On constructing the largest and smallest uninorms on bounded lattices

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Abstract

Uninorms on the unit interval are a common extension of triangular norms (t-norms) and triangular conorms (t-conorms). As important aggregation operators, uninorms play a very important role in fuzzy logic and expert systems. Recently, several researchers have studied constructions of uninorms on more general bounded lattices. In particular, Çaylı (2019) gave two methods for constructing uninorms on a bounded lattice L with $e \in L \setminus \{0, 1\}$, which is based on a t-norm T_e on $[0, e]$ and a t-conorms S_e on $[e, 1]$ that satisfy strict boundary conditions. In this paper, we propose two new methods for constructing uninorms on bounded lattices. Our constructed uninorms are indeed the largest and the smallest among all uninorms on L that have the same restrictions T_e and S_e on $[0, e]$ and, respectively, $[e, 1]$. Moreover, our constructions does not require the boundary condition, and thus completely solved an open problem raised by Çaylı.

Keywords: Bounded lattices; Aggregation operators; Uninorms; Neutral elements.

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1. Introduction

Uninorms on the unit interval $[0, 1]$, introduced by Yager and Rybalov [15], are an extension of triangular norms (t-norms) and triangular conorms (t-conorms) [13]. It has been widely recognized that uninorms are important aggregation operators in fuzzy logic, expert systems, neural networks and so on.

Noticing that bounded lattices are more general than the unit interval $[0, 1]$, several researchers [4–8, 11] have studied constructions of uninorms on bounded lattices. Very

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recently, Çaylı [5] gave two novel methods for constructing uninorms on bounded lattices. The following is the methods given by Çaylı.

Suppose $(L, \leq, 0, 1)$ is a bounded lattice with $e \in L \setminus \{0, 1\}$. Denote $x \parallel y$ if x and y are incomparable and use I_e for the set of elements that are incomparable with e.

Theorem 1.1 ([5])**.** *Suppose $(L, \leq, 0, 1)$ is a bounded lattice and $e \in L \setminus \{0, 1\}$. Given t-norm T_e on $[0, e]$ and t-conorm S_e on $[e, 1]$ such that $T_e(x, y) > 0$ for all $x, y \in (0, e]$ and $S_e(x, y) < 1$ for all $x, y \in [e, 1)$.*
(i) *If $x \parallel y$ for all $x \in I_e$ and $y \in [e, 1)$, then the function $U_1^e : L^2 \to L$ is a uninorm on L with the neutral element e, where*

$$U_1^e(x,y) = \begin{cases} T_e(x,y) & (x,y) \in [0,e]^2, \\ S_e(x,y) & (x,y) \in [e,1]^2, \\ x & (x,y) \in I_e \times [e,1) \cup I_e \times (0,e), \\ y & (x,y) \in [e,1) \times I_e \cup (0,e) \times I_e, \\ x \vee y & (x,y) \in I_e^2 \cup I_e \times \{1\} \cup \{1\} \times I_e \cup (0,e) \times \{1\} \cup \{1\} \times (0,e), \\ x \wedge y & \textit{otherwise}. \end{cases}$$

(ii) *If $x \parallel y$ for all $x \in I_e$ and $y \in (0, e]$, then U_2^e is a uninorm on L with neutral element e. The function $U_2^e : L^2 \to L$ is a uninorm on L with the neutral element e, where*

$$U_2^e(x,y) = \begin{cases} T_e(x,y) & (x,y) \in [0,e]^2, \\ S_e(x,y) & (x,y) \in [e,1]^2, \\ x & (x,y) \in I_e \times (e,1) \cup I_e \times (0,e], \\ y & (x,y) \in (e,1) \times I_e \cup (0,e] \times I_e, \\ x \wedge y & (x,y) \in I_e^2 \cup I_e \times \{0\} \cup \{0\} \times I_e \cup (e,1) \times \{0\} \cup \{1\} \times (e,1), \\ x \vee y & \textit{otherwise}. \end{cases}$$

In the above constructions of U_1^e and U_2^e, the underlying t-norm T_e and the t-conorm S_e are required to satisfy the strict boundary condition: $T_e(x, y) > 0$ for all $x, y > 0$ and $S_e(x, y) < 1$ for all $x, y < 1$. At the end of [5], Çaylı proposed an open problem: *when the assumptions on t-norm and t-conorm are removed, how is the structure of uninorms (especially idempotent uninorms) with the underlying t-norms and t-conorms on bounded lattices.*

In this work, based on the same incomparable condition, i.e., that x, y are incomparable for all $x \in I_e$ and all $y \in [e, 1)$ or that x, y are incomparable for all $x \in I_e$ and all $y \in (0, e]$, we address the problem above by giving two new methods for constructing uninorms on bounded lattices, which do not require that T_e and S_e satisfy the condition specified in Theorem 1.1. Moreover, given the same t-norm T_e and t-conorm S_e, we prove that the two uninorms defined in this work are, respectively, the largest and the smallest uninorms among all uninorms that have the same restrictions to $[0, e]$ and $[e, 1]$ (viz. T_e and S_e).

The remainder of this paper is organized as follows. Section 2 recalls basic concepts and results used in this paper and Section 3 describes our constructions. A brief conclusion is then given in Section 4.

2. Preliminaries

In this section, we recall some concepts and facts which will be used in the text.

Definition 2.1. [3] A lattice $(L, \leq)$ is called *bounded* if it has the top and bottom elements (written as 1 and 0, respectively), that is, $0 \leq x \leq 1$ for any $x \in L$.

Definition 2.2. [3] Let $(L, \leq, 0, 1)$ be a bounded lattice and $a, b \in L$ with $e \in L \setminus \{0, 1\}$.

(i) For $a, b \in L$ with $a \leq b$, $[a, b]$ is defined as $[a, b] = \{x | a \leq x \leq b\}$. Similarly, we can define $(a, b]$, $[a, b)$ and (a, b).

(ii) We write $A(e) = [0, e] \times [e, 1] \cup [e, 1] \times [0, e]$ and $I_e = \{x \in L \mid x \parallel e\}$.

Definition 2.3. [1, 9, 12, 14] Let $(L, \leq, 0, 1)$ be a bounded lattice.

(i) A function $T : L^2 \to L$ is called a *triangular norm* (t-norm for short) if it is commutative, associative, increasing with respect to both variables and has the neutral element 1 such that $T(1, x) = x$ for any $x \in L$.

(ii) A function $S : L^2 \to L$ is called a *triangular conorm* (t-conorm for short) if it is commutative, associative, increasing with respect to both variables and has the neutral element 0 such that $S(0, x) = x$ for any $x \in L$.

Two special t-norms and two special t-conorms are given below.

Example 2.1. Let $(L, \leq, 0, 1)$ be a bounded lattice. The smallest t-norm T_W (or t-conorm $S_\vee$) and the greatest t-norm $T_\wedge$ (or t-conorm S_W) are given as, respectively,

$$T_\wedge(x,y)=x\wedge y,\quad T_W(x,y)=\begin{cases} x & y=1,\\ y & x=1,\\ 0 & \text{otherwise},\end{cases}$$

$$S_\vee(x,y)=x\vee y,\quad S_W(x,y)=\begin{cases} x & y=0,\\ y & x=0,\\ 1 & \text{otherwise}.\end{cases}$$

Definition 2.4. [2, 10, 11] Let $(L,\leq,0,1)$ be a bounded lattice. A function $U:L^2\to L$ is called a *uninorm* on L if it is commutative, associative, increasing with respect to both variables and there exists neutral element $e\in L$ such that $U(e,x)=x$ for all $x\in L$.

Apparently, t-norms and t-conorms on L are special uninorms on L.

Definition 2.5. [5, 6] Let $(L,\leq,0,1)$ be a bounded lattice and U be a uninorm on L with neutral element $e\in L\setminus\{0,1\}$.

(i) An element $x\in L$ is called an idempotent element of U if $U(x,x)=x$.

(ii) U is called an idempotent uninorm if $U(x,x)=x$ for all $x\in L$.

Proposition 2.1. [5, 11] Let $(L,\leq,0,1)$ be a bounded lattice and U be a uninorm on L with neutral element $e\in L\setminus\{0,1\}$. Suppose $T_e:[0,e]^2\to[0,e]$ is the restriction of U on $[0,e]$ and $S_e:[e,1]^2\to[e,1]$ the restriction of U on $[e,1]$. Then T_e is a t-norm on $[0,e]$ and S_e is a t-conorm on $[e,1]$.

We call T_e (S_e) the underlying t-norm (t-conorm) of U.

3. Uninorms on bounded lattice

In this section, we recall some concepts and facts which will be used in this paper.

Theorem 3.1. *Let $(L,\leq,0,1)$ be a bounded lattice with $e\in L\setminus\{0,1\}$. Given t-norm T_e on $[0,e]$ and t-conorm S_e on $[e,1]$, if $x\parallel y$ for all $x\in I_e$ and $y\in[e,1)$, then the function $U_{1,e}:L^2\to L$ is a uninorm on L with the neutral element e, where*

$$U_{1,e}(x,y)=\begin{cases} T_e(x,y) & (x,y)\in[0,e]^2,\\ S_e(x,y) & (x,y)\in[e,1]^2,\\ y & (x,y)\in[0,e]\times I_e,\\ x & (x,y)\in I_e\times[0,e],\\ 1 & (x,y)\in(e,1]\times I_e\cup I_e\times(e,1]\cup I_e^2,\\ x\vee y & \textit{otherwise}.\end{cases}$$

Proof. See Appendix A. □

We next give a simple example.

Example 3.1. Let $L_1 = \{0, a, b, c, d, e, f, 1\}$ be the bounded lattice depicted by the Hasse diagram in Figure 1. Obviously, the condition in Theorem 3.1 is satisfied. Take $T_e = T_\wedge$ on $[0, e]$ and $S_e = S_W$ on $[e, 1]$. Then the function $U_{1,e}$ on L_1, shown in Table 1, is a uninorm on L_1 with the neutral element e.

It is worth pointing out that if the condition that $x \parallel y$ for all $x \in I_e$ and $y \in [e, 1)$ in Theorem 3.1 is not satisfied, then $U_{1,e}$ may not be a uninorm on L. See the following counterexample.

Example 3.2. Let $L_2 = \{0, a, b, c, d, e, 1\}$ be the bounded lattice depicted by the Hasse diagram in Figure 2, where $b \in I_e$, $c > e$, $d > e$, and b is comparable with c and d. Let $S_e = S_\vee$ on $[e, 1]$ and $T_e = T_W$ on $[0, e]$. Then the function $U_{1,e}$ on L_2, shown in Table 2, is not a uninorm on L_2. In fact, the monotonicity is not satisfied, because we have $b < c$ on one hand and $U_{1,e}(b, d) = 1 > d = S_\vee(c, d) = U_{1,e}(c, d)$ on the other hand.

Interestingly, the uninorm $U_{1,e}$ constructed in Theorem 3.1 is indeed the largest one among all uninorms with the same restrictions on $[0, e]$ and $[e, 1]$.

Proposition 3.1. Let $(L, \leq, 0, 1)$ be any bounded lattice with $e \in L \setminus \{0, 1\}$ and $x \parallel y$ for all $x \in I_e$ and $y \in [e, 1)$. Suppose $U_{1,e}$ is the uninorm on L defined as in Theorem 3.1 with underlying t-norm T_e and t-conorm S_e, and U is any uninorm on L such that the restrictions of U to $[0, e]$ and $[e, 1]$ are the t-norm T_e and the t-conorm S_e on $[e, 1]$, respectively. Then $U_{1,e} \geq U$.

Proof. Since $U_{1,e}$ and U have the same t-norm T_e on $[0, e]^2$ and t-conorm S_e on $[e, 1]^2$, we need only to consider the cases such as on $[0, e] \times [e, 1]$, $[0, e] \times I_e$ and so on.

If $(x, y) \in [0, e] \times [e, 1] \cup [e, 1] \times [0, e]$, then $x \wedge y \leq U(x, y) \leq x \vee y = U_{1,e}(x, y)$.

If $(x, y) \in [0, e] \times I_e$, then $U(x, y) \leq U(e, y) = y = U_{1,e}(x, y)$.

If $(x, y) \in I_e \times [0, e]$, then $U(x, y) \leq U(x, e) = x = U_{1,e}(x, y)$.

If $(x, y) \in I_e \times I_e \cup I_e \times (e, 1] \cup (e, 1] \times I_e$, then $U_{1,e}(x, y) = 1 \geq U(x, y)$.

Consequently, it always holds that $U_{1,e}(x, y) \geq U(x, y)$. □

As a consequence, we have the following characterization of the largest uninorm on a bounded lattice.

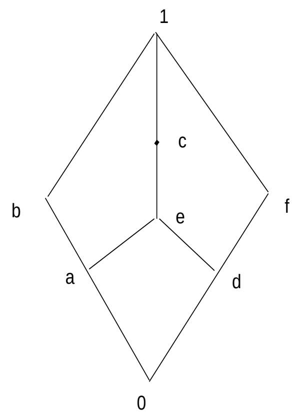

Figure 1: Bounded lattice L_1

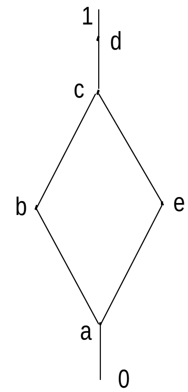

Figure 2: Bounded lattice L_2

$U_{1,e}$	0	a	b	c	d	e	f	1
0	0	0	b	c	0	0	f	1
a	0	a	b	c	0	a	f	1
b	b	b	1	1	b	b	1	1
c	c	c	1	1	c	c	1	1
d	0	0	b	c	d	d	f	1
e	0	a	b	c	d	e	f	1
f	f	f	1	1	f	f	1	1
1	1	1	1	1	1	1	1	1

Table 1: The function $U_{1,e}$ on the bounded lattice L_1 given in Figure 1

$U_{1,e}$	0	a	b	c	d	e	1
0	0	0	b	c	d	0	1
a	0	0	b	c	d	a	1
b	b	b	1	1	1	b	1
c	c	c	1	c	d	c	1
d	d	d	1	d	d	d	1
e	0	a	b	c	d	e	1
1	1	1	1	1	1	1	1

Table 2: The function $U_{1,e}$ on the bounded lattice L_2 given in Figure 2

Corollary 3.1. *Let $(L, \leq, 0, 1)$ be a bounded lattice, $e \in L \setminus \{0, 1\}$, $x \parallel y$ for all $x \in I_e$ and $y \in [e, 1)$. If we put $T_e = T_\wedge$ on $[0, e]^2$ and $S_e = S_W$ on $[e, 1]^2$ in Theorem 3.1, then the following U is the largest uninorm on L with the neutral element e, where*

$$U(x,y) = \begin{cases} T_\wedge(x,y) & (x,y) \in [0,e]^2, \\ S_W(x,y) & (x,y) \in [e,1]^2, \\ y & (x,y) \in [0,e] \times I_e, \\ x & (x,y) \in I_e \times [0,e], \\ 1 & (x,y) \in (e,1] \times I_e \cup I_e \times (e,1] \cup I_e^2, \\ x \vee y & \textit{otherwise}. \end{cases}$$

Proof. From Proposition 3.1, we can easily get the result. □

If $x \parallel y$ for all $x \in I_e$ and $y \in (0, e]$, then we have the following similar construction of uninorms on L.

Theorem 3.2. *Let $(L, \leq, 0, 1)$ be a bounded lattice with $e \in L \setminus \{0, 1\}$. If $x \parallel y$ for all $x \in I_e$ and $y \in (0, e]$, then the function $U_{2,e} : L^2 \to L$ is a uninorm on L with the neutral element* *e, where*

$$U_{2,e}(x,y)=\begin{cases} T_e(x,y) & (x,y)\in[0,e]^2,\\ S_e(x,y) & (x,y)\in[e,1]^2,\\ y & (x,y)\in[e,1]\times I_e,\\ x & (x,y)\in I_e\times[e,1],\\ 0 & (x,y)\in[0,e)\times I_e\cup I_e\times[0,e)\cup I_e^2,\\ x\wedge y & \textit{otherwise.}\end{cases}$$

Similarly, if the condition of $x \parallel y$ for all $x \in I_e$ and $y \in (0, e]$ in Theorem 3.2 is violated, then $U_{2,e}$ may not be a uninorm on L. See the counterexample below.

Example 3.3. Let $L_3 = \{0, a, b, c, d, e, 1\}$ be the bounded lattice depicted by the Hasse diagram in Figure 3, where $c \in I_e$, $a, b < e$, and c is comparable with a and b. Let $T_e = T_\wedge$ on $[0, e]$ and $S_e = S_W$ on $[e, 1]$. Consider the function $U_{2,e}$ on L_3 (shown in Table 3). Although $b < c$, we obtain $U_{2,e}(b, a) = b \wedge a = a > 0 = U_{2,e}(c, a)$. Hence the monotonicity does not hold and $U_{2,e}$ is not a uninorm on L_3.

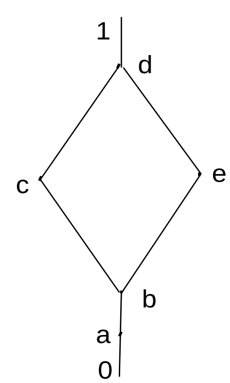

Figure 3: Bounded lattice L_3

Analogously, the uninorm $U_{2,e}$ constructed above is the smallest one among all uninorms with the same restrictions on $[0, e]$ and $[e, 1]$.

Proposition 3.2. Let $(L, \leq, 0, 1)$ be any bounded lattice with $e \in L \setminus \{0, 1\}$ and $x \parallel y$ for all $x \in I_e$ and $y \in (0, e]$. Suppose $U_{2,e}$ is the uninorm on L defined as in Theorem 3.2 and U

$U_{2,e}$	0	a	b	c	d	e	1
0	0	0	0	0	0	0	0
a	0	a	a	0	a	a	a
b	0	a	b	0	b	b	b
c	0	0	0	0	c	c	c
d	0	a	b	c	1	d	1
e	0	a	b	c	d	e	1
1	0	a	b	c	1	1	1

Table 3: The function $U_{2,e}$ on the bounded lattice L_3 given in Figure 3

is any uninorm on L such that the restrictions of U to $[0, e]$ and $[e, 1]$ are the t-norm T_e on $[0, e]$ and the t-conorm S_e on $[e, 1]$, respectively. Then $U_{2,e} \leq U$.

As a consequence, we have the following characterization for the smallest uninorm on L.

Corollary 3.2. *Let $(L, \leq, 0, 1)$ be a bounded lattice, $e \in L \setminus \{0, 1\}$, $x \parallel y$ for all $x \in I_e$ and $y \in (0, e]$. If we put $T_e = T_W$ on $[0, e]^2$ and $S_e = S_\vee$ on $[e, 1]^2$ in Theorem 3.1, then the following U is the smallest uninorm on L with the neutral element e, where*

$$U(x, y) = \begin{cases} T_W(x, y) & (x, y) \in [0, e]^2, \\ S_\vee(x, y) & (x, y) \in [e, 1]^2, \\ y & (x, y) \in [e, 1] \times I_e, \\ x & (x, y) \in I_e \times [e, 1], \\ 0 & (x, y) \in [0, e) \times I_e \cup I_e \times [0, e) \cup I_e^2, \\ x \wedge y & \textit{otherwise.} \end{cases}$$

Remark 3.1. (i) Compared to the construction of U_1^e (U_2^e, resp.) in [5], the construction of $U_{1,e}$ ($U_{2,e}$, resp.) has the same precondition, i.e., x, y are incomparable for all $x \in I_e$ and all $y \in [e, 1)$ (for all $x \in I_e$ and all $y \in (0, e]$, resp.). However, in our constructions of $U_{1,e}$ and $U_{2,e}$, there is no any requirement for the t-norm T_e and the t-conorm S_e. So we completely resolve the open problem raised in [5].

(ii) We can not obtain idempotent uninorms from Theorems 3.1 and 3.2 because, for any $x \in I_e$, we have $U_{1,e}(x,x) = 1$ or $U_{2,e}(x,x) = 0$, but not x.

(iii) If $I_e = \phi$, or specially $L = [0,1]$ in Theorems 3.1 and 3.2, then $U_{1,e} \in \mathscr{U}_{max}$ and $U_{2,e} \in \mathscr{U}_{min}$ [10].

4. Conclusion

Çaylı proposed in [5] two methods for constructing uninorms on bounded lattices, based on the assumption that x, y are incomparable for all $x \in I_e$ and all $y \in (0, e]$ (or all $y \in [e, 1)$). While in his construction the underlying t-norm T_e and t-conorm S_e have to satisfy the strict boundary condition $T_e(x, y) > 0$ for all $x, y \in (0, e]$ and $S_e(x, y) < 1$ for all $x, y \in [e, 1)$, this requirement is completely removed from our construction. Consequently, we completely resolved the open problem raised by Çaylı [5]. Given a bounded lattice L and a t-norm T on $[0, e]$ and a t-conorm S on $[e, 1]$, the uninorm constructed in Theorem 3.1 (Theorem 3.2, resp.) is the largest (smallest, resp.) among all uninorms on L which have restrictions T and S on $[0, e]$ and, respectively, $[e, 1]$.

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Appendix A. Proof of Theorem 3.1

Proof. First, we can easily check that $U_{1,e}$ is commutative and has e as the neutral element. We next check its monotonicity and associativity.

Monotonicity. Suppose $x, y, z \in L$ with $x \leq y$. We prove $U_{1,e}(x, z) \leq U_{1,e}(y, z)$.

1. $x \in [0, e]$.

1.1. $y \in [0, e]$.

1.1.1. $z \in [0, e]$.

$$U_{1,e}(x, z) = T_e(x, z) \leq T_e(y, z) = U_{1,e}(y, z).$$

1.1.2. $z \in (e, 1]$.

$$U_{1,e}(x, z) = x \vee z = z = y \vee z = U_{1,e}(y, z).$$

1.1.3. $z \in I_e$.

$$U_{1,e}(x, z) = z = U_{1,e}(y, z).$$

1.2. $y \in (e, 1]$.

1.2.1. $z \in [0, e]$.

$U_{1,e}(x, z) = T_e(x, z) \leq e < y = y \vee z = U_{1,e}(y, z)$.

1.2.2. $z \in (e, 1]$.

$U_{1,e}(x, z) = z \leq S_e(y, z) = U_{1,e}(y, z)$.

1.2.3. $z \in I_e$.

$U_{1,e}(x, z) = z \leq 1 = U_{1,e}(y, z)$.

1.3. $y \in I_e$.

1.3.1. $z \in [0, e]$.

$U_{1,e}(x, z) = T_e(x, z) \leq x \leq y = U_{1,e}(y, z)$.

1.3.2. $z \in (e, 1]$.

$U_{1,e}(x, z) = x \vee z = z \leq 1 = U_{1,e}(y, z)$.

1.3.3. $z \in I_e$.

$U_{1,e}(x, z) = z \leq 1 = U_{1,e}(y, z)$.

2. $x \in (e, 1]$. Then $y \in (e, 1]$.

2.1. $z \in [0, e]$.

$U_{1,e}(x, z) = x \vee z = x \leq y = y \vee z = U_{1,e}(y, z)$.

2.2. $z \in (e, 1]$.

$U_{1,e}(x, z) = S_e(x, z) \leq S_e(y, z) = U_{1,e}(y, z)$.

2.3. $z \in I_e$.

$U_{1,e}(x, z) = 1 = U_{1,e}(y, z)$.

3. $x \in I_e$. Then $y \in I_e$.

3.1. $z \in [0, e]$.

$U_{1,e}(x, z) = x \leq y = U_{1,e}(y, z)$.

3.2. $z \in (e, 1] \cup I_e$.

$U_{1,e}(x, z) = 1 = U_{1,e}(y, z)$.

Associativity. When $x = e$, or $y = e$, or $z = e$, the equation $U_{1,e}(U_{1,e}(x, y), z) = U_{1,e}(x, U_{1,e}(y, z))$ always holds. So we need only consider the case when $x \neq e$, $y \neq e$ and $z \neq e$.

1. $x \in [0, e)$.

1.1. $y \in [0, e)$.

1.1.1. $z \in [0, e)$.

$U_{1,e}(U_{1,e}(x, y), z) = T_e(T_e(x, y), z) = T_e(x, T_e(y, z)) = U_{1,e}(x, U_{1,e}(y, z))$.

1.1.2. $z \in (e, 1]$.

$$U_{1,e}(U_{1,e}(x,y),z)=U_{1,e}(T_e(x,y),z)=z=x\vee z=U_{1,e}(x,z)=U_{1,e}(x,U_{1,e}(y,z)).$$

1.1.3. $z\in I_e$.

$$U_{1,e}(U_{1,e}(x,y),z)=U_{1,e}(T_e(x,y),z)=z=U_{1,e}(x,z)=U_{1,e}(x,U_{1,e}(y,z)).$$

1.2. $y\in(e,1]$.

1.2.1. $z\in[0,e)$.

$$U_{1,e}(U_{1,e}(x,y),z)=U_{1,e}(x\vee y,z)=U_{1,e}(y,z)=y=U_{1,e}(x,y)=U_{1,e}(x,U_{1,e}(y,z)).$$

1.2.2. $z\in(e,1]$.

$$U_{1,e}(U_{1,e}(x,y),z)=U_{1,e}(y,z)=S_e(y,z)=U_{1,e}(x,S_e(y,z))=U_{1,e}(x,U_{1,e}(y,z)).$$

1.2.3. $z\in I_e$.

$$U_{1,e}(U_{1,e}(x,y),z)=U_{1,e}(y,z)=1=x\vee 1=U_{1,e}(x,1)=U_{1,e}(x,U_{1,e}(y,z)).$$

1.3. $y\in I_e$.

1.3.1. $z\in[0,e)$.

$$U_{1,e}(U_{1,e}(x,y),z)=U_{1,e}(y,z)=y=U_{1,e}(x,y)=U_{1,e}(x,U_{1,e}(y,z)).$$

1.3.2. $z\in(e,1]$.

$$U_{1,e}(U_{1,e}(x,y),z)=U_{1,e}(y,z)=1=x\vee 1=U_{1,e}(x,1)=U_{1,e}(x,U_{1,e}(y,z)).$$

1.3.3. $z\in I_e$.

$$U_{1,e}(U_{1,e}(x,y),z)=U_{1,e}(y,z)=1=x\vee 1=U_{1,e}(x,1)=U_{1,e}(x,U_{1,e}(y,z)).$$

2. $x\in(e,1]$.

2.1. $y\in[0,e)$.

2.1.1. $z\in[0,e)$.

$$U_{1,e}(U_{1,e}(x,y),z)=U_{1,e}(x,z)=x\vee z=x=x\vee T_e(y,z)=U_{1,e}(x,U_{1,e}(y,z)).$$

2.1.2. $z\in(e,1]$.

$$U_{1,e}(U_{1,e}(x,y),z)=U_{1,e}(x,z)=U_{1,e}(x,U_{1,e}(y,z)).$$

2.1.3. $z\in I_e$.

$$U_{1,e}(U_{1,e}(x,y),z)=U_{1,e}(x,z)=U_{1,e}(x,U_{1,e}(y,z)).$$

2.2. $y\in(e,1]$.

2.2.1. $z\in[0,e)$.

$$U_{1,e}(U_{1,e}(x,y),z)=S_e(x,y)\vee z=S_e(x,y)=U_{1,e}(x,y)=U_{1,e}(x,U_{1,e}(y,z)).$$

2.2.2. $z\in(e,1]$.

$$U_{1,e}(U_{1,e}(x,y),z)=S_e(S_e(x,y),z)=S_e(x,S_e(y,z))=U_{1,e}(x,U_{1,e}(y,z)).$$

2.2.3. $z\in I_e$.

$$U_{1,e}(U_{1,e}(x,y),z)=U_{1,e}(S_e(x,y),z)=1=S_e(x,1)=U_{1,e}(x,1)=U_{1,e}(x,U_{1,e}(y,z)).$$

2.3. $y\in I_e$.

2.3.1. $z\in[0,e)$.

$$U_{1,e}(U_{1,e}(x,y),z) = U_{1,e}(1,z) = 1 \vee z = 1 = U_{1,e}(x,y) = U_{1,e}(x,U_{1,e}(y,z)).$$

2.3.2. $z \in (e,1]$.

$$U_{1,e}(U_{1,e}(x,y),z) = U_{1,e}(1,z) = S_e(x,1) = 1 = U_{1,e}(x,1) = U_{1,e}(x,U_{1,e}(y,z)).$$

2.3.3. $z \in I_e$.

$$U_{1,e}(U_{1,e}(x,y),z) = U_{1,e}(1,z) = 1 = U_{1,e}(x,1) = U_{1,e}(x,U_{1,e}(y,z)).$$

3. $x \in I_e$.

3.1. $y \in [0,e)$.

3.1.1. $z \in [0,e)$.

$$U_{1,e}(U_{1,e}(x,y),z) = U_{1,e}(x,z) = x = U_{1,e}(x,T_e(y,z)) = U_{1,e}(x,U_{1,e}(y,z)).$$

3.1.2. $z \in (e,1]$.

$$U_{1,e}(U_{1,e}(x,y),z) = U_{1,e}(x,z) = U_{1,e}(x,y \vee z) = U_{1,e}(x,U_{1,e}(y,z)).$$

3.1.3. $z \in I_e$.

$$U_{1,e}(U_{1,e}(x,y),z) = U_{1,e}(x,z) = U_{1,e}(x,U_{1,e}(y,z)).$$

3.2. $y \in (e,1]$.

3.2.1. $z \in [0,e)$.

$$U_{1,e}(U_{1,e}(x,y),z) = U_{1,e}(1,z) = 1 \vee z = 1 = U_{1,e}(x,y) = U_{1,e}(x,U_{1,e}(y,z)).$$

3.2.2. $z \in (e,1]$.

$$U_{1,e}(U_{1,e}(x,y),z) = U_{1,e}(1,z) = S_e(1,z) = 1 = U_{1,e}(x,S_e(y,z)) = U_{1,e}(x,U_{1,e}(y,z)).$$

3.2.3. $z \in I_e$.

$$U_{1,e}(U_{1,e}(x,y),z) = U_{1,e}(1,z) = 1 = U_{1,e}(x,1) = U_{1,e}(x,U_{1,e}(y,z)).$$

3.3. $y \in I_e$.

3.3.1. $z \in [0,e)$.

$$U_{1,e}(U_{1,e}(x,y),z) = U_{1,e}(1,z) = 1 \vee z = 1 = U_{1,e}(x,y) = U_{1,e}(x,U_{1,e}(y,z)).$$

3.3.2. $z \in (e,1]$.

$$U_{1,e}(U_{1,e}(x,y),z) = U_1(1,z) = S_e(1,z) = 1 = U_{1,e}(x,1) = U_{1,e}(x,U_{1,e}(y,z)).$$

3.3.3. $z \in I_e$.

$$U_{1,e}(U_{1,e}(x,y),z) = U_{1,e}(1,z) = 1 = U_{1,e}(x,1) = U_{1,e}(x,U_{1,e}(y,z)).$$

Consequently, $U_{1,e}$ is a uninorm on L with the neutral element e. □

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