE COHERENCE AND SOURCE COHERENCE

In this section, we will recall the framework for quantifying the resource of coherence theory and then introduce two operational coherence monotones: accessible coherence and source coherence. Our idea comes from Schwaiger *et al.* [24] and Sauerwein *et al.* [30] in which the authors studied similar entanglement measures: accessible entanglement and source entanglement.

Baumgratz *et al.* [1] proposed a seminal framework for quantifying coherence as a resource. For a fixed basis $\{|i\rangle\}$, a functional C can be taken as a coherence measure if it satisfies the following four conditions:

(B1) $C(\rho) \geqslant 0$ for all quantum states, and $C(\rho) = 0$ if $\rho \in \mathcal{I}$, where $\mathcal{I}$ is the set of incoherence states which are diagonal in basis $\{|i\rangle\}$;

(B2) $C(\rho) \geqslant C[\Phi(\rho)]$ for all free operations Φ;

(B3) $C(\rho) \geqslant \sum_n p_n C(\rho_n)$, where $p_n = \mathrm{Tr}(K_n \rho K_n^\dagger)$, $\rho_n = \frac{1}{p_n} K_n \rho K_n^\dagger$, and K_n are the Kraus operators of an incoherent CPTP map $\Phi(\rho) = \sum_n K_n \rho K_n^\dagger$;

(B4) $\sum_i p_i C(\rho_i) \geqslant C(\rho)$ for $\rho = \sum_i p_i \rho_i$.

Similar to entanglement, the function C is a coherence monotone if it satisfies conditions (B1) and (B2).

Accessible coherence and source coherence

For two given states ρ and σ in a Hilbert space $\mathcal{H}$ with finite dimension d, we say that state ρ can reach state σ if there exists a free operator in the set $\mathcal{O}$ that transforms ρ into σ (deterministically). In this case, σ is accessible from ρ. We denote by $M_a^{\mathcal{O}}(\rho)$ the set of states that can be reached from ρ via free operations in the set $\mathcal{O}$ and denote by $M_s^{\mathcal{O}}(\rho)$ the set of states that can reach ρ via free operations in $\mathcal{O}$ (see Fig. 2).

We define two related magnitudes: the *accessible volume* $V_a^{\mathcal{O}}(\rho) = \mu[M_a^{\mathcal{O}}(\rho)]$, which quantifies the volume of states that can be reached by state ρ, and the *source volume* $V_s^{\mathcal{O}}(\rho) = \mu[M_s^{\mathcal{O}}(\rho)]$, which quantifies the volume of states that can reach ρ via free operations. Here, μ could be an arbitrary Lebesgue measure which maps the set of density matrices to non-negative real numbers such that $\mu[M_a^{\mathcal{O}}(\rho)] = 0$ and $\mu[M_s^{\mathcal{O}}(\rho)]$ reaches the maximally source volume if ρ is an incoherent state.

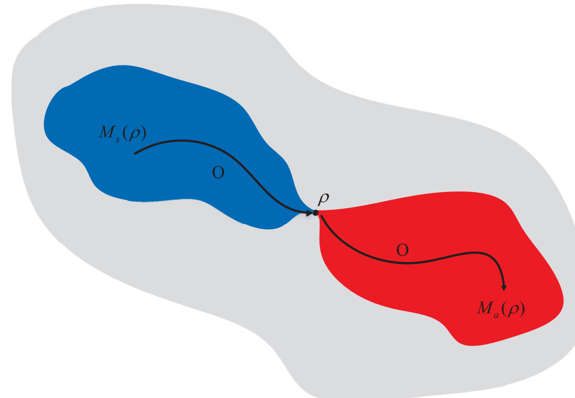

FIG. 2. In this schematic, the source set $M_s^{\mathcal{O}}(\rho)$ and the accessible set $M_a^{\mathcal{O}}(\rho)$ of state ρ are depicted. Any state in $M_s^{\mathcal{O}}(\rho)$ can be transformed to ρ via $\mathcal{O}$, and ρ can be transformed into any state in $M_a^{\mathcal{O}}(\rho)$ via $\mathcal{O}$.

The operational meaning is clear: If $M_a^{\mathcal{O}}(\rho)$ is larger than $M_a^{\mathcal{O}}(\rho')$, then state ρ could potentially be more useful than ρ' in quantum information-processing applications. On the other hand, if $M_s^{\mathcal{O}}(\rho)$ is too small, then not many states are useful than state ρ for any potential application, i.e, ρ is very useful than many other states in applying for the resource of coherence. We can then define the accessible coherence and the source coherence as follows:

$$C_a^{\mathcal{O}}(\rho) = \frac{V_a^{\mathcal{O}}(\rho)}{V_a^{\sup,\mathcal{O}}}, \tag{10}$$

and

$$C_s^{\mathcal{O}}(\rho) = 1 - \frac{V_s^{\mathcal{O}}(\rho)}{V_s^{\sup,\mathcal{O}}}, \tag{11}$$

where $V_a^{\sup,\mathcal{O}}$ ($V_s^{\sup,\mathcal{O}}$) denotes the maximal accessible (source) volume according to the measure μ.

We will now show that both accessible coherence and source coherence are coherence monotones.

Theorem 10. Both accessible coherence and source coherence satisfy conditions (B1) and (B2), thus they are coherence monotones.

Proof. It is evident that, if a state ρ is an incoherent state, then $\mu[M_a^{\mathcal{O}}(\rho)] = 0$ and $\mu[M_s^{\mathcal{O}}(\rho)]$ reaches the maximal source volume, i.e., $C_a^{\mathcal{O}}(\rho) = C_s^{\mathcal{O}}(\rho) = 0$ if $\rho \in \mathcal{I}$. We also have $C_a^{\mathcal{O}}(\rho) \geqslant 0$ and $C_s^{\mathcal{O}}(\rho) \geqslant 0$ for all quantum states. Condition (B1) therefore holds.

Let σ_ρ be the state that can be reached from ρ via free operations $\mathcal{O}$. By definition, $M_a^{\mathcal{O}}(\sigma_\rho) \subset M_a^{\mathcal{O}}(\rho)$, which means that $C_a^{\mathcal{O}}(\rho) \geqslant C_a^{\mathcal{O}}(\sigma_\rho)$. On the other hand, let ρ be the state that can be reached from ρ_σ via free operations $\mathcal{O}$. Then, $M_s^{\mathcal{O}}(\rho) \subset M_s^{\mathcal{O}}(\rho_\sigma)$, which means that $C_s^{\mathcal{O}}(\rho_\sigma) \geqslant C_s^{\mathcal{O}}(\rho)$. Condition (B2) therefore holds. For conditions (B3) and (B4), we can construct examples to show they do not hold, and we refer the interested readers to the Appendix for the detailed calculation. ■

Remark. Just as the in case in entanglement theory [24,30], we would also be interested in the transformations between specific classes of coherent states (e.g., the pure states transformation), and the specific volumes $V_a^{\mathcal{O}}(\rho)$ and $V_s^{\mathcal{O}}(\rho)$ are only supported in these classes. For example, for single-qubit states, any state can be represented as a point on (or in) the Bloch sphere. We can choose the superficial area on the sphere as the specific volumes $V_a^{\mathcal{O}}(\rho)$ and $V_s^{\mathcal{O}}(\rho)$ for this pure state transformation. Meanwhile, If a pure state $|\psi\rangle$ can be transformed to $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$ via free operations $\mathcal{O}$, there will be a channel $\Phi^{\mathcal{O}} = \sum_i p_i \Phi_i^{\mathcal{O}}$ corresponding to $\mathcal{O}$, where $\Phi_i^{\mathcal{O}}(|\psi\rangle) = |\psi_i\rangle$ for any i. It means that the more pure states we obtain, the more generic states will be obtained. Thus, in this sense, the proficiency of a pure state to generate other pure states characterizes its accessible coherence.

V. SOURCE COHERENCE AND ACCESSIBLE COHERENCE FOR A PURE STATE TRANSFORMS VIA LSICC AND LICC

In the following, we will derive explicit formulas for source coherence of pure state transforms via $\mathcal{O} \in \{\text{LSICC, LICC}\}$. We consider representatives of LIU classes. To obtain the source coherence, we have as follows:

Theorem 11. The source coherence of a bipartite state $|\phi\rangle = \sum_{i=1}^d \sqrt{\lambda_i}\,|ii\rangle$ with sorted Schmidt vector $\lambda(\phi_{AB}) = (\lambda_1, \lambda_2, \ldots, \lambda_d)$ is given by

$$C_s^{\mathcal{O}}(|\psi\rangle) = 1 - \sum_{\pi\in\Sigma_d} \frac{\left[\sum_{k=1}^d \pi(k)\lambda_k - \frac{d+1}{2}\right]^{d-1}}{\prod_{k=1}^{d-1} \pi(k) - \pi(k+1)}, \tag{12}$$

where $\mathcal{O} \in \{\text{LSICC, LICC}\}$.

Proof. In this section, μ is chosen as a measure on the set of LIU equivalence classes. In the proof of Lemma 3 since the IC operators Alice and Bob used are full rank. Thus, the local IC operators are local SIO operators. Then, we have as follows:

First, we will show the explicit formula of source coherence for pure states transforms via $\mathcal{O} \in \{\text{LSICC, LICC}\}$. From Lemma 2, we can see the source set of $|\psi\rangle$ is given by

$$M_s^{\mathcal{O}}(\psi) = \{|\phi\rangle \in \mathcal{H} \text{ such that } \lambda(\phi) \prec \lambda(\psi)\}, \tag{13}$$

because any pure states in a LIU equivalence class can be seen as the vector $\lambda(\psi)$, we can associate the set given in Eq. (13) the following set of sorted vectors in $\mathcal{R}^d$:

$$\mathcal{M}_s^{\mathcal{O}}(\psi) = \{\lambda^{\downarrow} \in \mathcal{R}^d \text{ such that } \lambda^{\downarrow} \prec \lambda(\psi)\}, \tag{14}$$

where d denotes the Schmidt number of $|\psi\rangle$.

The set given in Eq. (14) is a convex polytope, and as shown in Ref. [30], the source set Eq. (14) is a simple polytope [31,32]. The simple polytope of the set $\mathcal{M}_s^{\mathcal{O}}(\psi)$ is the same as the polytope of the source set of entanglement as shown in Ref. [30], thus the volume of $\mathcal{M}_s^{\mathcal{O}}(\psi)$ is

$$V_s^{\mathcal{O}}(|\psi\rangle) = \frac{1}{d!}\frac{\sqrt{d}}{(d-1)!}\sum_{\pi\in\Sigma_d} \frac{\left[\sum_{k=1}^d \pi(k)\lambda_k - \frac{d+1}{2}\right]^{d-1}}{\prod_{k=1}^{d-1} \pi(k) - \pi(k+1)}, \tag{15}$$

where π denotes an element of the permutation group Σ_d of d elements and $\mathcal{O} \in \{\text{LSICC, LICC}\}$. Note that for the incoherent state $|\psi_{\text{incoh}}\rangle$, the vector $\lambda(\psi_{\text{incoh}}) = (1, 0, 0, \ldots, 0)$ can be obtained from any other state via $\mathcal{O} \in \{\text{LSICC, LICC}\}$, and therefore its source volume is the maximum, i.e., $V_s^{\mathcal{O}}(|\psi_{\text{incoh}}\rangle) = \sup_\phi V_s^{\mathcal{O}}(|\phi\rangle)$. The volume is $V_s^{\mathcal{O}}(|\psi_{\text{incoh}}\rangle) = \frac{\sqrt{d}}{d!(d-1)!}$. For a maximal correlated state $|\psi^+\rangle$, the corresponding vector is $\lambda(\psi^+) = \frac{1}{\sqrt{d}}(1, 1, 1, \ldots, 1)$. It is straightforward to see that the volume is $V_s^{\mathcal{O}}(|\psi^+\rangle) = 0$.

Thus, the source coherence of a pure state $|\phi\rangle$ with sorted vector $\lambda(\phi)$ is given by

$$C_s^{\mathcal{O}}(|\psi\rangle) = 1 - \sum_{\pi\in\Sigma_d} \frac{\left[\sum_{k=1}^d \pi(k)\lambda_k - \frac{d+1}{2}\right]^{d-1}}{\prod_{k=1}^{d-1} \pi(k) - \pi(k+1)}, \tag{16}$$

where $\mathcal{O} \in \{\text{LSICC, LICC}\}$. ■

For a bipartite state $|\phi\rangle$ with Schmidt decomposition $|\phi\rangle = \sum_{i=1}^d \sqrt{\lambda_i}\,|\phi_i^A\rangle\,|\phi_i^B\rangle$, if the Schmidt bases $|\phi_i^A\rangle$ and $|\phi_i^B\rangle$ are incoherent bases, both source entanglement and source coherence (via $\mathcal{O} \in \{\text{LSICC, LICC}\}$) will have identical volumes. However, the source entanglement and source coherence (via $\mathcal{O} \in \{\text{LSICC, LICC}\}$) will have different volumes if the Schmidt bases $|\phi_i^A\rangle$ and $|\phi_i^B\rangle$ are not incoherent bases.

Note that, for a state $|\psi\rangle = \sum_i \sqrt{\lambda_i}\,|i\rangle$ and its "maximally correlated" state $|\psi'\rangle = \sum_i \sqrt{\lambda_i}\,|ii\rangle$, $\lambda^{\downarrow}[\Delta(\psi)] = \lambda^{\downarrow}(\psi')$. This implies that both accessible coherence and source coherence are the same for those two states.

Next, we will give two examples to derive the formula of accessible coherence and source coherence in the case of LICC and LSICC transformations.

Example [LICC and LSICC transformations of qitrit-qutrit pure states].

Consider the following two pure states $|\psi\rangle = \sqrt{a}\,|00\rangle + \sqrt{b}\,|11\rangle + \sqrt{c}\,|22\rangle$ and $|\phi\rangle = \sqrt{a}\,|00\rangle + \sqrt{b}\,|01\rangle + \sqrt{c}\,|02\rangle$ with $a \geqslant b \geqslant c \geqslant 0$ and $c = 1 - a - b$. Observe that $|\psi\rangle$ can be represented as a point on the x-y plane where the x axis represents the basis $|00\rangle$ and the y axis represents the basis vector $|01\rangle$. Similarly, $|\phi\rangle$ can be represented as a point on the x-z plane where in addition the z axis represents the basis vector $|11\rangle$. The accessible volume and the source volume are

$$V_a^{\mathcal{O}}(|\psi\rangle) = V_a^{\mathcal{O}}(|\phi\rangle) = \tfrac{1}{2}[(1-a)^2 - b^2], \tag{17}$$

$$V_s^{\mathcal{O}}(|\psi\rangle) = V_s^{\mathcal{O}}(|\phi\rangle) = \tfrac{1}{2}[(a+b)^2 - b^2], \tag{18}$$

where $\mathcal{O} \in \{\text{LSICC, LICC}\}$. Thus, the accessible coherence and the source coherence are

$$C_a^{\mathcal{O}}(|\psi\rangle) = C_a^{\mathcal{O}}(|\phi\rangle) = (1-a)^2 - b^2, \tag{19}$$

$$C_s^{\mathcal{O}}(|\psi\rangle) = C_s^{\mathcal{O}}(|\psi\rangle) = 1 - (a+b)^2 + b^2. \tag{20}$$

Figure 3 shows the accessible set and the source set are different between $|\psi\rangle$ and $|\phi\rangle$. ■

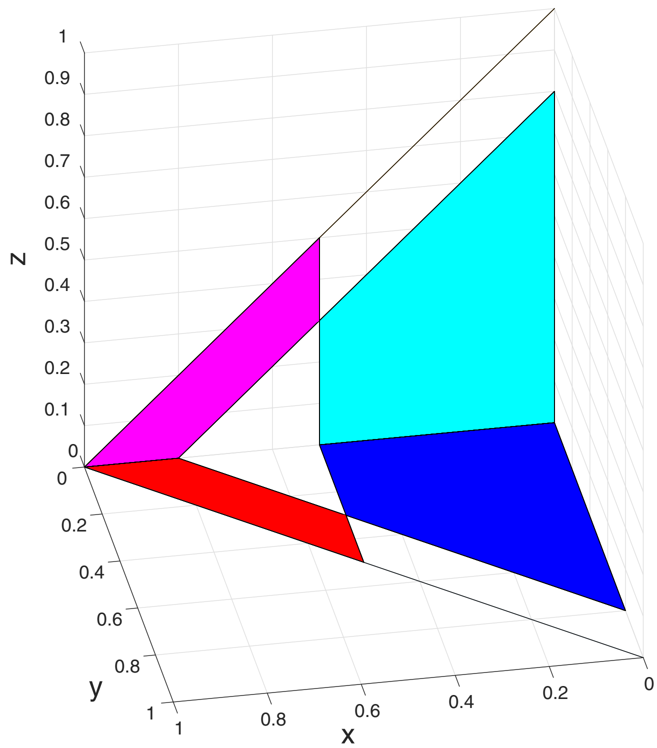

FIG. 3. The source set (blue) $M_s(|\psi\rangle)$ and the accessible set (red) $M_a(|\psi\rangle)$ of state $|\psi\rangle$ are depicted. Meanwhile, the source set (cyan) $M_s(|\phi\rangle)$ and the accessible set (magenta) $M_a(|\phi\rangle)$ of state $|\phi\rangle$ are depicted. Here, the x axis represents the basis vectors $|00\rangle$, the y axis represents the basis vectors $|01\rangle$, and the z axis represents the basis vectors $|11\rangle$. In the figure, $|\psi\rangle = \sqrt{0.5}\,|00\rangle + \sqrt{0.3}\,|11\rangle + \sqrt{0.2}\,|22\rangle$ and $|\phi\rangle = \sqrt{0.5}\,|00\rangle + \sqrt{0.3}\,|01\rangle + \sqrt{0.2}\,|02\rangle$. The source sets (accessible sets) are indeed different between these two states.

Example [LICC and LSICC transformations of two-qubit pure states].

Let $|\psi\rangle = \sqrt{a}\,|01\rangle + \sqrt{b}\,|10\rangle$, where $a + b = 1$. We consider state $|\psi'\rangle = \sqrt{a}\,|00\rangle + \sqrt{b}\,|11\rangle$ with Schmidt coefficients a and b. It is then straightforward to find that the accessible volume and source volume are

$$V_a^{\mathcal{O}}(|\psi\rangle) = \sqrt{2}\big(x - \tfrac{1}{2}\big), \tag{21}$$

$$V_s^{\mathcal{O}}(|\psi\rangle) = \sqrt{2}(1 - x), \tag{22}$$

where $x \geqslant y$, $x, y \in \{a, b\}$, and $\mathcal{O} \in \{\text{LICC, LSICC}\}$. Figure 4 shows this state transformation, thus the accessible coherence and source coherence are

$$C_a^{\mathcal{O}}(|\psi\rangle) = C_s^{\mathcal{O}}(|\psi\rangle) = 2(1 - x). \tag{23}$$

■

VI. SOURCE COHERENCE AND ACCESSIBLE COHERENCE FOR PURE STATE TRANSFORMS VIA PIO, SIO, AND IC

First, we will discuss source coherence for pure state transformation via $\mathcal{O} \in \{\text{SIO, IC}\}$. The method of this proof is similar to the proof of Theorem 11. Note that μ is chosen as

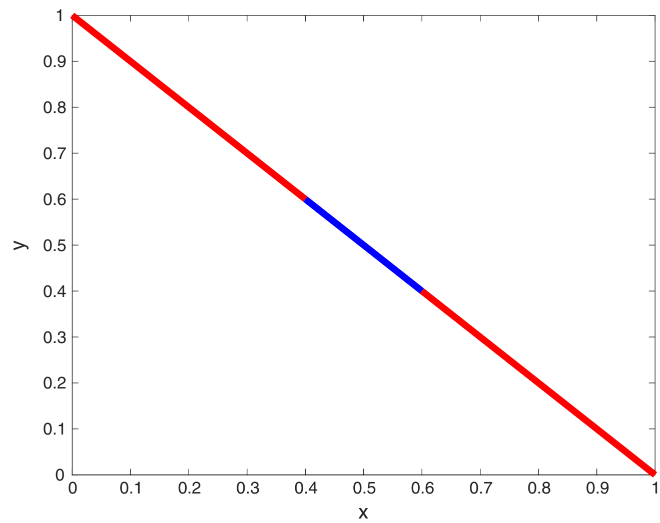

FIG. 4. The source set (blue) $M_s(|\psi\rangle)$ and the accessible set (red) $M_a(|\psi\rangle)$ of state $|\psi\rangle$ are depicted. Any state in $M_s(|\psi\rangle)$ can be transformed to $|\psi\rangle$ via LSICC and LICC, and $|\psi\rangle$ can be transformed into any state in $M_a(|\psi\rangle)$ via LSICC and LICC. In the figure, $|\psi\rangle = \sqrt{0.6}\,|01\rangle + \sqrt{0.4}\,|10\rangle$.

a measure on the set of IU equivalence classes. From Lemma 2, the source set of $|\psi\rangle$ is given by

$$M_s^{\mathcal{O}}(\psi) = \{|\phi\rangle \in \mathcal{H} \text{ such that } \lambda[\Delta(\phi)] \prec \lambda[\Delta(\psi)]\}. \tag{24}$$

Because any pure states in an IU equivalence class can be seen as the vector $\lambda[\Delta(\psi)]$, we can associate the set given in Eq. (24) as the following sets of sorted vectors in $\mathcal{R}^d$:

$$\mathcal{M}_s^{\mathcal{O}}(\psi) = \{\lambda^{\downarrow} \in \mathcal{R}^d \text{ such that } \lambda^{\downarrow} \prec \lambda[\Delta(\psi)]\}, \tag{25}$$

where d denotes the rank of $\Delta(\psi)$; these sets are hence supported on states of the same dimensions as $\Delta(\psi)$.

Then, the set given in Eqs. (25) is also a simple polytope [30]. Thus, the volume of $\mathcal{M}_s^{\mathcal{O}}(\psi)$ is

$$V_s^{\mathcal{O}}(|\psi\rangle) = \frac{1}{d!}\frac{\sqrt{d}}{(d-1)!}\sum_{\pi\in\Sigma_d}\frac{\left[\sum_{k=1}^{d}\pi(k)\lambda_k - \frac{d+1}{2}\right]^{d-1}}{\prod_{k=1}^{d-1}\pi(k) - \pi(k+1)}, \tag{26}$$

where π denotes an element of the permutation group Σ_d of d elements and $\mathcal{O} \in \{\text{SIO, IC}\}$. Note that for incoherent state $|\psi_{\text{incoh}}\rangle$, the vector $\lambda[\Delta(\psi_{\text{incoh}})] = (1, 0, 0, \ldots, 0)$ can be obtained from any other states via $\mathcal{O} \in \{\text{SIO, IC}\}$, and therefore its source volume is the maximum, i.e., $V_s^{\mathcal{O}}(|\psi_{\text{incoh}}\rangle) = \sup_\phi V_s^{\mathcal{O}}(|\phi\rangle)$. The volume is $V_s^{\mathcal{O}}(|\psi_{\text{incoh}}\rangle) = \frac{\sqrt{d}}{d!(d-1)!}$. For a maximally coherent state $|\psi^+\rangle$, the corresponding vector $\lambda[\Delta(\psi^+)] = \frac{1}{\sqrt{d}}(1, 1, 1, \ldots, 1)$. It is straightforward to see that the volume is $V_s^{\mathcal{O}}(|\psi^+\rangle) = 0$.

Thus, we have following result for a pure state with sorted vector $\lambda[\Delta(\phi)] \in \mathcal{C}^d$:

Theorem 12. The source coherence of a pure state $|\phi\rangle$ with sorted vector $\lambda[\Delta(\phi)]$ is given by

$$C_s^{\mathcal{O}}(|\psi\rangle) = 1 - \sum_{\pi\in\Sigma_d}\frac{\left[\sum_{k=1}^{d}\pi(k)\lambda_k - \frac{d+1}{2}\right]^{d-1}}{\prod_{k=1}^{d-1}\pi(k) - \pi(k+1)}, \tag{27}$$

where $\mathcal{O} \in \{\text{SIO, IC}\}$.

Next, we will explicitly calculate the accessible coherence and the source coherence for low-dimensional states.

For a single-qubit state, we calculate the coherence monotones by choosing different free operations, i.e., PIO, SIO, and IC.

Example [Single-qubit state transformation via SIO and IC].

If the set measure μ denotes the volume of these transform ranges above, for $\mathcal{O} \in \{\text{SIO}, \text{IC}\}$, we find that $C_a^{\mathcal{O}}(\rho)$ and $C_s^{\mathcal{O}}(\rho)$ in the qubit case is as follows:

$$V_a^{\mathcal{O}}(\rho) = 2\sqrt{\frac{r_x^2 + r_y^2}{1 - r_z^2}} \arcsin\sqrt{1 - r_z^2} + 2|r_z|\sqrt{r_x^2 + r_y^2}, \quad (28)$$

and

$$V_s^{\mathcal{O}}(\rho) = \begin{cases} 2 \arcsin\sqrt{1 - r_x^2 - r_y^2} - 2\sqrt{1 - r_x^2 - r_y^2}\sqrt{1 - r_z^2} - 2\sqrt{\frac{r_x^2+r_y^2}{1-r_z^2}} \arcsin|r_z| + 2|r_z|\sqrt{r_x^2 + r_y^2}, & r_x^2 + r_y^2 + r_z^2 \neq 1, \\ 2 \arcsin|r_z| - 2|r_z|\sqrt{1 - r_z^2}, & r_x^2 + r_y^2 + r_z^2 = 1. \end{cases} \quad (29)$$

Note that $\sqrt{r_x^2 + r_y^2}$ is the l_1-norm coherence of ρ. The supremum of $V_a^{\mathcal{O}}(\rho)$ is in the case $z_0 = 0$, thus $V_a^{\text{sup},\mathcal{O}}(\rho) = \pi$. Combining V_a^{sup} with V_a, we have

$$C_a^{\mathcal{O}}(\rho) = \frac{2}{\pi}\left(\sqrt{\frac{r_x^2 + r_y^2}{1 - r_z^2}} \arcsin\sqrt{1 - r_z^2} + |r_z|\sqrt{r_x^2 + r_y^2}\right). \quad (30)$$

If ρ is pure state, we have

$$C_a^{\mathcal{O}}(\rho) = \frac{2}{\pi}\left(\arcsin\sqrt{1 - r_z^2} + |r_z|\sqrt{1 - r_z^2}\right). \quad (31)$$

We also have $V_s^{\text{sup}} = \pi$, and note that $1 - \arcsin|r_z| = \arcsin\sqrt{1 - r_z^2}$, thus,

$$C_s^{\mathcal{O}}(\rho) = \begin{cases} 1 - \frac{2}{\pi}\left(\arcsin\sqrt{1 - r_x^2 - r_y^2} - \sqrt{1 - r_x^2 - r_y^2}\sqrt{1 - r_z^2} - \sqrt{\frac{r_x^2+r_y^2}{1-r_z^2}} \arcsin|r_z| + |r_z|\sqrt{r_x^2 + r_y^2}\right), & r_x^2 + r_y^2 + r_z^2 \neq 1, \\ \frac{2}{\pi}\left(\arcsin\sqrt{1 - r_z^2} + |r_z|\sqrt{1 - r_z^2}\right), & r_x^2 + r_y^2 + r_z^2 = 1. \end{cases} \quad (32)$$

For SIO and IC, accessible coherence and source coherence are consistent in the pure state case (see Fig. 5). ■

Example [Single-qubit state transformation via PIO].

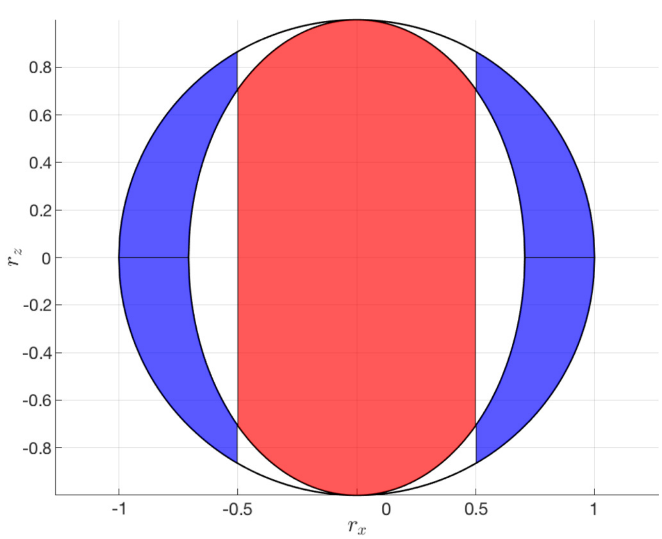

FIG. 5. As shown in the figure, the source set (blue color) $M_s(\rho)$ and the accessible set (red color) $M_a(\rho)$ of state ρ are depicted. Any state in $M_s(\rho)$ can be transformed to ρ via IC and SIO, and ρ can be transformed into any state in $M_a(\rho)$ via IC and SIO. In the figure, the single-qubit ρ has the Bloch vector $(\frac{1}{2}, 0, \frac{\sqrt{2}}{2})$.

As shown in Ref. [2], a single-qubit $\rho = \frac{1}{2}\begin{pmatrix} 1 + r_z & r_x + ir_y \\ r_x - ir_y & 1 - r_z \end{pmatrix}$ converted into $\sigma = \frac{1}{2}\begin{pmatrix} 1 + s_z & s_x + is_y \\ s_x - is_y & 1 - s_z \end{pmatrix}$ via PIO should satisfy the following equalities:

$$s_x^2 + s_y^2 \leqslant r_x^2 + r_y^2, \quad (33)$$

$$\frac{s_x^2 + s_y^2}{r_x^2 + r_y^2} \leqslant \frac{(s_z - 1)^2}{(1 - r_z)^2}, \quad (34)$$

and

$$\frac{s_x^2 + s_y^2}{r_x^2 + r_y^2} \leqslant \frac{(s_z + 1)^2}{(1 - r_z)^2}. \quad (35)$$

Since μ is chosen as a measure on the IU equivalence class, we can consider the transform range on the x-z plane, which is a convex hexagon with six vertices: $(\pm z, \pm\sqrt{r_x^2 + r_y^2})$ and $(\pm 1, 0)$ (more precisely, the IU equivalence classes of states are considered on the first quadrant, but we consider the x-z plane here since the symmetry does not affect the ratio $\frac{V^{\mathcal{O}(\rho)}}{V^{\text{sup}}}$).

The volume of accessible set $V_a^{\text{PIO}}(\rho)$ is the square of these six vertices,

$$V_a^{\text{PIO}}(\rho) = 2|r_z|\sqrt{r_x^2 + r_y^2} + 2\sqrt{r_x^2 + r_y^2}. \quad (36)$$

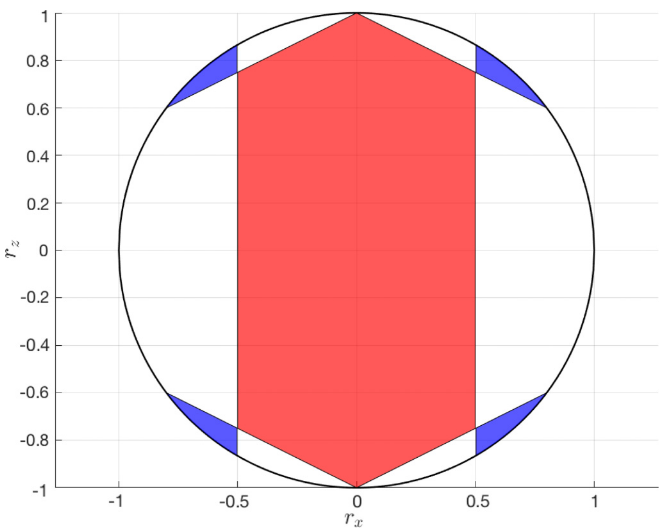

FIG. 6. As shown in the figure, the source set (blue color) $M_s(\rho)$ and the accessible set (red color) $M_a(\rho)$ of state ρ are depicted. Any state in $M_s(\rho)$ can be transformed to ρ via PIO, and ρ can be transformed into any state in $M_a(\rho)$ via PIO. In the figure, the single-qubit ρ has the Bloch vector $(\frac{1}{2}, 0, \frac{3}{4})$.

Since $2|r_z|\sqrt{r_x^2+r_y^2}+2\sqrt{r_x^2+r_y^2} \leqslant r_x^2+r_y^2+r_z^2+2\sqrt{r_x^2+r_y^2}$, the maximal accessible volume can be reached with $r_x^2+r_y^2=r_z^2=\frac{1}{2}$ and $V_a^{\text{sup,PIO}}(\rho)=1+\sqrt{2}$. Thus, we have as follows:

$$C_a^{\text{PIO}}(\rho)=\frac{2|r_z|\sqrt{r_x^2+r_y^2}+2\sqrt{r_x^2+r_y^2}}{1+\sqrt{2}}. \tag{37}$$

On the other hand, a single-qubit $\rho=\frac{1}{2}\begin{pmatrix}1+r_z & r_x+ir_y\\ r_x-ir_y & 1-r_z\end{pmatrix}$ can be transformed from $\sigma=\frac{1}{2}\begin{pmatrix}1+s_z & s_x+is_y\\ s_x-is_y & 1-s_z\end{pmatrix}$ via PIO and should satisfy the following equalities:

$$s_x^2+s_y^2 \geqslant r_x^2+r_y^2, \tag{38}$$

$$\frac{s_x^2+s_y^2}{r_x^2+r_y^2} \geqslant \frac{(s_z-1)^2}{(1-r_z)^2}, \tag{39}$$

$$\frac{s_x^2+s_y^2}{r_x^2+r_y^2} \geqslant \frac{(s_z+1)^2}{(1-r_z)^2}, \tag{40}$$

and

$$s_x^2+s_y^2+s_z^2 \leqslant 1. \tag{41}$$

The volume of source set $V_a^{\text{PIO}}(\rho)$ is as follows:

$$V_s^{\text{PIO}}(\rho)=\begin{cases}2Q_1+2\left[\sin 2\arcsin t-\sin 2\arcsin\sqrt{r_x^2+r_y^2}\right]-S_1, & \sqrt{r_x^2+r_y^2}+r_z^2\geqslant 1,\\ 2Q_2-2\sqrt{r_x^2+r_y^2}\sqrt{1-\left(r_x^2+r_y^2\right)}-S_2, & \sqrt{r_x^2+r_y^2}+r_z^2\leqslant 1, \qquad r_x^2+r_y^2\neq 0,\\ \pi, & r_x^2+r_y^2=0,\end{cases} \tag{42}$$

where $Q_1=\arcsin t-\arcsin\sqrt{r_x^2+r_y^2}$, $Q_2=\frac{\pi}{2}-\arcsin\sqrt{r_x^2+r_y^2}$, $t=\frac{2\sqrt{r_x^2+r_y^2}(1-|r_z|)}{(r_x^2+r_y^2)+(1-|r_z|)^2}$, $S_1=2(|r_z|+\frac{(r_x^2+r_y^2)-(|r_z|-1)^2}{(r_x^2+r_y^2)+(|r_z|-1)^2})$ $(\frac{2\sqrt{r_x^2+r_y^2}(1-|r_z|)}{(r_x^2+r_y^2)+(1-|r_z|)^2}-\sqrt{r_x^2+r_y^2})$, and $S_2=2\frac{|r_z|}{1-|r_z|}\sqrt{r_x^2+r_y^2}$.

Obviously, when $r_x^2+r_y^2=0$, $V_s^{\text{PIO}}(\rho)=V_s^{\text{sup,PIO}}=\pi$. Thus, we have as follows:

$$C_s^{\text{PIO}}(\rho)=\begin{cases}\frac{2}{\pi}Q_1+\frac{2}{\pi}\left[\sin\,(2\arcsin t)-\sin\left(2\arcsin\sqrt{r_x^2+r_y^2}\right)\right]-\frac{1}{\pi}S_1, & \sqrt{r_x^2+r_y^2}+r_z^2\geqslant 1,\\ \frac{2}{\pi}\left[\frac{\pi}{2}-\arcsin\sqrt{r_x^2+r_y^2}\right]-\frac{2}{\pi}\sqrt{r_x^2+r_y^2}\sqrt{1-\left(r_x^2+r_y^2\right)}-\frac{1}{\pi}S_2, & \sqrt{r_x^2+r_y^2}+r_z^2\leqslant 1, \qquad r_x^2+r_y^2\neq 0,\\ 1, & r_x^2+r_y^2=0,\end{cases} \tag{43}$$

where $Q_1=\arcsin t-\arcsin\sqrt{r_x^2+r_y^2}$ (see Fig. 6). ■

Example [Two-qubit pure state transformation via SIO and IC].

Consider a two-qubit pure state $|\psi\rangle=\sqrt{x_0}\,|00\rangle+\sqrt{x_1}\,|01\rangle+\sqrt{x_2}\,|10\rangle$ with $x_0+x_1+x_2=1$. As shown in Fig. 7 since every qutrit pure state can be represented as a point on the x-y plane, the accessible volume and source volume are

$$V_a^{\mathcal{O}}(|\psi\rangle)=\tfrac{1}{2}[(1-x)^2-y^2], \tag{44}$$

$$V_s^{\mathcal{O}}(|\psi\rangle)=\tfrac{1}{2}[(x+y)^2-y^2], \tag{45}$$

where $x\geqslant y\geqslant z$, $x,y,z\in\{x_0,x_1,x_2\}$, and $\mathcal{O}\in\{\text{SIO},\text{IC}\}$.

The accessible coherence and source coherence are

$$C_a^{\mathcal{O}}(|\psi\rangle)=(1-x)^2-y^2, \tag{46}$$

$$C_s^{\mathcal{O}}(|\psi\rangle)=1-(x+y)^2+y^2. \tag{47}$$

■

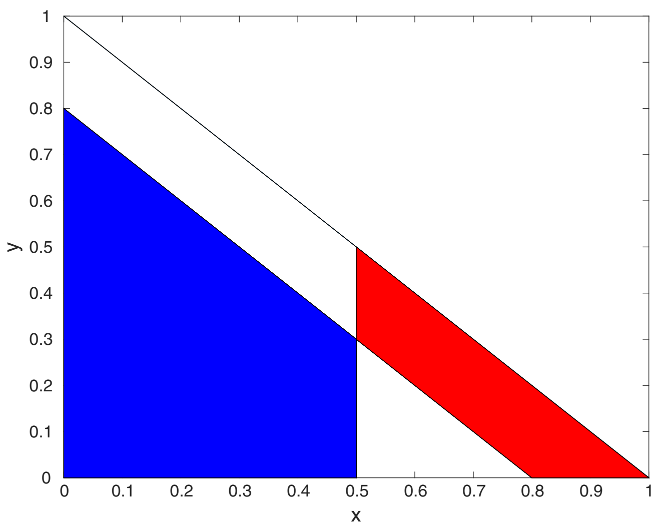

FIG. 7. As shown in the figure, the source set (blue) $M_s(|\psi\rangle)$ and the accessible set (red) $M_a(|\psi\rangle)$ of state ρ are depicted. Any state in $M_s(|\psi\rangle)$ can be transformed to $|\psi\rangle$ via SIO and IC, and $|\psi\rangle$ can be transformed into any state in $M_a(|\psi\rangle)$ via SIO and IC. In the figure, $|\psi\rangle = \sqrt{0.5}\,|00\rangle + \sqrt{0.3}\,|01\rangle + \sqrt{0.2}\,|10\rangle$.

VII. CONCLUSION

In this paper, we have generalized important results in multipartite entanglement theory to their counterparts in quantum coherence theory. First, we gave a necessary and sufficient condition for when two pure multipartite states are equivalent under LICC, i.e., two states can be deterministically transformed to each other under LICC operations. Next, we investigated and gave the conditions in which such a transformation succeeds only stochastically. Different from the entanglement case for two-qubit states, we find that the sLICC equivalence classes are infinite. Thus, it is possible that there are some classes of states in multipartite entanglement that can convert into each other, whereas they cannot convert into each other in multipartite coherence. These results above may help us understand the structure of multipartite coherence states and help us to know how to use coherence as a resource in the multipartite coherence systems.

The other contribution of our paper is as follows: In order to show the difference among sLICC classes, we introduced accessible coherence and source coherence as two coherence monotones. These coherence monotones have straightforward operational interpretations and can be applied in many scenarios (such as PIO, IC, LICC, and LSICC). We also analyzed pure (or mixed) states via IC and derived explicit formulas for the source coherence. We also showed how the accessible coherence can be computed numerically and gave examples. Moreover, we would like to connect these monotones with applications. An interesting option is to study the role of these monotones as figures of merit for known quantum information protocols. This could then lead to the identification of the most relevant multipartite states and maybe allow us to devise new applications of multipartite coherence. Finally, we hope these operational monotones will assist with understanding general quantum resource theories.

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APPENDIX: $C_a^{\mathcal{O}}(\rho)$ AND $C_s^{\mathcal{O}}(\rho)$ ARE COHERENCE MONOTONES THAT DO NOT SATISFY CONDITIONS (B3) AND (B4)

In this Appendix, we will first discuss the accessible coherence $C_a^{\mathcal{O}}(\rho)$. For condition (B1), we can see that, if $\rho \in \mathcal{I}$, then $C_a^{\mathcal{O}}(\rho) = 0$. For condition (B2), it is easy to see that $M_a^{\mathcal{O}}[\Phi(\rho)] \subset M_a^{\mathcal{O}}(\rho)$ for any incoherent operations Φ, thus $C_a^{\mathcal{O}}(\rho) \geqslant C_a^{\mathcal{O}}[\Phi(\rho)]$.

Now, we will show that $C_a^{\mathcal{O}}(\rho)$ does not satisfy condition (B4). A single-qubit state ρ can be represented as $\rho = \frac{1}{2}\begin{pmatrix} 1+z & te^{-i\theta} \\ te^{i\theta} & 1-z \end{pmatrix}$, where $-1 \leqslant z \leqslant 1$, $0 \leqslant t \leqslant 1$, and $0 \leqslant \theta \leqslant \pi$. Since accessible coherence $C_a^{\mathcal{O}}(\rho)$ satisfies condition (B2), without loss of generality, we can always consider the state as $\rho = \frac{1}{2}\begin{pmatrix} 1+z & t \\ t & 1-z \end{pmatrix}$, where $t^2 + z^2 \leqslant 1$. We can see $\rho = \lambda_1 |\lambda_1\rangle\langle\lambda_1| + \lambda_2 |\lambda_2\rangle\langle\lambda_2|$, where $\lambda_1 = \frac{1+\sqrt{t^2+z^2}}{2}$, $\lambda_2 = \frac{1-\sqrt{t^2+z^2}}{2}$, and

$$|\lambda_1\rangle = \frac{t}{2}\frac{z + \sqrt{t^2+z^2}}{t^2+z^2+z\sqrt{t^2+z^2}}|0\rangle + \frac{t}{2}\frac{t}{t^2+z^2+z\sqrt{t^2+z^2}}|1\rangle,$$

$$|\lambda_2\rangle = \frac{t}{2}\frac{z - \sqrt{t^2+z^2}}{t^2+z^2-z\sqrt{t^2+z^2}}|0\rangle + \frac{t}{2}\frac{t}{t^2+z^2-z\sqrt{t^2+z^2}}|1\rangle.$$

Let $t = z = 0.1$, then we can find $C_a^{\mathcal{O}}(\rho) - \lambda_1 C_a^{\mathcal{O}}(|\lambda_1\rangle) - \lambda_2 C_a^{\mathcal{O}}(|\lambda_2\rangle) = 0.0994$. Thus, $C_a^{\mathcal{O}}$ is not convex.

For condition (B3), consider a general amplitude damping channel [25] with $E_0 = \sqrt{p}\begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{pmatrix}$, $E_1 = \sqrt{p}\begin{pmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{pmatrix}$, $E_2 = \sqrt{1-p}\begin{pmatrix} \sqrt{1-\gamma} & 0 \\ 0 & 1 \end{pmatrix}$, and $E_3 = \sqrt{1-p}\begin{pmatrix} 0 & 0 \\ \sqrt{\gamma} & 0 \end{pmatrix}$. Let $p = 0.99$ and $\gamma = t = z = 0.5$, then we can find $C_a^{\mathcal{O}}(\rho) - \sum_n C_a^{\mathcal{O}}(\rho_n) = -0.1912$, thus condition (B3) does not hold.

We will now show that $C_s^{\mathcal{O}}(\rho)$ does not satisfy condition (B4). Let $t = z = 0.1$, then we can find $C_s^{\mathcal{O}}(\rho) - \lambda_1 C_s^{\mathcal{O}}(|\lambda_1\rangle) - \lambda_2 C_s^{\mathcal{O}}(|\lambda_2\rangle) = 0.6930$. Thus, $C_s^{\mathcal{O}}$ is not convex. For condition (B3), consider a general amplitude damping channel [25] with $E_0 = \sqrt{p}\begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{pmatrix}$,

$E_1 = \sqrt{p}\begin{pmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{pmatrix}$, $E_2 = \sqrt{1-p}\begin{pmatrix} \sqrt{1-\gamma} & 0 \\ 0 & 1 \end{pmatrix}$, and $E_3 = \sqrt{1-p}\begin{pmatrix} 0 & 0 \\ \sqrt{\gamma} & 0 \end{pmatrix}$. Let $p = 0.99$ and $\gamma = 0.8$, $t = z = 0.4$, then we find that $C_s^{\mathcal{O}}(\rho) - \sum_n C_s^{\mathcal{O}}(\rho_n) = -0.2123$, thus condition (B3) does not hold.

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