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Generalized Hamacher Aggregation Operators for Intuitionistic Uncertain Linguistic Sets: Multiple Attribute Group Decision Making Methods

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Abstract: In this paper, we consider multiple attribute group decision making (MAGDM) problems in which the attribute values take the form of intuitionistic uncertain linguistic variables. Based on Hamacher operations, we developed several Hamacher aggregation operators, which generalize the arithmetic aggregation operators and geometric aggregation operators, and extend the algebraic aggregation operators and Einstein aggregation operators. A number of special cases for the two operators with respect to the parameters are discussed in detail. Also, we developed an intuitionistic uncertain linguistic generalized Hamacher hybrid weighted average operator to reflect the importance degrees of both the given intuitionistic uncertain linguistic variables and their ordered positions. Based on the generalized Hamacher aggregation operator, we propose a method for MAGDM for intuitionistic uncertain linguistic sets. Finally, a numerical example and comparative analysis with related decision making methods are provided to illustrate the practicality and feasibility of the proposed method.

Keywords: intuitionistic uncertain linguistic sets; Hamacher aggregation operator; multiple attribute group decision making

1. Introduction

In many situations, decision information includes alternative performance measures that are shown in the form of vague variables rather than precise numerical ones. There are many reasons for this, such as time pressure, the level of technical knowledge, people's limited expertise related to the problem domain, and so on. As a result, the information involved in many decision making processes includes a large amount of fuzzy information, for example, the fuzzy linguistic approach intuitionistic fuzzy sets (Atanassov [1], interval-valued intuitionistic fuzzy sets (Atanassov and Gargov [2]), type-2 fuzzy sets (Dubois and Prade [3], Mizumoto and Tanaka [4]), type-n fuzzy sets (Dubois and Prade [3]), fuzzy multisets (Yager [5]), hesitant fuzzy sets (Torra and Narukawa [6]), Pythagorean fuzzy sets (Yager [7,8]), and uncertain variable sets. Especially, since the introduction of intuitionistic fuzzy sets and interval-valued intuitionistic fuzzy sets, intuitionistic fuzzy set theory has been extensively researched and applied in a wide range of fields, including similarity measures (Li and Cheng [9], Xu and Yager [10]), distance measures (Merigó et al. [11–13], Szmidt et al. [14], Tian et al. [15]), linguistic decision making (Yu et al. [16]), aggregation operators (Li [17], Hayat [18], Malik [19], Wang et al. [20], Wang and Liu [21]), and models for multiple attribute group decision making (MAGDM) (Li [22], Nie et al. [23], Peng et al. [24], Yu et al. [25], Zeng et al. [26,27], Morente-Molinera et al. [28], Idrus et al. [29], Bagga et al. [30], Carrasco et al. [31]). On the other hand, decision information with uncertain linguistic information has also attracted a great deal of attention over the last decades. Xu [32] developed the uncertain linguistic OWA (ordered weighted averaging) operator and uncertain linguistic hybrid aggregation operator. Wei [33] presented a method based on the uncertain linguistic hybrid geometric mean operator. Suo et al. [34] studied the DEMATEL method under uncertain linguistic environments. Peng and Ye [35] developed a number of induced uncertain pure linguistic geometric aggregation operators. For further research on this topic, refer to Peng et al. [36], Tao et al. [37], Wang et al. [38], Pamučar et al. [39–41], Liu et al. [42], Si et al. [43], and Stanković et al. [44]. An interesting generalization of the intuitionistic fuzzy sets with linguistic variable information, called the intuitionistic linguistic set was developed by Wang and Li [45]. The intuitionistic linguistic set is the generalization of the intuitionistic fuzzy set and the linguistic label variable to more accurately describe the fuzzy information. Furthermore, Liu and Jin [11] and Liu and Zhang [46] introduced the concept of intuitionistic uncertain linguistic sets and intuitionistic uncertain linguistic numbers, which take uncertainty on the basis of intuitionistic linguistic sets into consideration.

The intuitionistic uncertain linguistic variables have the following important advantages. Firstly, the intuitionistic uncertain linguistic variables express fuzzy information more accurately than the uncertain linguistic variables. Generally, the uncertain linguistic variables are an easy way to express the qualitative information. We can select an uncertain linguistic term from the uncertain linguistic set to show the evaluation information alternatives. In application, however, this method is not accurate because the uncertain linguistic variables only provide a range of options. Therefore, at this point we can use the intuitionistic uncertain linguistic variables to express the alternatives more accurately by selecting an uncertain linguistic term that is closer to the evaluation information. Then, we give the membership degree and non-membership degree to the uncertain linguistic term. For example, we want to evaluate the performance of a computer by $S = \{s_0 = \textit{very poor}, s_1 = \textit{poor}, s_2 = \textit{medium}, s_3 = \textit{good}, s_4 = \textit{very good}\}$, which is a given linguistic set. In this situation, we think the performance evaluation result is higher than s_2 and lower than s_4, that is between "medium" and "very good". In general, we can use the uncertain linguistic variables $[s_2, s_3]$ and $[s_3, s_4]$ to express the result. However, the range is not accurate. Thus, we can use an intuitionistic uncertain linguistic number for the computer evaluation. For the uncertain linguistic variable $[s_3, s_4]$, we can give the membership degree 0.9 and non-membership degree 0.1 to $[s_3, s_4]$. Therefore, the intuitionistic uncertain linguistic number is $< [s_3, s_4], (0.9, 0.1) >$. Of course, if we use the uncertain linguistic variable $[s_2, s_3]$, it cannot express the evaluation information adequately. The intuitionistic uncertain linguistic numbers consider not only the uncertain nature of decision information, but take the membership degree and non-membership degree of the uncertain linguistic variables into consideration, therefore, it can express fuzzy information more accurately than uncertain linguistic variables. Clearly, the intuitionistic uncertain linguistic variables can express fuzzy information more directly than intuitionistic fuzzy information. In terms of distance measures and aggregation operators, Bi et al. [47] presented various distance measures for intuitionistic uncertain linguistic sets and applied them to group decision making. Wang et al. [48] introduced an intuitionistic uncertain linguistic new aggregation operator. Combining the interval-valued intuitionistic fuzzy sets and linguistic information, Peng and Ye [49] further proposed several methods for aggregating interval-valued intuitionistic pure linguistic information.

The intuitionistic fuzzy, intuitionistic linguistic and intuitionistic uncertain linguistic aggregation operators mentioned above are based on the algebraic operational rules of intuitionistic fuzzy numbers. The algebraic product and algebraic sum are important components of the algebraic operation, which is one type of operation that can be used for modelling the intersection and union of intuitionistic fuzzy numbers (Tan et al. [50]). Xia et al. [51] proposed the operations of intuitionistic fuzzy sets based on Archimedean s-norm and t-norm, which are generalizations of many other s-norms and t-norms, such as algebraic, Einstein, Hamacher, and Frank s-norms and t-norms. They developed intuitionistic fuzzy aggregation operators based on Archimedean s-norm and t-norm. Wang and

Liu [52] developed intuitionistic fuzzy aggregation operators based on Einstein operations. Motivated by these ideas, Tao et al. [53] introduced a closed operational law for 2-tuple linguistic terms, which operates without loss of information. In view of the fact that Hamacher s-norm and t-norm are the generalization of algebraic and Einstein s-norm and t-norm, Liu [54] developed aggregation operators for interval-valued intuitionistic fuzzy numbers based on Hamacher t-norm and s-norm. Tan et al. [50] presented hesitant fuzzy Hamacher aggregation operators for multicriteria decision making. Rong et al. [55] presented a group decision making method based on the intuitionistic fuzzy generalized Hamacher aggregation operator.

However, in the existing literature few authors have researched MAGDM problems in an intuitionistic uncertain linguistic environment based on Hamacher operations. Thus, this paper presents an approach to MAGDM based on generalized Hamacher aggregation operators for intuitionistic uncertain linguistic sets. The rest of this paper is structured as follows. In Section 2, we introduce the concept of intuitionistic uncertain linguistic sets (IULSs) and intuitionistic uncertain linguistic numbers (IULNs), the Hamacher t-norm and s-norm, and special cases. In Section 3, we establish Hamacher operations of IULSs and their characteristics based on Hamacher t-norm and s-norm, and further develop several Hamacher aggregation operators for IULSs including ULGHWA (uncertain linguistic generalized Hamacher weighted average), IULGHOWA (intuitionistic uncertain linguistic generalized Hamacher ordered weighted average), HIULWGA (Hamacher intuitionistic uncertain linguistic weighted geometric average), HIULOWA (Hamacher intuitionistic uncertain linguistic ordered weighted average), HIULOWGA (Hamacher intuitionistic uncertain linguistic ordered weighted geometric average) and IULGHHWA (intuitionistic uncertain linguistic generalized Hamacher hybrid weighted average) operators. In Section 4, we propose a method for MAGDM with intuitionistic uncertain linguistic information based on the generalized Hamacher aggregation operator. Section 5 gives an illustrative example to show the feasibility and practicability of the developed approach, and provides a comparative analysis to discuss the effect on the results when the parameters λ and ζ taking different values also include other related group decision-making methods. Finally, the main conclusions of this paper are summarized in Section 6.

2. Preliminaries

2.1. Intuitionistic Uncertain Linguistic Sets

Let $S = \{s_\alpha | \alpha = 0, 1, 2, \cdots, 2t\}$ be a finite and totally ordered discrete term set, where s_α represents a possible value for a linguistic variable. For example, S can be defined as:

$$S = \{s_0 = \textit{very poor}, s_1 = \textit{poor}, s_2 = \textit{slightly poor}, s_3 = \textit{medium}, \\ s_4 = \textit{slightly good}, s_5 = \textit{good}, s_6 = \textit{very good}\},$$

which satisfies the following characteristics (Herrera et al. [56,57]):

(1) The set is ordered: $s_\alpha > s_\beta$ if and only if $\alpha > \beta$;
(2) There is the negative operator: $neg(s_\alpha) = s_{2t-\alpha}$. Especially, $neg(s_0) = s_0$;
(3) Max operator: $\max(s_\alpha, s_\beta) = s_\alpha$ if $s_\alpha > s_\beta$;
(4) Min operator: $\min(s_\alpha, s_\beta) = s_\alpha$ if $s_\alpha < s_\beta$.

To preserve all the decision information, Herrera et al. [56] and Xu [32] extended the discrete label set S to a continuous label set $\widetilde{S} = \left\{ s_\alpha \middle| \alpha \in [0, 2q] \right\}$, satisfied the characteristics above, and $q(q > t)$ is a sufficiently large positive number. We call s_α the original linguistic label when $s_\alpha \in S$, otherwise, we call it the virtual linguistic label. The original linguistic labels are used to evaluate attributes and alternatives while the virtual linguistic labels may appear in the calculation.

Xu [32] proposed the concept of uncertain linguistic variable, which can be defined as follows:

Definition 1. *Let* $\widetilde{s} = [s_\alpha, s_\beta]$, $s_\alpha, s_\beta \in \widetilde{S}$, s_α *and* s_β *are the lower and the upper limits, respectively, then* $\widetilde{s}$ *is called an uncertain linguistic variable. However,* $\widetilde{s}$ *is a real linguistic variable, if* $s_\alpha = s_\beta$.

Based on the concept of the intuitionistic linguistic set (ILS) (Wang and Li [45]) and uncertain linguistic variables (Xu [32]), Liu and Jin [11] presented the intuitionistic uncertain linguistic set (IULS), which can be defined as follows:

Definition 2. *An intuitionistic uncertain linguistic set in X is given as:*

$$A = \left\{ < x[[s_{\alpha(x)}, s_{\beta(x)}], (\mu_A(x), \nu_A(x))] > | x \in X \right\} \tag{1}$$

where $[s_{\alpha(x)}, s_{\beta(x)}] \in \widetilde{S}$, $\mu_A(x) : X \to [0,1]$ *and* $\nu_A(x) : X \to [0,1]$ *with the condition of* $0 \le \mu_A(x) + \nu_A(x) \le 1$, $\forall x \in X$, $\mu_A(x)$ *and* $\nu_A(x)$ *represent the membership degree and non-membership degree of the element x to the uncertain linguistic variable* $[s_{\alpha(x)}, s_{\beta(x)}]$, *respectively.* $\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$, $\forall x \in X$, *is called the degree of indeterminacy of the element x to the uncertain linguistic variable* $[s_{\alpha(x)}, s_{\beta(x)}]$.

For an intuitionistic uncertain linguistic set A, Liu and Jin [11] further defined the intuitionistic uncertain linguistic variables, also called the intuitionistic uncertain linguistic numbers (IULNs), which can be expressed as the quaternion $< [s_{\alpha(x)}, s_{\beta(x)}], (\mu_A(x), \nu_A(x)) >$. Therefore, the intuitionistic uncertain linguistic set A can be denoted by $A = \left\{ < [s_{\alpha(x)}, s_{\beta(x)}], (\mu_A(x), \nu_A(x)) > | x \in X \right\}$. At this point, the intuitionistic uncertain linguistic set A can be viewed as a collection of intuitionistic uncertain linguistic variables.

For any two intuitionistic uncertain linguistic variables α_1 and α_2, the operational laws are defined as follows (Liu and Jin [11], Liu and Zhang [46]):

$$\lambda\alpha_1 = < [s_{\lambda\alpha(x_1)}, s_{\lambda\beta(x_1)}], (1 - (1 - \mu_A(x_1))^\lambda, (\nu_A(x_1))^\lambda) >, \ \lambda \ge 0$$

$$\alpha_1^\lambda = < [s_{(\alpha(x_1))^\lambda}, s_{(\beta(x_1))^\lambda}], ((\mu_A(x_1))^\lambda, 1 - (1 - \nu_A(x_1))^\lambda) >, \lambda \ge 0 \tag{2}$$

$$\alpha_1 \oplus \alpha_2 = < [s_{\alpha(x_1)+\alpha(x_2)}, s_{\beta(x_1)+\beta(x_2)}], (\mu_A(x_1) + \mu_A(x_2) - \mu_A(x_1)\mu_A(x_2), \nu_A(x_1)\nu_A(x_2)) >$$

$$\alpha_1 \otimes \alpha_2 = < [s_{\alpha(x_1)\times\alpha(x_2)}, s_{\beta(x_1)\times\beta(x_2)}], (\mu_A(x_1)\mu_A(x_2), \nu_A(x_1) + \nu_A(x_2) - \nu_A(x_1)\nu_A(x_2)) > \tag{3}$$

Recently, Wang et al. [48] defined several new operational laws of intuitionistic uncertain linguistic variables based on Archimedean t-norm and s-norm as follows:

$$\lambda\alpha_1 = < [s_{\alpha(x_1)}, s_{\beta(x_1)}], (1 - (1 - \mu_A(x_1))^\lambda, (\nu_A(x_1))^\lambda) >, \ \lambda \ge 0$$

$$\alpha_1^\lambda = < [s_{\alpha(x_1)}, s_{\beta(x_1)}], ((\mu_A(x_1))^\lambda, 1 - (1 - \nu_A(x_1))^\lambda) >, \ \lambda \ge 0 \tag{4}$$

$$\begin{aligned} \alpha_1 \oplus \alpha_2 = < [& s_{\sum_{i=1}^2 \frac{\alpha(x_i)\ln[(1-\mu_A(x_i))\nu_A(x_i)]}{\sum_{i=1}^2 \ln[(1-\mu_A(x_i))\nu_A(x_i)]}}, s_{\sum_{i=1}^2 \frac{\beta(x_i)\ln[(1-\mu_A(x_i))\nu_A(x_i)]}{\sum_{i=1}^2 \ln[(1-\mu_A(x_i))\nu_A(x_i)]}}], \\ & (\mu_A(x_1) + \mu_A(x_2) - \mu_A(x_1)\mu_A(x_2), \nu_A(x_1)\nu_A(x_2)) > \end{aligned}$$

$$\begin{aligned} \alpha_1 \otimes \alpha_2 = < [& s_{\sum_{i=1}^2 \frac{\alpha(x_i)\ln[(1-\nu_A(x_i))\mu_A(x_i)]}{\sum_{i=1}^2 \ln[(1-\nu_A(x_i))\mu_A(x_i)]}}, s_{\sum_{i=1}^2 \frac{\beta(x_i)\ln[(1-\nu_A(x_i))\mu_A(x_i)]}{\sum_{i=1}^2 \ln[(1-\nu_A(x_i))\mu_A(x_i)]}}], \\ & (\mu_A(x_1)\mu_A(x_2), \nu_A(x_1) + \nu_A(x_2) - \nu_A(x_1)\nu_A(x_2)) > \end{aligned} \tag{5}$$

Comparing the operational laws of intuitionistic uncertain linguistic variables Equations (4)–(5) above, we find the intuitionistic parts in the quaternion are the same as Equations (2)–(3) while the uncertain linguistic parts are quite different. The main advantages of Equations (4)–(5) are that they take the Archimedean t-norm and s-norm into consideration, and also overcome the transgression of the uncertain linguistic parts in the quaternion. The results of Equations (4)–(5) are more in line with decision makers' intuition. Therefore, Equations (4)–(5) are superior to Equations (2)–(3) in uncertain decision environments, especially in terms of closure. Moreover, Wang et al. [48] defined the operational laws of any two intuitionistic uncertain linguistic variables, such as commutativity and linearity.

2.2. Hamacher T-Norm and S-Norm

Archimedean t-norm and t-conorm (or s-norm) are generalizations of other t-norms and s-norms, such as Algebraic, Einstein, Hamacher and Frank t-norms and s-norms. Among them, the Hamacher t-norm and s-norm can be viewed as a generalization of the algebraic sum and product, and Einstein t-norm and s-norm. The Hamacher t-norm and s-norm [58] can be defined as follows:

$$T_{\zeta}(x,y)=\frac{xy}{\zeta+(1-\zeta)(x+y-xy)},\ \zeta>0 \tag{6}$$

$$S_{\zeta}(x,y)=\frac{x+y-xy-(1-\zeta)xy}{1-(1-\zeta)xy},\ \zeta>0 \tag{7}$$

In particular, when $\zeta=1$, then the Hamacher t-norm and s-norm reduce to Algebraic t-norm $T(x,y)=xy$ and s-norm $S(x,y)=x+y-xy$; when $\zeta=2$, then the Hamacher t-norm and s-norm reduce to Einstein t-norm $T(x,y)=\frac{xy}{1+(1-x)(1-y)}$ and s-norm $S(x,y)=\frac{x+y}{1+xy}$, respectively.

Suppose $\widetilde{a}_1=(a_1,b_1)$ and $\widetilde{a}_2=(a_2,b_2)$ are two IFVs, in fact, as a wide application of Algebraic t-norm and s-norm, the algebraic product $\widetilde{a}_1\otimes\widetilde{a}_2$ and the algebraic sum $\widetilde{a}_1\oplus\widetilde{a}_2$ on two intuitionistic fuzzy numbers $\widetilde{a}_1$ and $\widetilde{a}_2$ can be obtained by defining t-norm and s-norm. When $T(x,y)=xy$ and $S(x,y)=x+y-xy$, we have the operational laws between the two IFVs given by Atanassov [1], Xu [59] and Xu and Yager [60].

3. Hamacher Operations of Intuitionistic Uncertain Linguistic Sets

In the Section, we introduce the operational rules and aggregation operators based on Hamacher t-norm and s-norm with intuitionistic uncertain linguistic information.

3.1. The Operational Rules Based on Hamacher T-Norm and S-Norm

Based on the operational laws of intuitionistic uncertain linguistic variables Equations (4)–(5) and Hamacher t-norm and s-norm Equations (6) and (7), we propose the operational rules of two intuitionistic uncertain linguistic variables.

If $\alpha_1=<[s_{\alpha(x_1)},s_{\beta(x_1)}],(\mu(x_1),\nu(x_1))>$ and $\alpha_2=<[s_{\alpha(x_2)},s_{\beta(x_2)}],(\mu(x_2),\nu(x_2))>$ are two intuitionistic uncertain linguistic variables, $\zeta>0$, we can define the operational rules based on the Hamacher t-norm and s-norm as follows:

$$\begin{aligned}\lambda\alpha_1=\Big\langle&[s_{\alpha(x_1)},s_{\beta(x_1)}],\big(\tfrac{(1+(\zeta-1)\mu(x_1))^{\lambda}-(1-\mu(x_1))^{\lambda}}{(1+(\zeta-1)\mu(x_1))^{\lambda}+(\zeta-1)(1-\mu(x_1))^{\lambda}},\\&\tfrac{\zeta(\nu(x_1))^{\lambda}}{(1+(\zeta-1)(1-\nu(x_1)))^{\lambda}+(\zeta-1)(\nu(x_1))^{\lambda}}\big)\Big\rangle,\ \lambda\geq 0\end{aligned} \tag{8}$$

$$\begin{aligned}\alpha_1^{\lambda}=\Big\langle&[s_{\alpha(x_1)},s_{\beta(x_1)}],\big(\tfrac{\zeta(\mu(x_1))^{\lambda}}{(1+(\zeta-1)(1-\mu(x_1)))^{\lambda}+(\zeta-1)(\mu(x_1))^{\lambda}},\\&\tfrac{(1+(\zeta-1)\nu(x_1))^{\lambda}-(1-\nu(x_1))^{\lambda}}{(1+(\zeta-1)\nu(x_1))^{\lambda}+(\zeta-1)(1-\nu(x_1))^{\lambda}}\big)\Big\rangle,\ \lambda\geq 0\end{aligned} \tag{9}$$

$$\begin{aligned}\alpha_1\oplus_h\alpha_2=\Big\langle&[s_{\sum_{i=1}^{2}\frac{\alpha(x_i)\ln[(1-\mu(x_i))\nu(x_i)]}{\sum_{i=1}^{2}\ln[(1-\mu(x_i))\nu(x_i)]}},s_{\sum_{i=1}^{2}\frac{\beta(x_i)\ln[(1-\mu(x_i))\nu(x_i)]}{\sum_{i=1}^{2}\ln[(1-\mu(x_i))\nu(x_i)]}}],\\&\big(\tfrac{\mu(x_1)+\mu(x_2)-\mu(x_1)\mu(x_2)-(1-\zeta)\mu(x_1)\mu(x_2)}{1-(1-\zeta)\mu(x_1)\mu(x_2)},\tfrac{\nu(x_1)\nu(x_2)}{\zeta+(1-\zeta)(\nu(x_1)+\nu(x_2)-\nu(x_1)\nu(x_2))}\big)\Big\rangle\end{aligned} \tag{10}$$

$$\alpha_1 \otimes_h \alpha_2 = \Big\langle [s_{\sum_{i=1}^{2} \frac{\alpha(x_i)\ln[(1-\nu(x_i))\mu(x_i)]}{\sum_{i=1}^{2}\ln[(1-\nu(x_i))\mu(x_i)]}}, s_{\sum_{i=1}^{2} \frac{\beta(x_i)\ln[(1-\nu(x_i))\mu(x_i)]}{\sum_{i=1}^{2}\ln[(1-\nu(x_i))\mu(x_i)]}}], \\ (\frac{\mu(x_1)\mu(x_2)}{\zeta+(1-\zeta)(\mu(x_1)+\mu(x_2)-\mu(x_1)\mu(x_2))}, \frac{\nu(x_1)+\nu(x_2)-\nu(x_1)\nu(x_2)-(1-\zeta)\nu(x_1)\nu(x_2)}{1-(1-\zeta)\nu(x_1)\nu(x_2)}) \Big\rangle \tag{11}$$

in which, $\alpha_1 \oplus_h \alpha_2$ and $\alpha_1 \otimes_h \alpha_2$ represent the Hamacher sum and Hamacher product of two intuitionistic uncertain linguistic variables, respectively.

Theorem 1. *Let* $\alpha_j =< [s_{\alpha(x_j)}, s_{\beta(x_j)}], (\mu(x_j), \nu(x_j)) >$, $j = 1, 2$ *are two intuitionistic uncertain linguistic variables,* $\zeta > 0$, *we have:*

$$(1)\ \alpha_1 \oplus_h \alpha_2 = \alpha_2 \oplus_h \alpha_1 \tag{12}$$

$$(2)\ \alpha_1 \otimes_h \alpha_2 = \alpha_2 \otimes_h \alpha_1 \tag{13}$$

$$(3)\ \lambda(\alpha_1 \oplus_h \alpha_2) = \lambda\alpha_1 \oplus_h \lambda\alpha_2,\ \lambda > 0 \tag{14}$$

$$(4)\ \lambda_1\alpha_1 \oplus_h \lambda_2\alpha_1 = (\lambda_1 + \lambda_2)\alpha_1,\ \lambda_1, \lambda_2 > 0 \tag{15}$$

$$(5)\ {\alpha_1}^{\lambda} \otimes_h {\alpha_2}^{\lambda} = (\alpha_1 \otimes_h \alpha_2)^{\lambda},\ \lambda > 0 \tag{16}$$

$$(6)\ {\alpha_1}^{\lambda_1} \otimes_h {\alpha_1}^{\lambda_2} = (\alpha_1)^{\lambda_1+\lambda_2},\ \lambda_1, \lambda_2 > 0 \tag{17}$$

The desirable properties of the Hamacher sum and Hamacher product of intuitionistic uncertain linguistic variables above, can be proved as follows.

Proof. According to Equations (10) and (11), we have (1) and (2). For (3), according to Equations (8) and (10), we have

$$\lambda(\alpha_1 \oplus_h \alpha_2) = \Big\langle [s_{\sum_{i=1}^{2} \frac{\alpha(x_i)\ln[(1-\mu(x_i))\nu(x_i)]}{\sum_{i=1}^{2}\ln[(1-\mu(x_i))\nu(x_i)]}}, s_{\sum_{i=1}^{2} \frac{\beta(x_i)\ln[(1-\mu(x_i))\nu(x_i)]}{\sum_{i=1}^{2}\ln[(1-\mu(x_i))\nu(x_i)]}}],$$

$$\lambda(\alpha_1 \oplus_h \alpha_2) = \Big\langle [s_{\sum_{i=1}^{2} \frac{\alpha(x_i)\ln[(1-\mu(x_i))\nu(x_i)]}{\sum_{i=1}^{2}\ln[(1-\mu(x_i))\nu(x_i)]}}, s_{\sum_{i=1}^{2} \frac{\beta(x_i)\ln[(1-\mu(x_i))\nu(x_i)]}{\sum_{i=1}^{2}\ln[(1-\mu(x_i))\nu(x_i)]}}], \\ (\frac{(1+(\zeta-1)(\mu(x_1)+\mu(x_2)-\mu(x_1)\mu(x_2)))^{\lambda}-(1-(1-\zeta)\mu(x_1)\mu(x_2))^{\lambda}}{(1+(\zeta-1)(1-(1-\zeta)\mu(x_1)\mu(x_2)))^{\lambda}+(\zeta-1)(1-(1-(1-\zeta)\mu(x_1)\mu(x_2)))^{\lambda}}, \\ \frac{\zeta(\nu(x_1)\nu(x_2))^{\lambda}}{(1+(\zeta-1)(1-(\zeta+(1-\zeta)(\nu(x_1)+\nu(x_2)-\nu(x_1)\nu(x_2)))))^{\lambda}+(\zeta-1)(\zeta+(1-\zeta)(\nu(x_1)+\nu(x_2)-\nu(x_1)\nu(x_2)))^{\lambda}}) \Big\rangle \\ = \Big\langle [s_{\alpha(x_1)}, s_{\beta(x_1)}], (\frac{(1+(\zeta-1)\mu(x_1))^{\lambda}-(1-\mu(x_1))^{\lambda}}{(1+(\zeta-1)\mu(x_1))^{\lambda}+(\zeta-1)(1-\mu(x_1))^{\lambda}}, \\ \frac{\zeta(\nu(x_1))^{\lambda}}{(1+(\zeta-1)(1-\nu(x_1)))^{\lambda}+(\zeta-1)(\nu(x_1))^{\lambda}}) \Big\rangle$$

$$\lambda(\alpha_1 \oplus_h \alpha_2) = \Big\langle [s_{\sum_{i=1}^{2} \frac{\alpha(x_i)\ln[(1-\mu(x_i))\nu(x_i)]}{\sum_{i=1}^{2}\ln[(1-\mu(x_i))\nu(x_i)]}}, s_{\sum_{i=1}^{2} \frac{\beta(x_i)\ln[(1-\mu(x_i))\nu(x_i)]}{\sum_{i=1}^{2}\ln[(1-\mu(x_i))\nu(x_i)]}}], \\ (\frac{(1+(\zeta-1)(\mu(x_1)+\mu(x_2)-\mu(x_1)\mu(x_2)))^{\lambda}-(1-(1-\zeta)\mu(x_1)\mu(x_2))^{\lambda}}{(1+(\zeta-1)(1-(1-\zeta)\mu(x_1)\mu(x_2)))^{\lambda}+(\zeta-1)(1-(1-(1-\zeta)\mu(x_1)\mu(x_2)))^{\lambda}}, \\ \frac{\zeta(\nu(x_1)\nu(x_2))^{\lambda}}{(1+(\zeta-1)(1-(\zeta+(1-\zeta)(\nu(x_1)+\nu(x_2)-\nu(x_1)\nu(x_2)))))^{\lambda}+(\zeta-1)(\zeta+(1-\zeta)(\nu(x_1)+\nu(x_2)-\nu(x_1)\nu(x_2)))^{\lambda}}) \Big\rangle \\ = \Big\langle [s_{\alpha(x_1)}, s_{\beta(x_1)}], (\frac{(1+(\zeta-1)\mu(x_1))^{\lambda}-(1-\mu(x_1))^{\lambda}}{(1+(\zeta-1)\mu(x_1))^{\lambda}+(\zeta-1)(1-\mu(x_1))^{\lambda}}, \\ \frac{\zeta(\nu(x_1))^{\lambda}}{(1+(\zeta-1)(1-\nu(x_1)))^{\lambda}+(\zeta-1)(\nu(x_1))^{\lambda}}) \Big\rangle \\ \oplus_h \Big\langle [s_{\alpha(x_2)}, s_{\beta(x_2)}], (\frac{(1+(\zeta-1)\mu(x_2))^{\lambda}-(1-\mu(x_2))^{\lambda}}{(1+(\zeta-1)\mu(x_2))^{\lambda}+(\zeta-1)(1-\mu(x_2))^{\lambda}}, \\ \frac{\zeta(\nu(x_2))^{\lambda}}{(1+(\zeta-1)(1-\nu(x_2)))^{\lambda}+(\zeta-1)(\nu(x_2))^{\lambda}}) \Big\rangle = \lambda\alpha_1 \oplus_h \lambda\alpha_2$$

□

For (4) and (5–6), according to Equations (8) and (10), Equations (9) and (11), the proofs are similar to that of (3), and are not replicated here.

3.2. The Intuitionistic Uncertain Linguistic Generalized Hamacher Aggregation Operators

Definition 3. *Let* $\alpha_j =< [s_{\alpha(x_j)}, s_{\beta(x_j)}], (\mu(x_j), \nu(x_j)) >$, $j = 1, 2, \cdots, n$ *be a collection of intuitionistic uncertain linguistic variables, and IULGHWA:* $\Omega^n \to \Omega$, then

$$IULGHWA(\alpha_1, \alpha_2, \cdots, \alpha_n) = \left(\overset{n}{\underset{j=1}{\oplus_h}} (\omega_j \alpha_j^{\lambda})\right)^{1/\lambda} \tag{18}$$

where Ω *is the set of intuitionistic uncertain linguistic variables,* $\lambda > 0$, $\omega = (\omega_1, \omega_2, \cdots, \omega_n)^T$ *is the weighting vector of the* α_j, *and* $\omega_j \in [0, 1]$, $\sum\limits_{j=1}^{n} \omega_j = 1$, *then IULGHWA is called the intuitionistic uncertain linguistic generalized Hamacher weighted average (IULGHWA) operator.*

Definition 4. *Let* $\alpha_j =< [s_{\alpha(x_j)}, s_{\beta(x_j)}], (\mu(x_j), \nu(x_j)) >$, $j = 1, 2, \cdots, n$ *be a collection of intuitionistic uncertain linguistic variables, and IULGHOWA:* $\Omega^n \to \Omega$ *that has an associated n vector* $w = (w_1, w_2, \cdots, w_n)^T$ *such that* $w_j \in [0, 1]$, $\sum\limits_{j=1}^{n} w_j = 1$ *and*

$$IULGHOWA(\alpha_1, \alpha_2, \cdots, \alpha_n) = \left(\overset{n}{\underset{j=1}{\oplus_h}} (w_j \alpha_{\sigma_j}^{\lambda})\right)^{1/\lambda} \tag{19}$$

where Ω *is the set of intuitionistic uncertain linguistic variables,* $\lambda > 0$, $(\sigma_1, \sigma_2, \cdots, \sigma_n)$ *is a permutation of* $(1, 2, \cdots, n)$, *such that* $\alpha_{\sigma_{j-1}} \geq \alpha_{\sigma_j}$ *for* $\forall j$, *then IULGHOWA is called the intuitionistic uncertain linguistic generalized Hamacher ordered weighted average (IULGHOWA) operator.*

Remark 1. *The IULGHWA and IULGHOWA operators have desirable properties such as boundary, idempotency and commutativity; the proof process is omitted here.*

As we can see, the two aggregation operators above are very suitable for dealing with situations where the input arguments are represented in intuitionistic uncertain linguistic numbers (IULNs). At the same time, both the operators have two parameters λ and ζ, therefore, in the following, some special cases for the two operators with respect to the parameters λ and ζ can be discussed in detail.

(1) If $\lambda = 1$, then the IULGHWA operator (18) will be reduced to the Hammer intuitionistic uncertain linguistic weighted averaging (HIULWA) operator, and

$$\begin{aligned} &HIULWA(\alpha_1, \alpha_2, \cdots, \alpha_n) \\ &= \left\langle \left[s_{\sum_{j=1}^{n} \omega_j \alpha(x_j)}, s_{\sum_{j=1}^{n} \omega_j \beta(x_j)} \right], \left(\frac{\prod\limits_{j=1}^{n} (1+(\zeta-1)\mu(x_j))^{\omega_j} - \prod\limits_{j=1}^{n} (1-\mu(x_j))^{\omega_j}}{\prod\limits_{j=1}^{n} (1+(\zeta-1)\mu(x_j))^{\omega_j} + (\zeta-1)\prod\limits_{j=1}^{n} (1-\mu(x_j))^{\omega_j}}, \right. \right. \\ &\left. \left. \frac{\zeta \prod\limits_{j=1}^{n} \nu(x_j)^{\omega_j}}{\prod\limits_{j=1}^{n} (1+(\zeta-1)(1-\nu(x_j)))^{\omega_j} + (\zeta-1)\prod\limits_{j=1}^{n} \nu(x_j)^{\omega_j}} \right) \right\rangle \end{aligned} \tag{20}$$

When $\zeta = 1$, Equation (20) will be reduced to the intuitionistic uncertain linguistic weighted averaging (IULWA) operator, which is defined by Wang et al. [48] and Liu and Zhang [46]. By using Equation (20), we have

$$IULWA(\alpha_1, \alpha_2, \cdots, \alpha_n) = \left\langle \left[s_{\sum_{j=1}^{n} \omega_j \alpha(x_j)}, s_{\sum_{j=1}^{n} \omega_j \beta(x_j)} \right], (1 - \prod_{j=1}^{n} (1 - \mu(x_j))^{\omega_j}, \prod_{j=1}^{n} \nu(x_j)^{\omega_j}) \right\rangle \tag{21}$$

- When $\zeta = 2$, Equation (20) will be reduced to the Einstein intuitionistic uncertain linguistic weighted averaging (EIULWA) operator:

$$\begin{gathered} EIULWA(\alpha_1, \alpha_2, \cdots, \alpha_n) = \left\langle \left[s_{\sum_{j=1}^{n} \omega_j \alpha(x_j)}, s_{\sum_{j=1}^{n} \omega_j \beta(x_j)} \right], \right. \\ \left. (\frac{\prod_{j=1}^{n} (1+\mu(x_j))^{\omega_j} - \prod_{j=1}^{n} (1-\mu(x_j))^{\omega_j}}{\prod_{j=1}^{n} (1+\mu(x_j))^{\omega_j} + \prod_{j=1}^{n} (1-\mu(x_j))^{\omega_j}}, \frac{2\prod_{j=1}^{n} \nu(x_j)^{\omega_j}}{\prod_{j=1}^{n} (2-\nu(x_j))^{\omega_j} + \prod_{j=1}^{n} \nu(x_j)^{\omega_j}}) \right\rangle \end{gathered} \tag{22}$$

(2) If $\lambda \to 0$, then the IULGHWA operator (18) will be reduced to the Hammer intuitionistic uncertain linguistic weighted geometric averaging (HIULWGA) operator, and

$$\begin{gathered} HIULWGA(\alpha_1, \alpha_2, \cdots, \alpha_n) \\ = \left\langle \left[s_{\sum_{j=1}^{n} \omega_j \alpha(x_j)}, s_{\sum_{j=1}^{n} \omega_j \beta(x_j)} \right], (\frac{\zeta \prod_{j=1}^{n} \mu(x_j)^{\omega_j}}{\prod_{j=1}^{n} (1+(\zeta-1)(1-\mu(x_j)))^{\omega_j} + (\zeta-1)\prod_{j=1}^{n} \mu(x_j)^{\omega_j}}, \right. \\ \left. \frac{\prod_{j=1}^{n} (1+(\zeta-1)\nu(x_j))^{\omega_j} - \prod_{j=1}^{n} (1-\nu(x_j))^{\omega_j}}{\prod_{j=1}^{n} (1+(\zeta-1)\nu(x_j))^{\omega_j} + (\zeta-1)\prod_{j=1}^{n} (1-\nu(x_j))^{\omega_j}}) \right\rangle \end{gathered} \tag{23}$$

- When $\zeta = 1$, Equation (23) will be reduced to the intuitionistic uncertain linguistic weighted geometric averaging (IULWGA) operator which is defined by Liu and Jin [11]. By using Equation (23), we have

$$IULWGA(\alpha_1, \alpha_2, \cdots, \alpha_n) = \left\langle \left[s_{\sum_{j=1}^{n} \omega_j \alpha(x_j)}, s_{\sum_{j=1}^{n} \omega_j \beta(x_j)} \right], (\prod_{j=1}^{n} \mu(x_j)^{\omega_j}, 1 - \prod_{j=1}^{n} (1 - \nu(x_j))^{\omega_j}) \right\rangle \tag{24}$$

- When $\zeta = 2$, Equation (24) will be reduced to the Einstein intuitionistic uncertain linguistic weighted geometric averaging (EIULWGA) operator:

$$\begin{gathered} EIULWGA(\alpha_1, \alpha_2, \cdots, \alpha_n) = \left\langle \left[s_{\sum_{j=1}^{n} \omega_j \alpha(x_j)}, s_{\sum_{j=1}^{n} \omega_j \beta(x_j)} \right], \right. \\ \left. (\frac{2\prod_{j=1}^{n} \mu(x_j)^{\omega_j}}{\prod_{j=1}^{n} (2-\mu(x_j))^{\omega_j} + \prod_{j=1}^{n} \mu(x_j)^{\omega_j}}, \frac{\prod_{j=1}^{n} (1+\nu(x_j))^{\omega_j} - \prod_{j=1}^{n} (1-\nu(x_j))^{\omega_j}}{\prod_{j=1}^{n} (1+\nu(x_j))^{\omega_j} + \prod_{j=1}^{n} (1-\nu(x_j))^{\omega_j}}) \right\rangle \end{gathered} \tag{25}$$

(3) If $\lambda = 1$, then the IULGHOWA operator (19) will be reduced to the Hammer intuitionistic uncertain linguistic ordered weighted averaging (HIULOWA) operator, and

$$HIULOWA(\alpha_1, \alpha_2, \cdots, \alpha_n) = \left\langle \left[s_{\sum_{j=1}^n w_j \alpha(x_{\sigma_j})}, s_{\sum_{j=1}^n w_j \beta(x_{\sigma_j})} \right], \left(\frac{\prod_{j=1}^n (1+(\zeta-1)\mu(x_{\sigma_j}))^{w_j} - \prod_{j=1}^n (1-\mu(x_{\sigma_j}))^{w_j}}{\prod_{j=1}^n (1+(\zeta-1)\mu(x_{\sigma_j}))^{w_j} + (\zeta-1)\prod_{j=1}^n (1-\mu(x_{\sigma_j}))^{w_j}}, \frac{\zeta \prod_{j=1}^n \nu(x_{\sigma_j})^{w_j}}{\prod_{j=1}^n (1+(\zeta-1)(1-\nu(x_{\sigma_j})))^{w_j} + (\zeta-1)\prod_{j=1}^n \nu(x_{\sigma_j})^{w_j}} \right) \right\rangle \tag{26}$$

- When $\zeta = 1$, Equation (26) will be reduced to the intuitionistic uncertain linguistic ordered weighted averaging (IULOWA) operator which is defined by Liu and Zhang [46]. By using Equation (26), we have

$$IULOWA(\alpha_1, \alpha_2, \cdots, \alpha_n) = \left\langle \left[s_{\sum_{j=1}^n w_j \alpha(x_{\sigma_j})}, s_{\sum_{j=1}^n w_j \beta(x_{\sigma_j})} \right], \left(1 - \prod_{j=1}^n (1-\mu(x_{\sigma_j}))^{w_j}, \prod_{j=1}^n \nu(x_{\sigma_j})^{w_j}\right) \right\rangle \tag{27}$$

- When $\zeta = 2$, Equation (26) will be reduced to the Einstein intuitionistic uncertain linguistic ordered weighted averaging (EIULOWA) operator:

$$EIULOWA(\alpha_1, \alpha_2, \cdots, \alpha_n) = \left\langle \left[s_{\sum_{j=1}^n w_j \alpha(x_{\sigma_j})}, s_{\sum_{j=1}^n w_j \beta(x_{\sigma_j})} \right], \left(\frac{\prod_{j=1}^n (1+\mu(x_{\sigma_j}))^{w_j} - \prod_{j=1}^n (1-\mu(x_{\sigma_j}))^{w_j}}{\prod_{j=1}^n (1+\mu(x_{\sigma_j}))^{w_j} + \prod_{j=1}^n (1-\mu(x_{\sigma_j}))^{w_j}}, \frac{2\prod_{j=1}^n \nu(x_{\sigma_j})^{w_j}}{\prod_{j=1}^n (2-\nu(x_{\sigma_j}))^{w_j} + \prod_{j=1}^n \nu(x_{\sigma_j})^{w_j}} \right) \right\rangle \tag{28}$$

(4) If $\lambda \to 0$, then the IULGHOWA operator (19) will be reduced to the Hammer intuitionistic uncertain linguistic ordered weighted geometric averaging (HIULOWGA) operator, and

$$HIULOWGA(\alpha_1, \alpha_2, \cdots, \alpha_n) = \left\langle \left[s_{\sum_{j=1}^n w_j \alpha(x_{\sigma_j})}, s_{\sum_{j=1}^n w_j \beta(x_{\sigma_j})} \right], \left(\frac{\zeta \prod_{j=1}^n \mu(x_{\sigma_j})^{w_j}}{\prod_{j=1}^n (1+(\zeta-1)(1-\mu(x_{\sigma_j})))^{w_j} + (\zeta-1)\prod_{j=1}^n \mu(x_{\sigma_j})^{w_j}}, \frac{\prod_{j=1}^n (1+(\zeta-1)\nu(x_{\sigma_j}))^{w_j} - \prod_{j=1}^n (1-\nu(x_{\sigma_j}))^{w_j}}{\prod_{j=1}^n (1+(\zeta-1)\nu(x_{\sigma_j}))^{w_j} + (\zeta-1)\prod_{j=1}^n (1-\nu(x_{\sigma_j}))^{w_j}} \right) \right\rangle \tag{29}$$

- When $\zeta = 1$, Equation (29) will be reduced to the intuitionistic uncertain linguistic ordered weighted geometric averaging (IULOWGA) operator which is defined by Liu and Jin [11]. By using Equation (29), we have

$$IULOWGA(\alpha_1, \alpha_2, \cdots, \alpha_n) = \left\langle \left[s_{\sum_{j=1}^n w_j \alpha(x_{\sigma_j})}, s_{\sum_{j=1}^n w_j \beta(x_{\sigma_j})} \right], \left(\prod_{j=1}^n \mu(x_{\sigma_j})^{w_j}, 1 - \prod_{j=1}^n (1-\nu(x_{\sigma_j}))^{w_j} \right) \right\rangle \tag{30}$$

- When $\zeta = 2$, Equation (29) will be reduced to the Einstein intuitionistic uncertain linguistic ordered weighted geometric averaging (EIULOWGA) operator:

$$EIULOWGA(\alpha_1, \alpha_2, \cdots, \alpha_n) = \left\langle \left[s_{\sum_{j=1}^{n} w_j \alpha(x_{\sigma_j})}, s_{\sum_{j=1}^{n} w_j \beta(x_{\sigma_j})} \right], \left(\frac{2\prod_{j=1}^{n} \mu(x_{\sigma_j})^{w_j}}{\prod_{j=1}^{n} (2-\mu(x_{\sigma_j}))^{w_j} + \prod_{j=1}^{n} \mu(x_{\sigma_j})^{w_j}}, \frac{\prod_{j=1}^{n} (1+\nu(x_{\sigma_j}))^{w_j} - \prod_{j=1}^{n} (1-\nu(x_{\sigma_j}))^{w_j}}{\prod_{j=1}^{n} (1+\nu(x_{\sigma_j}))^{w_j} + \prod_{j=1}^{n} (1-\nu(x_{\sigma_j}))^{w_j}} \right) \right\rangle \tag{31}$$

Through the analysis above, we know that the IULGHWA operator weights the intuitionistic uncertain linguistic variables while the IULGHOWA operator weights the ordered positions of the intuitionistic uncertain linguistic variables instead of weighting the variables themselves. That is, weights represent different aspects in both of the two operators. Therefore, we propose a generalized Hamacher hybrid operator to overcome this drawback, which is defined as follows:

Definition 5. *Let $\alpha_j =< [s_{\alpha(x_j)}, s_{\beta(x_j)}], (\mu(x_j), \nu(x_j)) >$, $j = 1, 2, \cdots, n$ be a collection of the intuitionistic uncertain linguistic variables, and IULGHHWA: $\Omega^n \to \Omega$ that has an associated n vector $w = (w_1, w_2, \cdots, w_n)^T$ such that $w_j \in [0, 1]$, $\sum_{j=1}^{n} w_j = 1$ and*

$$IULGHHWA(\alpha_1, \alpha_2, \cdots, \alpha_n) = \mathop{\oplus_h}_{j=1}^{n} (w_j \beta_{\sigma_j}) \tag{32}$$

where Ω is the set of intuitionistic uncertain linguistic variables, β_{σ_j} is the jth largest of the β_j value ($\beta_j = n\omega_j \alpha_j =< [s_{\widetilde{\alpha}(x_j)}, s_{\widetilde{\beta}(x_j)}], (\widetilde{\mu}(x_j), \widetilde{\nu}(x_j)) >, j = 1, 2, \cdots, n$), $\omega = (\omega_1, \omega_2, \cdots, \omega_n)^T$ is the weighting vector of the α_j, and $\omega_j \in [0, 1]$, $\sum_{j=1}^{n} \omega_j = 1$, n is the balancing coefficient. $(\sigma_1, \sigma_2, \cdots, \sigma_n)$ is a permutation of $(1, 2, \cdots, n)$, such that $\alpha_{\sigma_{j-1}} \geq \alpha_{\sigma_j}$ for $\forall j$, then IULGHHWA is called the intuitionistic uncertain linguistic generalized Hamacher hybrid weighted average (IULGHHWA) operator.

Theorem 2. *Let $\alpha_j =< [s_{\alpha(x_j)}, s_{\beta(x_j)}], (\mu(x_j), \nu(x_j)) >$, $j = 1, 2, \cdots, n$ be a collection of the intuitionistic uncertain linguistic variables, then*

$$\begin{aligned} &IULGHHWA(\alpha_1, \alpha_2, \cdots, \alpha_n) \\ &= \left\langle \left[s_{\sum_{j=1}^{n} w_j \widetilde{\alpha}(x_{\sigma_j})}, s_{\sum_{j=1}^{n} w_j \widetilde{\beta}(x_{\sigma_j})} \right], \left(\frac{\zeta \left(\prod_{j=1}^{n} \widetilde{\phi}_{\sigma_j}^{w_j} - \prod_{j=1}^{n} \widetilde{\varphi}_{\sigma_j}^{w_j} \right)^{1/\lambda}}{\left(\prod_{j=1}^{n} \widetilde{\phi}_{\sigma_j}^{w_j} + (\zeta^2-1) \prod_{j=1}^{n} \widetilde{\varphi}_{\sigma_j}^{w_j} \right)^{1/\lambda} + (\zeta-1) \left(\prod_{j=1}^{n} \widetilde{\phi}_{\sigma_j}^{w_j} - \prod_{j=1}^{n} \widetilde{\varphi}_{\sigma_j}^{w_j} \right)^{1/\lambda}}, \right.\right. \\ &\left.\left. \frac{\left(\prod_{j=1}^{n} \widetilde{\vartheta}_{\sigma_j}^{w_j} + (\zeta^2-1) \prod_{j=1}^{n} \widetilde{\psi}_{\sigma_j}^{w_j} \right)^{1/\lambda} - \left(\prod_{j=1}^{n} \widetilde{\vartheta}_{\sigma_j}^{w_j} - \prod_{j=1}^{n} \widetilde{\psi}_{\sigma_j}^{w_j} \right)^{1/\lambda}}{\left(\prod_{j=1}^{n} \widetilde{\vartheta}_{\sigma_j}^{w_j} + (\zeta^2-1) \prod_{j=1}^{n} \widetilde{\psi}_{\sigma_j}^{w_j} \right)^{1/\lambda} + (\zeta-1) \left(\prod_{j=1}^{n} \widetilde{\vartheta}_{\sigma_j}^{w_j} - \prod_{j=1}^{n} \widetilde{\psi}_{\sigma_j}^{w_j} \right)^{1/\lambda}} \right) \right\rangle \end{aligned} \tag{33}$$

in which, $(\sigma_1, \sigma_2, \cdots, \sigma_n)$ is a permutation of $(1, 2, \cdots, n)$, such that $\alpha_{\sigma_{j-1}} \geq \alpha_{\sigma_j}$ for $\forall j$, $\zeta > 0$,

$$\widetilde{\phi}_j = (1 + (\zeta - 1)(1 - \widetilde{\mu}(x_j)))^\lambda + (\zeta^2 - 1)\widetilde{\mu}(x_j)^\lambda,$$

$$\widetilde{\varphi}_j = (1 + (\zeta - 1)(1 - \widetilde{\mu}(x_j)))^\lambda - \widetilde{\mu}(x_j)^\lambda, \widetilde{\vartheta}_j = (1 + (\zeta - 1)\widetilde{\nu}(x_j))^\lambda + (\zeta^2 - 1)(1 - \widetilde{\nu}(x_j))^\lambda,$$

$$\widetilde{\psi}_j = (1 + (\zeta - 1)\widetilde{\nu}(x_j))^\lambda - (1 - \widetilde{\nu}(x_j))^\lambda.$$

$$\widetilde{\mu}(x_j) = \frac{\zeta \mu(x_j)^{n\omega_j}}{(1 + (\zeta - 1)(1 - \mu(x_j)))^{n\omega_j} + (\zeta - 1)\mu(x_j)^{n\omega_j}},$$

$$\widetilde{\nu}(x_j) = \frac{(1+(\zeta-1)\nu(x_j))^{n\omega_j} - (1-\nu(x_j))^{n\omega_j}}{(1+(\zeta-1)\nu(x_j))^{n\omega_j} + (\zeta-1)(1-\nu(x_j))^{n\omega_j}}.$$

Theorem 2 can be proved by mathematical induction, shown as follows:

Proof.

(1) When $n = 1$, we can know $w_1 = 1$, $\omega_1 = 1$, for the left side of Equation (33),

$$IULGHHWA(\alpha_1, \alpha_2, \cdots, \alpha_n) = \beta_{\sigma_1} = \alpha_{\sigma_1} = \alpha_1 =< [s_{\alpha(x_1)}, s_{\beta(x_1)}], (\mu(x_1), \nu(x_1)) >$$

and for the right side of Equation (33), we have

$$\left\langle \left[s_{\alpha(x_1)}, s_{\beta(x_1)}\right], \left(\frac{\zeta(\phi_1-\varphi_1)^{1/\lambda}}{(\phi_1+(\zeta^2-1)\varphi_1)^{1/\lambda}+(\zeta-1)(\phi_1-\varphi_1)^{1/\lambda}},\right.\right.$$

$$\left.\left.\frac{\left(\vartheta_1+(\zeta^2-1)\psi_1\right)^{1/\lambda}-(\vartheta_1-\psi_1)^{1/\lambda}}{(\vartheta_1+(\zeta^2-1)\psi_1)^{1/\lambda}+(\zeta-1)(\vartheta_1-\psi_1)^{1/\lambda}}\right)\right\rangle =< [s_{\alpha(x_1)}, s_{\beta(x_1)}], (\mu(x_1), \nu(x_1)) >,$$

in which,

$$\phi_1 = (1+(\zeta-1)(1-\mu(x_1)))^{\lambda} + (\zeta^2-1)\mu(x_1)^{\lambda},$$
$$\varphi_1 = (1+(\zeta-1)(1-\mu(x_1)))^{\lambda} - \mu(x_1)^{\lambda}, \vartheta_1 = (1+(\zeta-1)\nu(x_1))^{\lambda} + (\zeta^2-1)(1-\nu(x_1))^{\lambda},$$
$$\psi_1 = (1+(\zeta-1)\nu(x_1))^{\lambda} - (1-\nu(x_1))^{\lambda}.$$

Therefore, Equation (33) holds for $n = 1$.

(2) Assume that Equation (33) holds for $n = k$, we have

$$IULGHHWA(\alpha_1, \alpha_2, \cdots, \alpha_k)$$
$$= \left\langle \left[s_{\sum_{j=1}^k w_j\widetilde{\alpha}(x_{\sigma_j})}, s_{\sum_{j=1}^k w_j\widetilde{\beta}(x_{\sigma_j})}\right], \left(\frac{\zeta\left(\prod_{j=1}^k \widetilde{\phi}_{\sigma_j}^{w_j} - \prod_{j=1}^k \widetilde{\varphi}_{\sigma_j}^{w_j}\right)^{1/\lambda}}{\left(\prod_{j=1}^k \widetilde{\phi}_{\sigma_j}^{w_j} + (\zeta^2-1)\prod_{j=1}^k \widetilde{\varphi}_{\sigma_j}^{w_j}\right)^{1/\lambda} + (\zeta-1)\left(\prod_{j=1}^k \widetilde{\phi}_{\sigma_j}^{w_j} - \prod_{j=1}^k \widetilde{\varphi}_{\sigma_j}^{w_j}\right)^{1/\lambda}},\right.\right.$$

$$\left.\left.\frac{\left(\prod_{j=1}^k \widetilde{\vartheta}_{\sigma_j}^{w_j} + (\zeta^2-1)\prod_{j=1}^k \widetilde{\psi}_{\sigma_j}^{w_j}\right)^{1/\lambda} - \left(\prod_{j=1}^k \widetilde{\vartheta}_{\sigma_j}^{w_j} - \prod_{j=1}^k \widetilde{\psi}_{\sigma_j}^{w_j}\right)^{1/\lambda}}{\left(\prod_{j=1}^k \widetilde{\vartheta}_{\sigma_j}^{w_j} + (\zeta^2-1)\prod_{j=1}^k \widetilde{\psi}_{\sigma_j}^{w_j}\right)^{1/\lambda} + (\zeta-1)\left(\prod_{j=1}^k \widetilde{\vartheta}_{\sigma_j}^{w_j} - \prod_{j=1}^k \widetilde{\psi}_{\sigma_j}^{w_j}\right)^{1/\lambda}}\right)\right\rangle.$$

when $n = k+1$,

$$IULGHHWA(\alpha_1, \alpha_2, \cdots, \alpha_k, \alpha_{k+1}) = IULGHHWA(\alpha_1, \alpha_2, \cdots, \alpha_k) \oplus_h (w_{k+1}\beta_{k+1})$$

$$= \left\langle \left[s_{\sum_{j=1}^k w_j\widetilde{\alpha}(x_{\sigma_j})}, s_{\sum_{j=1}^k w_j\widetilde{\beta}(x_{\sigma_j})}\right], \left(\frac{\zeta\left(\prod_{j=1}^k \widetilde{\phi}_{\sigma_j}^{w_j} - \prod_{j=1}^k \widetilde{\varphi}_{\sigma_j}^{w_j}\right)^{1/\lambda}}{\left(\prod_{j=1}^k \widetilde{\phi}_{\sigma_j}^{w_j} + (\zeta^2-1)\prod_{j=1}^k \widetilde{\varphi}_{\sigma_j}^{w_j}\right)^{1/\lambda} + (\zeta-1)\left(\prod_{j=1}^k \widetilde{\phi}_{\sigma_j}^{w_j} - \prod_{j=1}^k \widetilde{\varphi}_{\sigma_j}^{w_j}\right)^{1/\lambda}},\right.\right.$$

$$\frac{\left(\prod_{j=1}^{k}\widetilde{\vartheta}_{\sigma_j}^{w_j}+(\zeta^2-1)\prod_{j=1}^{k}\widetilde{\psi}_{\sigma_j}^{w_j}\right)^{1/\lambda}-\left(\prod_{j=1}^{k}\widetilde{\vartheta}_{\sigma_j}^{w_j}-\prod_{j=1}^{k}\widetilde{\psi}_{\sigma_j}^{w_j}\right)^{1/\lambda}}{\left(\prod_{j=1}^{k}\widetilde{\vartheta}_{\sigma_j}^{w_j}+(\zeta^2-1)\prod_{j=1}^{k}\widetilde{\psi}_{\sigma_j}^{w_j}\right)^{1/\lambda}+(\zeta-1)\left(\prod_{j=1}^{k}\widetilde{\vartheta}_{\sigma_j}^{w_j}-\prod_{j=1}^{k}\widetilde{\psi}_{\sigma_j}^{w_j}\right)^{1/\lambda}})\Big\rangle$$

$$\oplus_h\Big\langle[s_{\widetilde{\alpha}(x_{k+1})},s_{\widetilde{\beta}(x_{k+1})}],(\frac{(1+(\zeta-1)\widetilde{\mu}(x_{k+1}))^{w_{k+1}}-(1-\widetilde{\mu}(x_{k+1}))^{w_{k+1}}}{(1+(\zeta-1)\widetilde{\mu}(x_{k+1}))^{w_{k+1}}+(\zeta-1)(1-\widetilde{\mu}(x_{k+1}))^{w_{k+1}}},$$
$$\frac{\zeta(\widetilde{\nu}(x_{k+1}))^{w_{k+1}}}{(1+(\zeta-1)(1-\widetilde{\nu}(x_{k+1})))^{w_{k+1}}+(\zeta-1)(\widetilde{\nu}(x_{k+1}))^{w_{k+1}}})\Big\rangle$$

$$=\Big\langle\Big[s_{\sum_{j=1}^{k+1}w_j\widetilde{\alpha}(x_{\sigma_j})},s_{\sum_{j=1}^{k+1}w_j\widetilde{\beta}(x_{\sigma_j})}\Big],(\frac{\zeta\left(\prod_{j=1}^{k+1}\widetilde{\phi}_{\sigma_j}^{w_j}-\prod_{j=1}^{k+1}\widetilde{\varphi}_{\sigma_j}^{w_j}\right)^{1/\lambda}}{\left(\prod_{j=1}^{k+1}\widetilde{\phi}_{\sigma_j}^{w_j}+(\zeta^2-1)\prod_{j=1}^{k+1}\widetilde{\varphi}_{\sigma_j}^{w_j}\right)^{1/\lambda}+(\zeta-1)\left(\prod_{j=1}^{k+1}\widetilde{\phi}_{\sigma_j}^{w_j}-\prod_{j=1}^{k+1}\widetilde{\varphi}_{\sigma_j}^{w_j}\right)^{1/\lambda}},$$

$$\frac{\left(\prod_{j=1}^{k+1}\widetilde{\vartheta}_{\sigma_j}^{w_j}+(\zeta^2-1)\prod_{j=1}^{k+1}\widetilde{\psi}_{\sigma_j}^{w_j}\right)^{1/\lambda}-\left(\prod_{j=1}^{k+1}\widetilde{\vartheta}_{\sigma_j}^{w_j}-\prod_{j=1}^{k+1}\widetilde{\psi}_{\sigma_j}^{w_j}\right)^{1/\lambda}}{\left(\prod_{j=1}^{k+1}\widetilde{\vartheta}_{\sigma_j}^{w_j}+(\zeta^2-1)\prod_{j=1}^{k+1}\widetilde{\psi}_{\sigma_j}^{w_j}\right)^{1/\lambda}+(\zeta-1)\left(\prod_{j=1}^{k+1}\widetilde{\vartheta}_{\sigma_j}^{w_j}-\prod_{j=1}^{k+1}\widetilde{\psi}_{\sigma_j}^{w_j}\right)^{1/\lambda}})\Big\rangle.$$

Therefore, when $n=k+1$, Equation (33) holds.

(3) According to steps 1 and 2, we get Equation (33) which holds for any n.

□

Remark 2. *Let* $\omega=(\frac{1}{n},\frac{1}{n},\cdots,\frac{1}{n})^T$ *and* $w=(\frac{1}{n},\frac{1}{n},\cdots,\frac{1}{n})^T$ *, the IULGHWA and IULGHOWA operators are the special cases of the IULGHHWA operator, respectively.*

From the analysis above, we know that the IULGHHWA operator generalizes both the IULGHWA and IULGHOWA operators, and reflects the importance degrees of both the given intuitionistic uncertain linguistic variables and their ordered positions.

4. An Approach to MAGDM Based on the Generalized Hamacher Operator with Intuitionistic Uncertain Linguistic Information

In this Section, we consider a MAGDM problem based on the generalized Hamacher aggregation operator with intuitionistic uncertain linguistic information. Let $X=\{x_1,x_2,\cdots,x_n\}$ be a discrete set of alternatives, $G=\{g_1,g_2,\cdots,g_l\}$ be the set of attributes, and $\omega=(\omega_1,\omega_2,\cdots,\omega_l)^T$ be the weight vector of attributes, where $\omega_i\in[0,1]$, $i=1,2,\cdots,l$, $\sum_{i=1}^{l}\omega_i=1$. Let $d_k\in D=\{d_1, d_2, ..., d_m\}$ and $u=(u_1,u_2,\cdots,u_m)^T$ be the set of decision makers (DMs) and the weight vector of DMs, respectively, with the condition $u_k\geq 0$, $\sum_{k=1}^{m}u_k=1$. Suppose $A^{(k)}=(\alpha_{ij}^{(k)})_{l\times n}$ is the decision matrix, where $\alpha_{ij}^{(k)}=<[s_{\alpha_{x(ij)}^k},s_{\beta_{x(ij)}^k}],(\mu_{x(ij)}^k,\nu_{x(ij)}^k)>$ takes the form of intuitionistic uncertain linguistic variables, which is provided by the DMs $d_k(k=1,2,\cdots,m)$ over all the alternatives $x_j\in X(j=1,2,\cdots,n)$ with respect to the attribute $g_i\in G$, and $0\leq\mu_{x(ij)}^k\leq 1$, $0\leq\nu_{x(ij)}^k\leq 1$, $\mu_{x(ij)}^k+\nu_{x(ij)}^k\leq 1$, $[s_{\alpha_{x(ij)}^k},s_{\beta_{x(ij)}^k}]\in\widetilde{S}$. Next, we propose a method for MAGDM with intuitionistic uncertain linguistic information based on the generalized Hamacher aggregation operator as follows:

Step 1: First, we normalize the intuitionistic uncertain linguistic variable to unify all attributes, unless all the attributes have the same measurement unit or belong to the maximizing attribute. Suppose that the decision matrix $A^{(k)}=(\alpha_{ij}^{(k)})_{l\times n}$ is normalized into the corresponding matrix

$\hat{A}^{(k)} = (\hat{\alpha}_{ij}^{(k)})_{l\times n}$, $i = 1,2,\cdots,l$, $j = 1,2,\cdots,n$. For the minimizing attribute, the normalization formula is $\hat{\alpha}_{ij}^{(k)} =< [neg(s_{\hat{\beta}^k_{x(ij)}}), neg(s_{\hat{\alpha}^k_{x(ij)}})], (\hat{\nu}^k_{x(ij)}, \hat{\mu}^k_{x(ij)}) >$.

Step 2: Aggregate all the decision matrices $\hat{A}^{(k)} = (\hat{\alpha}_{ij}^{(k)})_{l\times n}$, $k = 1,2,\cdots,m$ into the collective intuitionistic uncertain linguistic decision matrix $\hat{A} = (\hat{\alpha}_{ij})_{l\times n}$ by using the IULGHHWA operator:

$$\hat{\alpha}_{ij} = IULGHHWA\Big(\hat{\alpha}_{ij}^{(1)}, \hat{\alpha}_{ij}^{(2)}, \cdots, \hat{\alpha}_{ij}^{(m)}\Big) \tag{34}$$

Step 3: By using the IULGHHWA operator to derive the collective overall preference values $\hat{\alpha}_j$ $(j = 1,2,\cdots,n)$:

$$\hat{\alpha}_j = IULGHHWA\Big(\hat{\alpha}_{1j}, \hat{\alpha}_{2j}, \cdots, \hat{\alpha}_{lj}\Big) \tag{35}$$

Step 4: Calculate the expected value $E(\hat{\alpha}_j)$ and accuracy function $H(\hat{\alpha}_j)$ (if any) of the collective overall preference values $\hat{\alpha}_j$ $(j = 1,2,\cdots,n)$. For an intuitionistic uncertain linguistic variable $\alpha_i =< [s_{\alpha(x_i)}, s_{\beta(x_i)}], (\mu(x_i), \nu(x_i)) >$, the expected value $E(\alpha_i)$ and accuracy function $H(\alpha_i)$ are defined as follows (Liu and Jin [11], Wang et al. [48]):

$$E(\alpha_i) = \frac{[\mu(x_i) + (1 - \nu(x_i))][\alpha(x_i) + \beta(x_i)]}{4} \tag{36}$$

$$H(\alpha_i) = \frac{[\mu(x_i) + \nu(x_i)][\alpha(x_i) + \beta(x_i)]}{2} \tag{37}$$

To compare two intuitionistic uncertain linguistic variables α_1 and α_2, if $E(\alpha_1) < E(\alpha_2)$, then $\alpha_1 < \alpha_2$; if $E(\alpha_1) = E(\alpha_2)$, then we need to further compare the accuracy function, if $H(\alpha_1) < H(\alpha_2)$, then $\alpha_1 < \alpha_2$ and if $H(\alpha_1) = H(\alpha_2)$, then $\alpha_1 = \alpha_2$.

Step 5: Rank all the alternatives $x_j \in X(j = 1,2,\cdots,n)$ and select the best one(s) in accordance with the expected value $E(\hat{\alpha}_j)$ and accuracy function $H(\hat{\alpha}_j)$.

Step 6: End.

5. Illustrative Example and Discussion

The choice of suppliers is an important decision in supply chain management. The purchasing department of an overseas multi-national corporation plans to choose a suitable supplier in order to improve the development of the organization. Therefore, the selection of an excellent supplier is a very important process. Suppose there are four suppliers (alternatives) denoted by $x_j(j = 1,2,3,4)$. The four possible alternatives are to be evaluated by means of the intuitionistic uncertain linguistic set A by three decision makers $d_k(k = 1,2,3)$. Meanwhile, the department must make a decision according to the following three attributes, g_1: risk analysis (risk identification, risk assessment and risk management strategy); g_2: development analysis (external management environment, enterprise inner quality and resource conditions; g_3: social-political impact analysis, which are weighted as $\omega = (0.4, 0.4, 0.2)^T$. After a heated discussion about the suppliers with regard to the three attributes mentioned above, the three decision makers came to a consensus on the decision matrix $A^{(k)}$, which the takes the form of intuitionistic uncertain linguistic numbers, as listed in Tables 1–3, respectively.

Table 1. Decision matrix $A^{(1)}$ with IULNs.

	x_1	x_2	x_3	x_4
g_1	$< [s_1, s_2], (0.5, 0.4) >$	$< [s_2, s_3], (0.4, 0.5) >$	$< [s_2, s_3], (0.5, 0.3) >$	$< [s_2, s_3], (0.5, 0.3) >$
g_2	$< [s_1, s_2], (0.5, 0.2) >$	$< [s_2, s_4], (0.6, 0.2) >$	$< [s_2, s_2], (0.4, 0.5) >$	$< [s_1, s_1], (0.2, 0.4) >$
g_3	$< [s_2, s_3], (0.4, 0.2) >$	$< [s_1, s_2], (0.2, 0.3) >$	$< [s_1, s_3], (0.2, 0.4) >$	$< [s_1, s_2], (0.6, 0.1) >$

Table 2. Decision matrix $A^{(2)}$ with IULNs.

	x_1	x_2	x_3	x_4
g_1	$<[s_1, s_1], (0.6, 0.4)>$	$<[s_1, s_2], (0.4, 0.3)>$	$<[s_2, s_2], (0.5, 0.2)>$	$<[s_2, s_2], (0.4, 0.3)>$
g_2	$<[s_2, s_3], (0.7, 0.1)>$	$<[s_1, s_2], (0.5, 0.2)>$	$<[s_2, s_3], (0.5, 0.5)>$	$<[s_2, s_3], (0.4, 0.3)>$
g_3	$<[s_2, s_2], (0.8, 0.2)>$	$<[s_1, s_2], (0.4, 0.3)>$	$<[s_2, s_2], (0.2, 0.4)>$	$<[s_3, s_4], (0.7, 0.1)>$

Table 3. Decision matrix $A^{(3)}$ with IULNs.

	x_1	x_2	x_3	x_4
g_1	$<[s_1, s_2], (0.6, 0.3)>$	$<[s_2, s_3], (0.5, 0.5)>$	$<[s_1, s_2], (0.6, 0.3)>$	$<[s_1, s_2], (0.4, 0.4)>$
g_2	$<[s_2, s_3], (0.7, 0.3)>$	$<[s_1, s_2], (0.7, 0.2)>$	$<[s_1, s_2], (0.5, 0.5)>$	$<[s_1, s_2], (0.3, 0.4)>$
g_3	$<[s_3, s_4], (0.5, 0.2)>$	$<[s_1, s_2], (0.2, 0.5)>$	$<[s_3, s_4], (0.6, 0.4)>$	$<[s_2, s_3], (0.6, 0.1)>$

5.1. The Decision Making Procedure Using the Developed Method

Using the decision information above, and in view of the intuitionistic uncertain linguistic MAGDM problem, by following the method developed in Section 4, the decision making procedure is solved based on the generalized Hamacher aggregation operator as follows:

Step 1: Because all the attributes have the same measurement unit and belong to the maximizing attribute, it is not necessary to normalize the decision information.

Step 2: By utilizing Equation (34), we aggregate the decision matrices $A^{(k)} = (\alpha_{ij}^{(k)})_{3\times4}$, $k = 1, 2, 3$ into the collective intuitionistic uncertain linguistic decision matrix $A = (\alpha_{ij})_{3\times4}$ by using the IULGHHWA operator (the weight vector $w = (0.2429, 0.5142, 0, 2429)^T$ associated with which is derived by the Gaussian distribution based method (Xu [61]). Without loss of generality, let $\lambda = 1$ and $\zeta = 2$.

$$A = (\alpha_{ij})_{3\times4} = \begin{bmatrix} \langle[s_1, s_{2.3}], (0.46, 0.28)\rangle & \langle[s_2, s_{3.6}], (0.45, 0.33)\rangle & \langle[s_2, s_{2.5}], (0.35, 0.49)\rangle & \langle[s_{1.3}, s_{1.8}], (0.36, 0.44)\rangle \\ \langle[s_{1.8}, s_{2.4}], (0.76, 0.2)\rangle & \langle[s_1, s_{2.1}], (0.43, 0.29)\rangle & \langle[s_{2.1}, s_{2.7}], (0.41, 0.45)\rangle & \langle[s_{2.3}, s_3], (0.48, 0.28)\rangle \\ \langle[s_2, s_3], (0.66, 0.32)\rangle & \langle[s_{1.3}, s_{2.4}], (0.58, 0.38)\rangle & \langle[s_{1.3}, s_{2.4}], (0.5, 0.49)\rangle & \langle[s_{1.2}, s_{2.3}], (0.38, 0.35)\rangle \end{bmatrix}$$

Step 3: Furthermore, calculating the collective overall preference values α_j $(j = 1, 2, 3, 4)$ by using the IULGHHWA operator (suppose the weight vector of DMs $u = (u_1, u_2, u_3)^T = (0.3, 0.5, 0.2)^T$, let $\lambda = 1$ and $\zeta = 2$), we can calculate:

$$\alpha_1 =<[s_{1.9}, s_{2.8}], (0.71, 0.28)>, \ \alpha_2 =<[s_{1.4}, s_{2.8}], (0.46, 0.38)>,$$
$$\alpha_3 =<[s_2, s_{2.6}], (0.39, 0.54)>, \ \alpha_4 =<[s_{2.2}, s_3], (0.4, 0.39)>.$$

Step 4: Calculate the expected value $E(\alpha_j)$ of the collective overall preference values α_j $(j = 1, 2, 3, 4)$:

$$E(\alpha_1) = 1.68, \ E(\alpha_2) = 1.13, \ E(\alpha_3) = 0.98, \ E(\alpha_4) = 1.31.$$

Step 5: Rank all the alternatives $x_j \in X(j = 1, 2, 3, 4)$ and select the best one(s) in accordance with the expected value $E(\alpha_j)$:

$$E(\alpha_1) \succ E(\alpha_4) \succ E(\alpha_2) \succ E(\alpha_3)$$

Thus, the corresponding order is $x_1 \succ x_4 \succ x_2 \succ x_3$, and the best supplier (alternative) is x_1. This is the supplier that the purchasing department of the overseas multi-national corporation should choose to maximize development.

5.2. Comparative Analysis and Discussion

We take $\lambda = 1$ and $\zeta = 1$ in steps 2 and 3 to rank the alternatives in the example above, as presently the intuitionistic uncertain linguistic generalized Hamacher hybrid weighted average (IULGHHWA) operator will be reduced to the intuitionistic uncertain linguistic hybrid weighted averaging (IULHWA) operator which is defined by Liu and Zhang [46]. When $\lambda = 1$ and $\zeta = 1$, by utilizing Equation (34), we can calculate the collective intuitionistic uncertain linguistic decision matrix as follows:

$$A = (\alpha_{ij})_{3\times 4} = \begin{bmatrix} \langle [s_1, s_{2.3}], (0.5, 0.24)\rangle & \langle [s_2, s_{3.6}], (0.53, 0.28)\rangle & \langle [s_2, s_{2.5}], (0.42, 0.42)\rangle & \langle [s_{1.3}, s_{1.8}], (0.38, 0.26)\rangle \\ \langle [s_{1.8}, s_{2.4}], (0.71, 0.17)\rangle & \langle [s_1, s_{2.1}], (0.48, 0.25)\rangle & \langle [s_{2.1}, s_{2.7}], (0.48, 0.38)\rangle & \langle [s_{2.3}, s_3], (0.47, 0.23)\rangle \\ \langle [s_2, s_3], (0.67, 0.27)\rangle & \langle [s_{1.3}, s_{2.4}], (0.62, 0.32)\rangle & \langle [s_{1.3}, s_{2.4}], (0.56, 0.42)\rangle & \langle [s_{1.2}, s_{2.3}], (0.39, 0.28)\rangle \end{bmatrix}$$

Furthermore, we can calculate the collective overall preference values α_j $(j = 1, 2, 3, 4)$:

$$\alpha_1 =< [s_{1.9}, s_{2.8}], (0.7, 0.22) >,\ \alpha_2 =< [s_{1.4}, s_{2.8}], (0.56, 0.29) >$$
$$\alpha_3 =< [s_2, s_{2.6}], (0.52, 0.41) >,\ \alpha_4 =< [s_{2.2}, s_3], (0.46, 0.26) >$$

Rank the alternatives $x_j \in X(j = 1, 2, 3, 4)$ in accordance with the expected value

$$E(\alpha_1) = 1.74,\ E(\alpha_2) = 1.33,\ E(\alpha_3) = 1.28,\ E(\alpha_4) = 1.56.$$

Therefore, we obtain the same result as found in Section 5.1, and the corresponding order is still $x_1 \succ x_4 \succ x_2 \succ x_3$, that is, the best supplier (alternative) is x_1.

Next, we take $\lambda \to 0$ for the Hammer geometric averaging operator in a similar way. At present, when $\zeta = 1$, the operators are intuitionistic uncertain linguistic ordered weighted geometric averaging (IULOWGA) (Liu and Jin [11]) and Algebraic aggregation operator, and when $\zeta = 2$, the operator is the Einstein aggregation operator. Moreover, we consider a comparative analysis with other related group decision making methods such as Xian et al. [62], Wang et al. [48], the corresponding order and the best alternative is x_1, which is the same as the result of the developed method.

In this section, we have discussed the effect on the results when the parameters λ and ζ take different values based on IULGHHWA operator. More concretely, it will be reduced to the IULGHWA and IULGHOWA operator when $\omega = (\frac{1}{n}, \frac{1}{n}, \cdots, \frac{1}{n})^T$ and $w = (\frac{1}{n}, \frac{1}{n}, \cdots, \frac{1}{n})^T$, respectively. The generalized Hamacher aggregation operators developed in this paper not only extend the existing methods that have been proposed by Liu and Zhang [46] and Liu and Jin [11], but also the generalized intuitionistic uncertain linguistic arithmetic\geometric averaging operators and algebraic\Einstein aggregation operators. They provide very general formulations that include a broad range of aggregation operators for intuitionistic uncertain linguistic information as special cases, including all arithmetic aggregation operators, geometric aggregation operators, the Algebraic aggregation operators IULWGA (Liu and Jin [11]), IULWA (Wang et al. [48] and Liu and Zhang [46]) and Einstein aggregation operators (EIULWA, EIULWGA, EIULOWA). Therefore, they can easily accommodate the environment in which the given arguments are different values.

6. Concluding Remarks

In this paper, we have presented the theory and application of Archimedean t-norm and s-norm based on intuitionistic uncertain linguistic sets. The existing aggregation operators for intuitionistic uncertain linguistic sets are mostly based on algebraic t-norm and s-norm. For further study, we have developed several new aggregation operators for intuitionistic uncertain linguistic sets based on Hamacher t-norm and s-norm, such as IULGHWA, IULGHOWA, HIULWGA, HIULOWA, HIULOWGA and IULGHHWA operators, and the properties of these operators have been investigated. Moreover, we have provided a numerical example and a comparative analysis with related decision-making methods to illustrate the practicality and feasibility of the developed method.

In future research, we expect to develop further extensions by considering new characteristics, including the distance measures in the problem and the application of such operational laws to other types of linguistic fuzzy information such as unbalanced linguistic information, and hesitant fuzzy linguistic sets. The application of the proposed methods in different fields such as data mining and pattern recognition is also a potential research topic.

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References

1. Atanassov, K.T. Intuitionistic fuzzy sets. *Fuzzy Sets Syst.* **1986**, *20*, 87–96. [CrossRef]
2. Atanassov, K.T.; Gargov, G. Interval valued intuitionistic fuzzy sets. *Fuzzy Sets Syst.* **1989**, *31*, 343–349. [CrossRef]
3. Dubois, D.; Prade, H. *Fuzzy Sets and Systems: Theory and Applications*; Academic Press: New York, NY, USA, 1980.
4. Mizumoto, M.; Tanaka, K. Fuzzy sets of type-2 under algebraic product and algebraic sum. *Fuzzy Sets Syst.* **1981**, *5*, 277–290. [CrossRef]
5. Yager, R.R. On ordered weighted averaging aggregation operators in multi-criteria decision making. *IEEE Trans. Syst. Man Cybern.* **1988**, *18*, 183–190. [CrossRef]
6. Torra, V.; Narukawa, Y. On hesitant fuzzy sets and decision. In Proceedings of the 18th IEEE International Conference on Fuzzy Systems, Jeju Island, Korea, 20–24 August 2009; pp. 1378–1382.
7. Yager, R.R. Pythagorean fuzzy subsets. In Proceedings of the Joint IFSA World Congress and NAFIPS Annual Meeting, Edmonton, AB, Canada, 24–28 June 2013; pp. 57–61.
8. Yager, R.R. Pythagorean membership grades in multicriteria decision making. *IEEE Trans. Fuzzy Syst.* **2014**, *22*, 958–965. [CrossRef]
9. Li, D.F.; Cheng, C.T. New similarity measures of intuitionistic fuzzy sets and application to pattern recognitions. *Pattern Recognit. Lett.* **2002**, *23*, 221–225.
10. Xu, Z.S.; Yager, R.R. Intuitionistic and interval-valued intutionistic fuzzy preference relations and their measures of similarity for the evaluation of agreement within a group. *Fuzzy Optim. Decis. Mak.* **2009**, *8*, 123–139. [CrossRef]
11. Liu, P.D.; Jin, F. Methods for aggregating intuitionistic uncertain linguistic variables and their application to group decision making. *Inf. Sci.* **2012**, *205*, 58–71. [CrossRef]
12. Merigó, J.M.; Yager, R.R. Generalized moving averages, distance measures and OWA operators. *Int. J. Uncertain. Fuzziness Knowl. Based Syst.* **2013**, *21*, 533–559. [CrossRef]
13. Merigó, J.M.; Gil-Lafuente, A.M.; Yager, R.R. An overview of fuzzy research with bibliometric indicators. *Appl. Soft Comput.* **2015**, *27*, 420–433. [CrossRef]
14. Szmidt, E.; Kacprzyk, J.; Bujnowski, P. How to measure the amount of knowledge conveyed by Atanassov's intuitionistic fuzzy set. *Inf. Sci.* **2014**, *257*, 276–285. [CrossRef]
15. Tian, Z.P.; Wang, J.Q.; Zhang, H.Y.; Wang, T.L. Signed distance-based consensus in multi-criteria group decision-making with multi-granular hesitant unbalanced linguistic information. *Comput. Ind. Eng.* **2018**, *124*, 125–138. [CrossRef]
16. Yu, D.J.; Li, D.F.; Merigó, J.M.; Fang, L. Mapping development of linguistic decision making studies. *J. Intell. Fuzzy Syst.* **2016**, *30*, 2727–2736. [CrossRef]
17. Li, Z.; Gao, H.; Wei, G. Methods for Multiple Attribute Group Decision Making Based on Intuitionistic Fuzzy Dombi Hamy Mean Operators. *Symmetry* **2018**, *10*, 574. [CrossRef]
18. Hayat, K.; Ali, M.I.; Cao, B.-Y.; Karaaslan, F.; Yang, X.-P. Another View of Aggregation Operators on Group-Based Generalized Intuitionistic Fuzzy Soft Sets: Multi-Attribute Decision Making Methods. *Symmetry* **2018**, *10*, 753. [CrossRef]
19. Malik, M.G.A.; Bashir, Z.; Rashid, T.; Ali, J. Probabilistic Hesitant Intuitionistic Linguistic Term Sets in Multi-Attribute Group Decision Making. *Symmetry* **2018**, *10*, 392. [CrossRef]
20. Wang, J.Q.; Yang, Y.; Li, L. Multi-criteria decision-making method based on single valued neutrosophic linguistic Maclaurin symmetric mean operators. *Neural Comput. Appl.* **2018**, *30*, 1529–1547. [CrossRef]
21. Wang, Y.; Liu, P. Linguistic Neutrosophic Generalized Partitioned Bonferroni Mean Operators and Their Application to Multi-Attribute Group Decision Making. *Symmetry* **2018**, *10*, 160. [CrossRef]

22. Li, D.F. Topsis-based nonlinear-programming methodology for multi-attribute decision making with interval-valued intuitionistic fuzzy sets. *IEEE Trans. Fuzzy Syst.* **2010**, *18*, 299–311.
23. Nie, R.X.; Tian, Z.P.; Wang, J.Q.; Zhang, H.Y.; Wang, T.L. Water security sustainability evaluation: Applying a multistage decision support framework in industrial region. *J. Clean. Prod.* **2018**, *196*, 1681–1704. [CrossRef]
24. Peng, H.G.; Zhang, H.Y.; Wang, J.Q. Cloud decision making support model for selecting hotels on TripAdvisor.com with probabilistic linguistic information. *Int. J. Hosp. Manag.* **2018**, *68*, 124–138. [CrossRef]
25. Yu, S.M.; Zhang, H.Y.; Wang, J.Q. Hesitant Fuzzy Linguistic Maclaurin Symmetric Mean Operators and their Applications to Multi-Criteria Decision-Making Problem. *Int. J. Intell. Syst.* **2017**, 953–982. [CrossRef]
26. Zeng, S.Z.; Mu, Z.M.; Baležentis, T. A novel aggregation method for Pythagorean fuzzy multiple attribute group decision making. *Int. J. Intell. Syst.* **2018**, *33*, 573–585. [CrossRef]
27. Zeng, S.Z.; Xiao, Y. A method based on topsis and distance measures for hesitant fuzzy multiple attribute decision making. *Technol. Econ. Dev. Econ.* **2018**, *24*, 905–919. [CrossRef]
28. Morente-Molinera, J.A.; Kou, G.; González-Crespo, R.; Corchado, J.M. Solving multi-criteria group decision making problems under environments with a high number of alternatives using fuzzy ontologies and multi-granular linguistic modelling methods. *Knowl. Based Syst.* **2017**, *137*, 54–64. [CrossRef]
29. Idrus, A.; Mahmoud, M.A.; MAhmad, S.; Yahya, A.; Husen, H. A Solution Generator Algorithm for Decision Making based Automated Negotiation in the Construction Domain. *Int. J. Interact. Multimed. Artif. Intell.* **2017**, *4*, 18–23. [CrossRef]
30. Bagga, P.; Joshi, A.; Hans, R. QoS based Web Service Selection and Multi-Criteria Decision Making Methods. *Int. J. Interact. Multimed. Artif. Intell.* **2019**, *5*, 113–121. [CrossRef]
31. Carrasco, R.A.; Blasco, M.F.; García-Madariaga, J.; Herrera-Viedma, E. A Fuzzy Linguistic RFM Model Applied to Campaign Management. *Int. J. Interact. Multimed. Artif. Intell.* **2019**, *5*, 21–27. [CrossRef]
32. Xu, Z.S. Uncertain linguistic aggregation operators based approach to multiple attribute group decision making under uncertain linguistic environment. *Inf. Sci.* **2004**, *168*, 171–184. [CrossRef]
33. Wei, G.W. Uncertain linguistic hybrid geometric mean operator and its application to group decision making under uncertain linguistic environment. *Int. J. Uncertain. Fuzziness Knowl. Based Syst.* **2009**, *17*, 251–267. [CrossRef]
34. Suo, W.L.; Feng, B.; Fan, Z.P. Extension of the DEMATEL method in an uncertain linguistic environment. *Soft Comput.* **2012**, *16*, 471–483. [CrossRef]
35. Peng, B.; Ye, C.M. Some induced uncertain geometric aggregation operators with pure linguistic information and their application to group decision making. *Int. J. Uncertain. Fuzziness Knowl. Based Syst.* **2013**, *21*, 723–742. [CrossRef]
36. Peng, B.; Ye, C.M.; Zeng, S.Z. Uncertain pure linguistic hybrid harmonic averaging operator and generalized interval aggregation operator based approach to group decision making. *Knowl. Based Syst.* **2012**, *36*, 175–181. [CrossRef]
37. Tao, Z.F.; Chen, H.Y.; Song, X.; Zhou, L.G.; Liu, J.P. Uncertain linguistic fuzzy soft sets and their applications in group decision making. *Appl. Soft Comput.* **2015**, *34*, 587–605. [CrossRef]
38. Wang, X.K.; Peng, H.G.; Wang, J.Q. Hesitant linguistic intuitionistic fuzzy sets and their application in multi-criteria decision-making problems. *Int. J. Uncertain. Quantif.* **2018**, *8*, 321–341. [CrossRef]
39. Pamučar, D.; Badi, I.; Sanja, K.; Obradovic, R. A Novel Approach for the Selection of Power-Generation Technology Using a Linguistic Neutrosophic CODAS Method: A Case Study in Libya. *Energies* **2018**, *11*, 2489. [CrossRef]
40. Pamučar, D.; Božanić, D.; Lukovac, V.; Komazec, N. Normalized weighted geometric Bonferroni mean operator of interval rough numbers—Application in interval rough DEMATEL-COPRAS model. *FU Mech. Eng.* **2018**, *16*, 171–191. [CrossRef]
41. Pamučar, D.; Sremac, S.; Stević, Ž.; Ćirović, G.; Tomić, D. New multi-criteria LNN WASPAS model for evaluating the work of advisors in the transport of hazardous goods. *Neural Comput. Appl.* **2019**, 1–24. [CrossRef]
42. Liu, F.; Aiwu, G.; Lukovac, V.; Vukic, M.D. A multicriteria model for the selection of the transport service provider: A single valued neutrosophic DEMATEL multicriteria model. *Decis. Mak. Appl. Manag. Eng.* **2018**, *1*, 121–130. [CrossRef]
43. Si, A.; Das, S.; Kar, S. An approach to rank picture fuzzy numbers for decision making problems. *Decis. Mak. Appl. Manag. Eng.* **2019**. [CrossRef]

44. Milan, S.; Gladović, P.; Popović, V. Determining the importance of the criteria of traffic accessibility using fuzzy AHP and rough AHP method. *Decis. Mak. Appl. Manag. Eng.* **2019**, *2*, 86–104.
45. Wang, J.Q.; Li, H.B. Multi-criteria decision-making method based on aggregation operators for intuitionisic linguistic fuzzy numbers. *Control Decis.* **2010**, *25*, 1571–1574.
46. Liu, P.D.; Zhang, X. Intuitionistic uncertain linguistic aggregation operators and their application to group decision making. *Syst. Eng. Theory Pract.* **2012**, *12*, 2704–2711.
47. Bi, J.X.; Lei, L.H.; Peng, B. Some distance measures for intuitionistic uncertain linguistic sets and their application to group decision making. *Econ. Comput. Econ. Cybern. Stud. Res.* **2015**, *49*, 287–304.
48. Wang, Z.X.; Chen, J.; Lan, J.B. Multi-attribute decision making method based on intuitionistic uncertain linguistic new aggregation operator. *Syst. Eng. Theory Pract.* **2016**, *36*, 1871–1878.
49. Peng, B.; Ye, C.M. Methods for aggregating interval-valued intuitionistic pure linguistic information and their application to group decision making. *Syst. Eng. Theory Pract.* **2016**, *36*, 1526–1535.
50. Tan, C.Q.; Yi, W.T.; Chen, X.H. Hesitant fuzzy Hamacher aggregation operators for multicriteria decision making. *Appl. Soft Comput.* **2015**, *26*, 325–349. [CrossRef]
51. Xia, M.M.; Xu, Z.S.; Zhu, B. Some issues on intuitionistic fuzzy aggregation operators based on Archimedean t-conorm and t-norm. *Knowl. Based Syst.* **2012**, *31*, 78–88. [CrossRef]
52. Wang, W.Z.; Liu, X.W. Intuitionistic Fuzzy Information Aggregation Using Einstein Operators. *IEEE Trans. Fuzzy Syst.* **2012**, *20*, 923–938. [CrossRef]
53. Tao, Z.F.; Chen, H.Y.; Zhou, L.G.; Liu, J.P. On new operational laws of 2-tuple linguistic information using Archimedean t-norm and s-norm. *Knowl. Based Syst.* **2014**, *66*, 156–165. [CrossRef]
54. Liu, P.D. Some Hamacher aggregation operators based on the interval-valued intuitionistic fuzzy numbers and their application to group decision making. *IEEE Trans. Fuzzy Syst.* **2014**, *22*, 83–97. [CrossRef]
55. Rong, L.L.; Liu, P.D.; Chu, Y.C. Multiple attribute group decision making methods based on intuitionistic fuzzy generalized Hamacher aggregation operator. *Econ. Comput. Econ. Cybern. Stud. Res.* **2016**, *50*, 211–230.
56. Herrera, F.; Herrera-Viedma, E.; Verdegay, J.L. A model of consensus in group decision making under linguistic assessments. *Fuzzy Sets Syst.* **1996**, *78*, 73–87. [CrossRef]
57. Herrera, F.; Herrera-Viedma, E.; Martínez, L. A fusion approach for managing multi-granularity linguistic term sets in decision making. *Fuzzy Sets Syst.* **2000**, *114*, 43–58. [CrossRef]
58. Hamachar, H. Uber logische verknunpfungenn unssharfer Aussagen undderen Zugenhorige Bewertungsfunktione. In *Progress in Cybernetics and Systems Research*; Hemisphere: Washington, DC, USA, 1978; Volume 3, pp. 276–288.
59. Xu, Z.S. Intuitionistic fuzzy aggregation operators. *IEEE Trans. Fuzzy Syst.* **2007**, *15*, 1179–1187.
60. Xu, Z.S.; Yager, R.R. Some geometric aggregation operators based on intuitionistic fuzzy sets. *Int. J. Gen. Syst.* **2006**, *35*, 417–433. [CrossRef]
61. Xu, Z.S. An overview of methods for determining OWA weights. *Int. J. Intell. Syst.* **2005**, *20*, 843–865. [CrossRef]
62. Xian, S.D.; Jing, N.; Xue, W.; Chai, J. A New Intuitionistic Fuzzy Linguistic Hybrid Aggregation Operator and Its Application for Linguistic Group Decision Making. *Int. J. Intell. Syst.* **2017**, 1332–1352. [CrossRef]