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Quantum Büchi Automata

Qisheng Wang

Department of Computer Science and Technology, Tsinghua University, China

wqs17@mails.tsinghua.edu.cn

Mingsheng Ying

Centre for Quantum Software and Information, University of Technology Sydney, Australia

State Key Laboratory of Computer Science, Institute of Software, Chinese Academy of Sciences, China

Department of Computer Science and Technology, Tsinghua University, China

Mingsheng.Ying@uts.edu.au

Abstract

This paper defines a notion of quantum Büchi automaton (QBA for short) with two different acceptance conditions for ω-words: non-disturbing and disturbing. Several pumping lemmas are established for QBAs. The relationship between the ω-languages accepted by QBAs and those accepted by classical Büchi automata are clarified with the help of the pumping lemmas. The closure properties of the languages accepted by QBAs are studied in the probable, almost sure and threshold semantics. The decidability of the emptiness problem for the languages accepted by QBAs is proved using the Tarski-Seidenberg elimination.

Keywords: Quantum computing, Büchi automata, ω-languages, pumping lemma, closure properties, decision problem

1. Introduction

As acceptors for infinite words (i.e. ω-words), Büchi automata [1] are widely applied in model-checking, program analysis and verification, reasoning about infinite games and decision problems for various logics. Many variants of Büchi automata have been defined in the literature, with acceptance conditions different from the original one in [1] (e.g. Muller, Rabin and Street conditions [2]). More recently, probabilistic generalisations of Büchi automata and other ω-automata have been systematically studied in [3, 4].

In quantum computing, quantum automata over finite words were introduced almost 20 years ago and have been extensively studied since then; see for example [5, 6, 7, 8]. To the best of our knowledge, however, quantum automata over infinite words were only very briefly considered in [9, 10] where Büchi, Street and Rabin acceptance conditions were defined. The only result obtained in [9] is an example ω-language accepted by a 2-way quantum automaton but not by any 1-way quantum automaton, and in our opinion, the definition of quantum Büchi automata given in [9, 10] is problematic (see Subsection 4.1).

The overall aim of this paper is to properly define the notion of quantum Büchi automata and systematically study their fundamental properties, with the expectation that the results obtained here can serve as the mathematical tools needed in the areas like model-checking quantum systems [11, 12, 13, 14, 15], semantics and verification of quantum programs and quantum cryptographic protocols [16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27] and analysis of quantum games [28, 29, 30, 31].

One of the *major differences* between classical and quantum automata is: in a classical automaton, checking the acceptance condition does not disturb the state of the system. In contrast, checking the acceptance condition in a quantum automaton may require to perform a quantum measurement which can disturb the state of the system. Thus, different decisions about where a quantum measurement be introduced lead to different languages accepted by the same quantum automaton. In the case of quantum automata over finite words, two modes of acceptance have been defined in the literature:

- *Measure-once* (MO): The measurement for checking acceptance condition is performed only at the end of the system's unitary evolution [7].

- *Measure-many* (MM): The measurement for checking acceptance condition is performed after each of the system's unitary transformations [5].

To define a quantum Büchi automaton, we consider an infinite sequence of check points in the evolution of the system over an infinite word, and then two scenarios naturally arise:

- *Non-disturbing acceptance* (ND): For a given state in the accepting space, we perform a measurement at a check point to see whether the system's state coincides with it, and then the system's post-measurement state is discarded (for a more detailed description, see Definition 3.1 and Remark 3.1).

- *Disturbing acceptance* (D): For a given state in the accepting space, we perform a measurement at a check point to see whether the system's state coincides with it, the system evolves from the post-measurement state and we perform the same measurement at the next check point. Thus, the system's state is disturbed by the measurements (see Definition 3.2 and Remark 3.2).

1.1. Contributions of this paper

Under both the non-disturbing and disturbing acceptance conditions, we define the probable, almost sure and threshold semantics for quantum Büchi automata, which generalise the corresponding semantics defined in [3, 4] for probabilistic Büchi automata to the quantum case. Within these three semantics, we study the closure properties of the ω-languages accepted by quantum Büchi automata under boolean operations and the emptiness decision problems for these languages. The main *technical contributions* include:

1. Pumping lemmas for both QBA|NDs (QBAs with non-disturbing acceptance) [Theorems 5.1 and 5.2] and QBA|Ds (QBAs with disturbing acceptance) [Theorems 5.3 and 5.4].
2. The precise threshold of QBA|NDs doesn't matter for the threshold semantics [Theorem 6.2].
3. There is a QBA|ND (resp. QBA|D) under the probable semantics that cannot be simulated by any QBA|ND (resp. QBA|D) under the threshold semantics [Theorem 6.4].
4. There is a language accepted by a QBA|NDs under the threshold semantics which is not ω-regular [Theorem 7.1].
5. There is a language accepted by a QBA|ND (resp. QBA|D) under the almost sure semantics that is not ω-context-free (resp. ω-regular) [Theorem 7.1 and Theorem 7.2].
6. The QBA|NDs under the probable semantics is closed under union but not under intersection and complement [Theorem 8.1].

7. The emptiness problem for the languages accepted by QBA|NDs or QBA|Ds is decidable [Theorem 9.2, Theorem 9.4 and Theorem 9.6].

1.2. Organisation of the paper

In Section 2, we review the notion of quantum finite automata from the previous literature. In Section 3, we define the acceptance conditions and various semantics for quantum Büchi automata. Section 4 establishes several basic properties of the accepting probabilities by QBAs. The pumping lemmas for both QBA|NDs and QBA|Ds are presented in Section 5. Sections 6 and 7 are devoted to examine the relationship between different semantics of QBAs, and the relationship between QBAs and classical Büchi automata. The closure properties of QBAs are given in Section 8. Several decision problems about emptiness of the languages accepted by QBAs are discussed in Section 9. A brief conclusion is drawn and in particular, several open problems are raised in Section 10. For readability, the proofs of all results are put into the Appendices (some of them are quite lengthy and tedious).

2. Quantum Finite Automata

For convenience of the reader, in this preliminary section we recall some basic notions of quantum finite automata from [5, 7].

2.1. Definition of Quantum Automata

Definition 2.1. *A quantum (finite) automaton (QFA) is a 5-tuple*

$$\mathcal{A} = (\mathcal{H}, |s_0\rangle, \Sigma, \{U_\sigma : \sigma \in \Sigma\}, F),$$

where

- $\mathcal{H}$ *is a finite-dimensional Hilbert space;*
- $|s_0\rangle$ *is a pure state in* $\mathcal{H}$*, called the initial state;*
- Σ *is a finite alphabet;*
- *For each* $\sigma \in \Sigma$*,* U_σ *is a unitary operator on* $\mathcal{H}$*;*
- F *is a subspace of* $\mathcal{H}$*, called the space of accepting states.*

For a finite word $w = \sigma_1\sigma_2 \dots \sigma_n \in \Sigma^*$, we write:

$$U_w = U_{\sigma_n} \dots U_{\sigma_2} U_{\sigma_1}.$$

Let P_F and $P_{F^\perp}$ be the projections onto F and $F^\perp$ (the ortho-complement of F), respectively. We further set $U'_{\sigma_i} = U_{\sigma_i} P_{F^\perp}$ and

$$U'_{w_i} = U'_{\sigma_i} \dots U'_{\sigma_2} U'_{\sigma_1}.$$

A major difference between classical or probabilistic automata and quantum automata is that the latter check the acceptance condition by a quantum measurement, which can disturb the state of the system. Therefore, acceptance of w by quantum automaton $\mathcal{A}$ can be defined in the following two different ways:

- *Measure-once* (MO): The probability that w is accepted by $\mathcal{A}$ in the MO scenario is:
$$f_{\mathcal{A}}^{\mathrm{MO}}(w) = \left\| P_F U_w \left| s_0 \right\rangle \right\|^2.$$

- *Measure-many* (MM): The probability that w is accepted by $\mathcal{A}$ in the MM scenario is:
$$f_{\mathcal{A}}^{\mathrm{MM}}(w) = \sum_{i=0}^{n} \left\| P_F U'_{w_i} \left| s_0 \right\rangle \right\|^2.$$

Intuitively, in the MO scenario, automaton $\mathcal{A}$ starts in state $|s_0\rangle$ and executes unitary transformations $U_{\sigma_1}, ..., U_{\sigma_n}$. At the end, $\mathcal{A}$ is in state $U_w |s_0\rangle$. Then we perform a yes/no measurement $M_F = \{M_{\mathrm{yes}}^F, M_{\mathrm{no}}^F\}$ with $M_{\mathrm{yes}}^F = P_F$ and $M_{\mathrm{no}}^F = P_{F^\perp}$ to see whether the state of $\mathcal{A}$ is in accepting space F, and $f_{\mathcal{A}}^{\mathrm{MO}}$ is the probability that the measurement outcome is "yes". In the MM scenario, $\mathcal{A}$ starts in state $|s_0\rangle$ and at each step, say step i, it executes unitary transformation U_{σ_i} and then perform measurement M_F: if the outcome is "yes", it terminates; otherwise it enters step $i+1$. Then $f_{\mathcal{A}}^{\mathrm{MM}}(w)$ is the probability that w is accepted within n steps.

2.2. Semantics of Quantum Automata

Similar to the case of probabilistic automata, we can define probable, almost sure and threshold semantics for quantum automata.

Definition 2.2. *Let $\mathcal{A}$ be a quantum automaton, $X \in \{\mathrm{MO}, \mathrm{MM}\}$ stand for the measure-once or measure-many scenario, and $\lambda \in (0, 1)$. The language accepted by $\mathcal{A}$ in the X scenario is:*

1. *Probable semantics:*

$$\mathcal{L}^{>0}(\mathcal{A}|X) = \left\{ w \in \Sigma^* | f_{\mathcal{A}}^X(w) > 0 \right\}.$$

2. *Almost sure semantics:*

$$\mathcal{L}^{=1}(\mathcal{A}|X) = \left\{ w \in \Sigma^* | f_{\mathcal{A}}^X(w) = 1 \right\}.$$

3. *Threshold semantics:*

$$\mathcal{L}^{>\lambda}(\mathcal{A}|X) = \left\{ w \in \Sigma^* | f_{\mathcal{A}}^X(w) > \lambda \right\};$$

$$\mathcal{L}^{\geq\lambda}(\mathcal{A}|X) = \left\{ w \in \Sigma^* | f_{\mathcal{A}}^X(w) \geq \lambda \right\}.$$

Definition 2.3. *The class of languages accepted by* QFA*s in the X scenario* (QFA$|X$*s for short) is:*

1. *Probable semantics:*

$$\mathbb{L}^{>0}(\mathrm{QFA}|X) = \left\{ \mathcal{L}^{>0}(\mathcal{A}|X) | \mathcal{A} \in \mathrm{QFA} \right\}.$$

2. *Almost sure semantics:*

$$\mathbb{L}^{=1}(\mathrm{QFA}|X) = \left\{ \mathcal{L}^{=1}(\mathcal{A}|X) | \mathcal{A} \in \mathrm{QFA} \right\}.$$

3. *Threshold semantics:*

$$\mathbb{L}^{>\lambda}(\mathrm{QFA}|X) = \left\{ \mathcal{L}^{>\lambda}(\mathcal{A}|X) | \mathcal{A} \in \mathrm{QFA} \right\};$$

$$\mathbb{L}^{\geq\lambda}(\mathrm{QFA}|X) = \left\{ \mathcal{L}^{\geq\lambda}(\mathcal{A}|X) | \mathcal{A} \in \mathrm{QFA} \right\}.$$

2.3. Operations of Quantum Automata

As in the case of classical and probabilistic automata, several operations can be defined for quantum automata.

Definition 2.4. *Suppose that*

$$\mathcal{A} = (\mathcal{H}^{\mathcal{A}}, \left|s_0^{\mathcal{A}}\right\rangle, \Sigma, \{U_\sigma^{\mathcal{A}} : \sigma \in \Sigma\}, F^{\mathcal{A}}),$$
$$\mathcal{B} = (\mathcal{H}^{\mathcal{B}}, \left|s_0^{\mathcal{B}}\right\rangle, \Sigma, \{U_\sigma^{\mathcal{B}} : \sigma \in \Sigma\}, F^{\mathcal{B}})$$

are quantum automata, and a, b *are two complex numbers with* $|a|^2 + |b|^2 = 1$.

1. *The weighted direct sum of* $\mathcal{A}$ *and* $\mathcal{B}$ *is:*

$$a\mathcal{A} \oplus b\mathcal{B} = (\mathcal{H}^{\mathcal{A}} \oplus \mathcal{H}^{\mathcal{B}}, a\left|s_0^{\mathcal{A}}\right\rangle \oplus b\left|s_0^{\mathcal{B}}\right\rangle, \Sigma, \{U_\sigma^{\mathcal{A}} \oplus U_\sigma^{\mathcal{B}} : \sigma \in \Sigma\}, F^{\mathcal{A}} \oplus F^{\mathcal{B}}).$$

2. *The tensor product of* $\mathcal{A}$ *and* $\mathcal{B}$ *is:*

$$\mathcal{A} \otimes \mathcal{B} = (\mathcal{H}^{\mathcal{A}} \otimes \mathcal{H}^{\mathcal{B}}, \left|s_0^{\mathcal{A}}\right\rangle \otimes \left|s_0^{\mathcal{B}}\right\rangle, \Sigma, \{U_\sigma^{\mathcal{A}} \otimes U_\sigma^{\mathcal{B}} : \sigma \in \Sigma\}, F^{\mathcal{A}} \otimes F^{\mathcal{B}}).$$

3. *The ortho-complement of* $\mathcal{A}$ *is:*

$$\mathcal{A}^{\perp} = (\mathcal{H}, |s_0\rangle, \Sigma, \{U_\sigma : \sigma \in \Sigma\}, F^{\perp}).$$

3. Basic Definitions of Quantum Büchi Automata

Now we start to define quantum Büchi automata. Recall that in a classical Büchi automaton, a run r is Büchi accepted if there is a state q in the accepting subset which appears infinitely many times in the run. In the quantum case, each state $|\psi\rangle \in \mathcal{H}$ defines a yes/no measurement:

$$M_\psi = \{M_{\text{yes}}^\psi, M_{\text{no}}^\psi\},$$

where $M_{\text{yes}}^\psi = |\psi\rangle\langle\psi|$ and $M_{\text{no}}^\psi = I - M_{\text{yes}}^\psi$. This measurement is used to check whether the system is in the state $|\psi\rangle$. However, such a measurement can disturb the state of the automaton whenever it is performed.

3.1. Non-disturbing Büchi Acceptance

Let us first consider the non-disturbing scenario. Assume that $\mathcal{A} = (\mathcal{H}, |s_0\rangle, \Sigma, \{U_\sigma : \sigma \in \Sigma\}, F)$ is a quantum automaton and $w = \sigma_1\sigma_2 \cdots \in \Sigma^\omega$ an infinite word. Then the non-disturbing run of $\mathcal{A}$ over w is the infinite sequence of states $r = |s_0\rangle, |s_1\rangle, |s_2\rangle, \ldots$ with $|s_n\rangle = U_{\sigma_n} |s_{n-1}\rangle$ for all $n \geq 1$.

Definition 3.1. *The probability that w is non-disturbingly Büchi accepted by $\mathcal{A}$ is:*

$$f_{\mathcal{A}}^{\mathrm{ND}}(w) = \sup_{|\psi\rangle \in F} \sup_{\{n_i\}} \inf_{i=1}^{\infty} |\langle \psi \mid s_{n_i} \rangle|^2, \tag{1}$$

where $|s_n\rangle$ is the non-disturbing run of $\mathcal{A}$ over w, $\{n_i\}$ ranges over all infinite sequences with $0 \leq n_1 < n_2 < \ldots$, and each n_i is called a checkpoint.

Intuitively, $|\langle \psi \mid s_{n_i} \rangle|^2$ in Eq. (1) can be understood as the similarity degree between states $|\psi\rangle$ and $|s_{n_i}\rangle$.

Remark 3.1. *The physical interpretation of the sequence $|\langle\psi|s_{n_i}\rangle|^2$, $i = 1, 2, ...$ is given by the following experiment. Take a system and perform measurement M_ψ on it after it runs n_1 steps, $|\langle\psi|s_{n_1}\rangle|^2$ is the probability that the measurement outcome is "yes", then discard the system. Take a second, identically prepared system, let it run n_2 steps, perform measurement M_ψ on it, $|\langle\psi|s_{n_2}\rangle|^2$ is the probability that the outcome is "yes", then discard the system. We can continue similarly for an arbitrary number of steps. In fact, this procedure was often adopted by physicists in studying recurrence behaviour of quantum systems (see [32] for example of quantum Markov chains).*

The following simple example can help us to see how an infinite word is non-disturbingly accepted by a quantum automaton.

Example 1. *Consider quantum automaton $\mathcal{A}$, where*

1. $\mathcal{H} = \mathrm{span}\{|0\rangle, |1\rangle\}$,
2. $|s_0\rangle = |0\rangle$,
3. $\Sigma = \{a, b\}$,
4. $F = \mathrm{span}\{|0\rangle\}$, *and*
5. $U_a = R_x(\sqrt{2}\pi)$ *and* $U_b = R_x(-\sqrt{2}\pi)$.

Here, $R_x(\cdot)$ stands for the rotation about the x axe of the Bloch sphere; that is,

$$R_x(\theta) = \cos\left(\frac{\theta}{2}\right) I - \mathrm{i}\sin\left(\frac{\theta}{2}\right) X = \begin{bmatrix} \cos(\theta/2) & -\mathrm{i}\sin(\theta/2) \\ -\mathrm{i}\sin(\theta/2) & \cos(\theta/2) \end{bmatrix}$$

for any real number θ, and

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

The non-disturbing run of $\mathcal{A}$ over $(ab)^\omega$ is $|s_{2k}\rangle = |0\rangle$ and

$$|s_{2k+1}\rangle = \cos\left(\frac{\sqrt{2}\pi}{2}\right)|0\rangle - \mathrm{i}\sin\left(\frac{\sqrt{2}\pi}{2}\right)|1\rangle$$

for all $k \in \mathbb{N}$. Since there is a sequence $n_i = 2i$, $i \in \mathbb{N}$ such that $|\langle 0 \mid s_{n_i}\rangle|^2 = 1$, we have $f_{\mathcal{A}}^{\mathrm{ND}}((ab)^\omega) = 1$.

3.2. Disturbing Büchi Acceptance

Now we turn to consider a different scenario. For every infinite sequence $\{n_i\}$ such that $0 \leq n_1 < n_2 < ...$, if we actually perform measurement M_ψ at each checkpoint n_i, then the disturbing run of $\mathcal{A}$ over an infinite word w under the measurement M_ψ with checkpoints $\{n_i\}$ is the infinite sequence of states $r = |s_0\rangle, |s_1\rangle, |s_2\rangle, \ldots$, where for any $i \in \mathbb{N}$:

- $|s_{n_i+1}\rangle = U_{\sigma_i+1}|\psi\rangle$,
- for all $n_i + 2 \leq n \leq n_{i+1}$, $|s_n\rangle = U_{\sigma_n}|s_{n-1}\rangle$.

Definition 3.2. *The probability that w is disturbingly Büchi accepted by $\mathcal{A}$ is:*

$$f_{\mathcal{A}}^{D}(w) = \sup_{|\psi\rangle \in F} \sup_{\{n_i\}} \inf_{i=1}^{\infty} |\langle \psi \mid s_{n_i}\rangle|^2$$

where $|s_n\rangle$ is the disturbing run of $\mathcal{A}$ over w under M_ψ with $\{n_i\}$, and $\{n_i\}$ ranges over all infinite sequences with $0 \leq n_1 < n_2 < \ldots$.

Remark 3.2. *The disturbing scenario is widely adopted in defining quantum walks with absorbing boundary [33] and in studying the recurrence of quantum Markov chains, see for example [34].*

The following example is a modification of Example 1, from which we can see the difference between non-disturbing and disturbing acceptance.

Example 2. *We consider a quantum automaton $\mathcal{A}$ similar to the one in Example 1. Let $\mathcal{A}$ be a quantum automaton, where*

1. $\mathcal{H} = \mathrm{span}\{|0\rangle, |1\rangle\}$,
2. $|s_0\rangle = |1\rangle$,
3. $\Sigma = \{a, b\}$,
4. $F = \mathrm{span}\{|0\rangle\}$, *and*
5. $U_a = R_x(\sqrt{2}\pi)$ *and* $U_b = R_x(-\sqrt{2}\pi)$.

Put word $w = (ab)^\omega$. Then we only need to consider measurement $M_0 = \left\{M^0_{\mathrm{yes}}, M^0_{\mathrm{no}}\right\}$ with $M^0_{\mathrm{yes}} = |0\rangle\langle 0|$ and $M^0_{\mathrm{no}} = |1\rangle\langle 1|$. One should notice that whichever the checkpoints $\{n_i\}$ is chosen, $|s_{n_1}\rangle$ is either $|1\rangle$ (when n_1 is even) or $R_x(\sqrt{2}\pi)|1\rangle$ (when n_1 is odd), where $|s_n\rangle$ is the disturbing run of $\mathcal{A}$ over w under M_0 with the checkpoints $\{n_i\}$. Once n_1 is fixed, and if it is even, we can choose $n_i = n_1 + 2(i-1)$ for all $i \geq 2$ and then $|s_{n_i}\rangle = |0\rangle$ for all $i \geq 2$. If we choose some odd n_1, and the remaining n_i can be set similarly, then the disturbing run is $|1\rangle, R_x(\sqrt{2}\pi)|1\rangle, |0\rangle, |0\rangle, \ldots,$ which leads to

$$f^{\mathrm{D}}_{\mathcal{A}}(w) = \cos^2 \frac{\pi}{\sqrt{2}}.$$

3.3. Semantics of Quantum Büchi Automata

The probable, almost sure and threshold semantics for quantum finite automata given in Definitions 2.2 and 2.3 can be easily generalised into the quantum case.

Definition 3.3. *Let $\mathcal{A}$ be a quantum automata, $X \in \{\mathrm{ND}, \mathrm{D}\}$ stand for non-disturbing or disturbing acceptance condition, and $\lambda \in (0,1)$. Then the language accepted by $\mathcal{A}$ with the X acceptance is defined as follows:*

1. *Probable semantics:*

$$\mathcal{L}^{>0}(\mathcal{A}|X) = \left\{w \in \Sigma^\omega | f^X_{\mathcal{A}}(w) > 0\right\}.$$

2. *Almost sure semantics:*

$$\mathcal{L}^{=1}(\mathcal{A}|X) = \left\{w \in \Sigma^\omega | f_{\mathcal{A}}^X(w) = 1\right\}.$$

3. *Threshold semantics:*

$$\mathcal{L}^{>\lambda}(\mathcal{A}|X) = \left\{w \in \Sigma^\omega | f_{\mathcal{A}}^X(w) > \lambda\right\};$$

$$\mathcal{L}^{\geq\lambda}(\mathcal{A}|X) = \left\{w \in \Sigma^\omega | f_{\mathcal{A}}^X(w) \geq \lambda\right\}.$$

Definition 3.4. *The class of languages accepted by quantum Büchi automata with the $X \in \{\mathrm{ND}, \mathrm{D}\}$ acceptance is defined as follows:*

1. *Probable semantics:*

$$\mathbb{L}^{>0}(\mathrm{QBA}|X) = \left\{\mathcal{L}^{>0}(\mathcal{A}|X) | \mathcal{A} \in \mathrm{QBA}\right\}.$$

2. *Almost sure semantics:*

$$\mathbb{L}^{=1}(\mathrm{QBA}|X) = \left\{\mathcal{L}^{=1}(\mathcal{A}|X) | \mathcal{A} \in \mathrm{QBA}\right\}.$$

3. *Threshold semantics:*

$$\mathbb{L}^{>\lambda}(\mathrm{QBA}|X) = \left\{\mathcal{L}^{>\lambda}(\mathcal{A}|X) | \mathcal{A} \in \mathrm{QBA}\right\};$$

$$\mathbb{L}^{\geq\lambda}(\mathrm{QBA}|X) = \left\{\mathcal{L}^{\geq\lambda}(\mathcal{A}|X) | \mathcal{A} \in \mathrm{QBA}\right\}.$$

Here, we use $\mathcal{A} \in \mathrm{QBA}$ to indicate that $\mathcal{A}$ is a quantum Büchi automaton.

The above two definitions are quantum generalisations of the corresponding semantics defined in [3, 4] for probabilistic ω-automata.

To conclude this section, let us see a simple example showing how the semantics define above can be actually used to describe certain behaviours of quantum systems.

Example 3. *Consider a quantum system with Hilbert space $\mathcal{H}_4 = \mathrm{span}\{|0\rangle, |1\rangle, |2\rangle, |3\rangle\}$ and initial state $|0\rangle$. It behaves as follows: repeatedly choose one of the two unitary operators:*

$$W_\pm = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 0 & \mp 1 \\ \pm 1 & \mp 1 & \pm 1 & 0 \\ 0 & 1 & 1 & \pm 1 \\ 1 & 0 & -1 & \pm 1 \end{bmatrix}$$

and apply it. Our question is whether the system's state can be arbitrarily close to $|2\rangle$ infinitely often. Formally, we construct quantum automaton

$$\mathcal{A} = \{\mathcal{H}_4, |0\rangle, \{+, -\}, \{W_+, W_-\}, \mathrm{span}\{|2\rangle\}\},$$

and then the question is: whether $\mathcal{L}^{=1}(\mathcal{A}|\mathrm{ND}) = \emptyset$?

4. Properties of Accepting Probabilities by Quantum Büchi Automata

In this section, we examine some basic properties of the accepting probabilities by quantum Büchi automata. These properties will serve as a step stone for studying the languages accepted by quantum Büchi automata.

4.1. An Alternative Definition of QBA|NDs

A Büchi non-disturbing acceptance condition for quantum automata was introduced in [9]. Using the notations introduced in this paper, it can be rephrased as the following:

Definition 4.1. *The probability that w is non-disturbingly Büchi accepted by $\mathcal{A}$ is:*

$$f_{\mathcal{A}}^{\mathrm{IR}}(w) = \sup_{\{n_i\}} \inf_{i=1}^{\infty} \|P_F |s_{n_i}\rangle\|^2,$$

where $|s_n\rangle$ is the non-disturbing run of $\mathcal{A}$ over w, P_F is the projector onto F, $\{n_i\}$ ranges over all infinite sequences with $0 \leq n_1 < n_2 < \ldots$, and each n_i is called a checkpoint.

Intuitively, for $0 \leq p \leq 1$, a state $|\varphi\rangle$ in $\mathcal{H}$ is said to be F_p-accepted if $\|P_F|\varphi\rangle\|^2 \geq p$. An infinite word $w \in \Sigma^\omega$ is Büchi accepted by $\mathcal{A}$ with probability p if there are in infinite sequence $\{n_i\}$ such that $0 \leq n_1 < n_2 < ...$ and $|s_{n_i}\rangle$ is F_p-accepted for every $i \geq 1$. At the first glance, this acceptance looks very different from Definition 3.1, and it is hard to be regarded as a quantum counterpart of Büchi acceptance because it only guarantees that F (as a subspace) is hit infinitely often, but Büchi acceptance requires that some (single) state in F is visited infinitely often. Actually, it is a quantum generalisation of reachability of F (see [35, 36, 37] for the definition of reachability in quantum Markov chains and Markov decision processes). But surprisingly, Definitions 3.1 and 4.1 are equivalent; more precisely, we have:

Proposition 4.1. *For any quantum automaton $\mathcal{A}$ and $w \in \Sigma^\omega$,*

$$f_{\mathcal{A}}^{\mathrm{ND}}(w) = f_{\mathcal{A}}^{\mathrm{IR}}(w).$$

4.2. Accepting Probability by a QBA as a Limit of Accepting Probabilities by QFAs

We recall that for any classical nondeterministic finite automaton $\mathcal{M}$, and for any $w \in \Sigma^\omega$, we have:

$$\chi_{\mathcal{M}}^{\mathrm{NBA}}(w) = \limsup_{n\to\infty} \chi_{\mathcal{M}}^{\mathrm{NFA}}(w_n),$$

where $\chi_{\mathcal{M}}^{\mathrm{NBA}}(\cdot)$, $\chi_{\mathcal{M}}^{\mathrm{NFA}}(\cdot)$ are the characteristic functions of the languages accepted by (nondeterministic) Büchi automata $\mathcal{M}$ and finite automata $\mathcal{M}$, respectively, and w_n stands for the prefix of w of length n for all $n \geq 0$. This conclusion can be generalised into the quantum case.

Proposition 4.2. *Suppose that $\mathcal{A}$ is a quantum automaton. Then for any $w \in \Sigma^\omega$:*

$$f_{\mathcal{A}}^{\mathrm{ND}}(w) = \limsup_{n\to\infty} f_{\mathcal{A}}^{\mathrm{MO}}(w_n).$$

The above proposition establishes a connection between quantum finite automata in the measure-once (MO) scenario and quantum Büchi automata with the non-disturbing acceptance. It will be extensively used to prove closure properties of quantum Büchi automata.

Corollary 4.3. *For any two quantum automata $\mathcal{A}$ and $\mathcal{B}$, we have:*

$$(\forall w \in \Sigma^\omega)\, f_{\mathcal{A}}^{\mathrm{MO}}(w) = f_{\mathcal{B}}^{\mathrm{MO}}(w) \Rightarrow (\forall w \in \Sigma^\omega)\, f_{\mathcal{A}}^{\mathrm{ND}}(w) = f_{\mathcal{B}}^{\mathrm{ND}}(w).$$

This corollary shows that the equivalence of two quantum automata in the measure-once (MO) scenario implies the equivalence of them as quantum Büchi automata with the non-disturbing acceptance. But the following example shows that its inverse is not true.

Example 4. *Let $\mathcal{A}$ be a quantum automaton, where*

1. $\mathcal{H} = \mathrm{span}\{|0\rangle, |1\rangle\}$,
2. $|s_0\rangle = |0\rangle$,
3. $\Sigma = \{a\}$,
4. $F = \mathrm{span}\{|0\rangle\}$, *and*
5. $U_a^{\mathcal{A}} = R_x(\sqrt{2}\pi)$,

and $\mathcal{B}$ is the same as $\mathcal{A}$ except for $U_a^{\mathcal{B}} = R_x(\sqrt{3}\pi)$. It is obvious that $f_{\mathcal{A}}^{\mathrm{MO}}(a^n) \neq f_{\mathcal{B}}^{\mathrm{MO}}(a^n)$ for any $n \geq 1$, but $f_{\mathcal{A}}^{\mathrm{ND}}(a^\omega) = f_{\mathcal{B}}^{\mathrm{ND}}(a^\omega) = 1$.

4.3. Accepting Probabilities by Operations of QBA|NDs

Several operations of quantum automata were reviewed in Subsection 2.3. Now we consider how do they accept infinite words. The following proposition establishes a relationship between the acceptance probabilities by two QBAs and the acceptance probability by their weighted direct sum and by their tensor product as well as a relationship between the acceptance probability by a QBA and that by its ortho-complement.

Proposition 4.4. *Let $\mathcal{A}$ and $\mathcal{B}$ be two quantum automata and a, b two complex numbers with $|a|^2 + |b|^2 = 1$. Then for any $w \in \Sigma^\omega$, we have:*

1. $|a|^2 f^{\mathrm{ND}}_{\mathcal{A}}(w) + |b|^2 f^{\mathrm{ND}}_{\mathcal{B}}(w) \geq f^{\mathrm{ND}}_{a\mathcal{A}\oplus b\mathcal{B}}(w) \geq \max\left\{|a|^2 f^{\mathrm{ND}}_{\mathcal{A}}(w), |b|^2 f^{\mathrm{ND}}_{\mathcal{B}}(w)\right\} \geq f^{\mathrm{ND}}_{a\mathcal{A}\oplus b\mathcal{B}}(w)/2$.
2. $f^{\mathrm{ND}}_{\mathcal{A}\otimes\mathcal{B}}(w) \leq f^{\mathrm{ND}}_{\mathcal{A}}(w) f^{\mathrm{ND}}_{\mathcal{B}}(w)$.
3. $f^{\mathrm{ND}}_{\mathcal{A}}(w) + f^{\mathrm{ND}}_{\mathcal{A}^\perp}(w) \geq 1$, *and the equality holds if and only if* $\lim_{n\to\infty} f^{\mathrm{MO}}_{\mathcal{A}}(w_n)$ *exists.*

Corollary 4.5. 1. *For any two complex number a, b with $|a|^2 + |b|^2 = 1$, and for any $w \in \Sigma^\omega$, we have: $f^{\mathrm{ND}}_{a\mathcal{A}\oplus b\mathcal{A}}(w) = f^{\mathrm{ND}}_{\mathcal{A}}(w)$.*

2. *for any $w \in \Sigma^\omega$ and any positive integer k, we have: $f^{\mathrm{ND}}_{\mathcal{A}^{\otimes k}}(w) = \left(f^{\mathrm{ND}}_{\mathcal{A}}(w)\right)^k$.*

The next proposition shows the existence of QBA's with their acceptance probabilities being that of a given QBA modified by a constant.

Proposition 4.6. 1. *For any $\lambda \in [0, 1]$, there is a quantum automaton $\mathcal{A}$ such that $f^{\mathrm{ND}}_{\mathcal{A}}(w) = \lambda$ for all $w \in \Sigma^\omega$.*

2. *For any quantum automaton $\mathcal{A}$ and $\lambda \in [0, 1]$, there is a quantum automaton $\mathcal{B}$ such that $f^{\mathrm{ND}}_{\mathcal{B}}(w) = \lambda f^{\mathrm{ND}}_{\mathcal{A}}(w)$ for all $w \in \Sigma^\omega$.*
3. *For any quantum automaton $\mathcal{A}$ and $\lambda \in [0, 1]$, there is a quantum automaton $\mathcal{B}$ such that $f^{\mathrm{ND}}_{\mathcal{B}}(w) = \lambda f^{\mathrm{ND}}_{\mathcal{A}}(w) + (1 - \lambda)$ for all $w \in \Sigma^\omega$.*

4.4. Accepting Probabilities by Operations of QBA|Ds

The behaviours of QBA|Ds are much more complicated than that of QBA|NDs. At this moment, we don't know as much about the accepting probabilities by the former as that by the latter. But we are able to prove the following:

Proposition 4.7. 1. *For any $\lambda \in [0, 1]$, there is a quantum automaton $\mathcal{A}$ such that $f^{\mathrm{D}}_{\mathcal{A}}(w) = \lambda$ for all $w \in \Sigma^\omega$.*

2. *Let $\mathcal{A}$ and $\mathcal{B}$ be two quantum automata and a, b two complex numbers with $|a|^2 + |b|^2 = 1$. Then for any $w \in \Sigma^\omega$, we have: $f^{\mathrm{D}}_{a\mathcal{A}\oplus b\mathcal{B}}(w) \geq \max\left\{|a|^2 f^{\mathrm{D}}_{\mathcal{A}}(w), |b|^2 f^{\mathrm{D}}_{\mathcal{B}}(w)\right\}$.*

5. Pumping Lemmas

In this section, we prove several pumping lemmas for quantum Büchi automata with non-disturbing acceptance (QBA|NDs for short) or disturbing acceptance (QBA|Ds for short). They will be used in Section 7 to prove non-inclusion between the ω-languages accepted by classical and quantum automata.

5.1. Pumping Lemmas for QBA|NDs

We first establish a pumping lemma for QBA|NDs in terms of their acceptance probabilities.

Theorem 5.1. *Let $\mathcal{A}$ be a quantum automaton. For any $w \in \Sigma^+$ and any $\varepsilon > 0$, there is a positive integer k such that*

$$\left|f^{\mathrm{ND}}_{\mathcal{A}}(uv) - f^{\mathrm{ND}}_{\mathcal{A}}(uw^k v)\right| \leq \varepsilon \tag{2}$$

for any $u \in \Sigma^*$ and $v \in \Sigma^\omega$. Moreover, if $\mathcal{A}$ is n-dimensional, there is a constant c such that $k \leq (c\varepsilon)^{-n}$.

The above theorem is an ω-generalisation of the pumping lemma for quantum automata over finite words given in [7]. The following pumping lemma is a corollary of the above theorem.

We can establish a pumping lemma for QBA|NDs in terms of their accepted words rather than acceptance probabilities. Occasionally, it is more convenient to use than Theorem 5.1 (see the proof of Theorem 6.2, Theorem 6.4, Theorem 7.1 and Theorem 8.1).

Theorem 5.2. *Let $L \in \mathbb{L}^{>\lambda}(\mathrm{QBA|ND})$ for some $\lambda \in [0, 1)$.*

1. *For any $w \in \Sigma^+$, $u \in \Sigma^*$ and $v \in \Sigma^\omega$, $uv \in L$ implies there are infinitely many positive integers k such that $uw^k v \in L$.*
2. *For any $v \in L$, there are infinitely many prefixes v_n of v such that $v_n w^\omega \in L$ for all $w \in \Sigma^+$.*

5.2. Pumping Lemmas for QBA|Ds

We are also able to prove several pumping lemmas for QBA|Ds.

Theorem 5.3. *Let $\mathcal{A}$ be a quantum automaton. For any $w \in \Sigma^+$ and any $\varepsilon > 0$, there is a positive integer k such that*

$$f^{\mathrm{D}}_{\mathcal{A}}(uw^k v) \geq f^{\mathrm{D}}_{\mathcal{A}}(uv) - \varepsilon \tag{3}$$

for any $u \in \Sigma^*$ and $v \in \Sigma^\omega$. Moreover, if $\mathcal{A}$ is n-dimensional, there is a constant c such that $k \leq (c\varepsilon)^{-n}$.

The above theorem is a counterpart of Theorem 5.1 in the case of disturbing acceptance. It is worthy noting the difference between Theorem 5.1 and Theorem 5.3. The inequality (2) in Theorem 5.1 implies both inequality (3) in Theorem 5.3 and

$$f^{\mathrm{ND}}_{\mathcal{A}}(uw^k v) \leq f^{\mathrm{ND}}_{\mathcal{A}}(uv) + \varepsilon. \tag{4}$$

However, inequality (4) does not hold for disturbing acceptance, and the following example presents a quantum automaton such that there are $\varepsilon > 0$, $w \in \Sigma^+$, $u \in \Sigma^*$ and $v \in \Sigma^\omega$ with $f^{\mathrm{D}}_{\mathcal{A}}(uw^k v) > f^{\mathrm{D}}_{\mathcal{A}}(uv) + \varepsilon$ for all $k \in \mathbb{N}$.

Example 5. *Consider quantum automaton $\mathcal{A}$, where*

1. $\mathcal{H} = \mathrm{span}\{|0\rangle, |1\rangle\}$,
2. $|s_0\rangle = |0\rangle$,
3. $\Sigma = \{a, b, c\}$,
4. $F = \mathrm{span}\{|1\rangle\}$, *and*
5. $U_a = R_x(\sqrt{2}\pi)$, $U_b = R_x(-\sqrt{2}\pi)$ *and* $U_c = I$.

Let $u = \epsilon$, $v = c^\omega$ and $w = ab$. It can be verified that $f^{\mathrm{D}}_{\mathcal{A}}(uv) = f^{\mathrm{D}}_{\mathcal{A}}(c^\omega) = 0$. We consider the word $uw^k v = (ab)^k c^\omega$ for all positive integer k. Let us construct a sequence $\{n_i\}$ as follows: $n_1 = 1$, $n_2 = 2k$ and $n_i = 2k + i - 2$ for all $i \geq 3$. We write $|s_k\rangle$ for the disturbing run of $\mathcal{A}$ over $uw^k v$. Then

$$|\langle 1 \mid s_{n_1}\rangle|^2 = \sin^2\left(\frac{\pi}{\sqrt{2}}\right) = 0.633127\ldots,$$

$$|\langle 1 \mid s_{n_2}\rangle|^2 = \cos^2\left(\frac{\pi}{\sqrt{2}}\right) = 0.366872\ldots,$$

and $|\langle 1 \mid s_{n_i}\rangle|^2 = 1$ for all $i \geq 3$. We always have $f^{\mathrm{D}}_{\mathcal{A}}(uw^k v) > 0.3$.

Theorem 5.4. *Let $L \in \mathbb{L}^{>\lambda}(\mathrm{QBA|D})$ for some $\lambda \in [0, 1)$.*

1. *For any $w \in \Sigma^+$, $u \in \Sigma^*$ and $v \in \Sigma^\omega$, $uv \in L$ implies there are infinitely many positive integers k such that $uw^k v \in L$.*
2. *For any $v \in L$, there are infinitely many prefixes v_n of v such that $v_n w^\omega \in L$ for all $w \in \Sigma^+$.*
3. *For any $n \in \mathbb{N}^+$, each $w \in L$ can be written as $w = xyz$, where $x, y \in \Sigma^*, z \in \Sigma^\omega$, and*
   (a) $|x| \geq n$;
   (b) $|y| \geq 1$;
   (c) $xy^k z \in L$ *for all* $k \in \mathbb{N}$.

Obviously, Clauses 1 and 2 of the above theorem is a counterpart of Theorem 5.2 in the case of disturbing acceptance.

6. Relationship between Different Semantics

In this section, we clarify the relationships between: (1) the languages accepted by QBAs and QFAs; (2) two different acceptance conditions QBA|NDs and QBA|Ds; and (3) the probable, almost sure and threshold semantics of QBA|NDs and QBA|Ds.

6.1. Relationship between QBAs and QFAs

First of all, we noticed that a simple relationship between the accepted languages by QBA|NDs and QFAs can be derived from Proposition 4.2. In this subsection, we establishes a connection between quantum finite automata in the measure-once (MO) scenario and quantum Büchi automata with the disturbing acceptance. Recall that for any class $\mathbb{L}$ of finite languages, its ω-Kleene closure is defined as follows:

$$\omega\text{-}\mathbb{L} = \left\{ \bigcup_{i=1}^{n} U_i V_i^\omega : U_i, V_i \in \mathbb{L}, n \in \mathbb{N} \right\}.$$

Especially, ω-RL and ω-CFL are the classes of ω-regular languages and ω-context-free languages, respectively. If L is an ω-regular language, then

$$L = \bigcup_{i=1}^{k} U_i V_i^\omega$$

for some positive integer k, where U_i and V_i are regular languages. This conclusion can also be generalised into the quantum case. Let $\mathcal{A} = (\mathcal{H}, |s_0\rangle, \Sigma, \{U_\sigma : \sigma \in \Sigma\}, F)$ be a quantum automaton. We define its modification: $\mathcal{A}_{u,v} = (\mathcal{H}, |u\rangle, \Sigma, \{U_\sigma : \sigma \in \Sigma\}, F_v)$, where $F_v = \text{span}\{|v\rangle\}$.

Proposition 6.1. *For any quantum automaton $\mathcal{A}$, $w \in \Sigma^\omega$ and $\lambda \in [0, 1)$, we have: $w \in \mathcal{L}^{>\lambda}(\mathcal{A}|\text{D})$ if and only if $w = u_0u_1u_2\ldots$, where:*

1. $u_0 \in \mathcal{L}^{>\lambda+\varepsilon}(\mathcal{A}_{s_0,\psi}|\text{MO})$;
2. $u_1, u_2, \cdots \in \mathcal{L}^{>\lambda+\varepsilon}(\mathcal{A}_{\psi,\psi}|\text{MO}) \setminus \{\epsilon\}$

for some real number $\varepsilon > 0$ and state $|\psi\rangle \in F$.

6.2. Relationship between Different Semantics of QBA|NDs

In this subsection, we examine the relationship between the probable, almost sure and threshold semantics of QBA|NDs.

Theorem 6.2. 1. *For any $\mu, \lambda \in (0, 1)$, we have:* $\mathbb{L}^{>0}(\text{QBA}|\text{ND}) \subseteq \mathbb{L}^{>\mu}(\text{QBA}|\text{ND}) = \mathbb{L}^{>\lambda}(\text{QBA}|\text{ND})$.
2. *For any $\mu, \lambda \in (0, 1)$, we have:* $\mathbb{L}^{=1}(\text{QBA}|\text{ND}) \subseteq \mathbb{L}^{\geq\mu}(\text{QBA}|\text{ND}) = \mathbb{L}^{\geq\lambda}(\text{QBA}|\text{ND})$.
3. *For any $\lambda \in [0, 1)$, it holds that* $\mathbb{L}^{=1}(\text{QBA}|\text{ND}) \not\subseteq \mathbb{L}^{>\lambda}(\text{QBA}|\text{ND})$.

Parts 1) and 2) of the above theorem indicates that in the threshold semantics, the concrete threshold value $\lambda > 0$ is not essential. Moreover, it is interesting to see from parts 2) and 3) that $\mathbb{L}^{=1}(\text{QBA}|\text{ND})$ is included in the non-strict threshold semantics $\mathbb{L}^{\geq\mu}(\text{QBA}|\text{ND})$ but not in the strict threshold semantics $\mathbb{L}^{>\mu}(\text{QBA}|\text{ND})$. But it is still not aware whether the inclusion $\mathbb{L}^{>0}(\text{QBA}|\text{ND}) \subseteq \mathbb{L}^{>\lambda}(\text{QBA}|\text{ND})$ with $\lambda > 0$ given in part 1) is proper or not.

6.3. Relationship between Different Semantics of QBA|Ds

The following theorem clarifies the relationship between the almost sure and threshold semantics of QBA|Ds.

Theorem 6.3. *For any $\lambda \in [0, 1)$, we have:* $\mathbb{L}^{=1}(\text{QBA}|\text{D}) \not\subseteq \mathbb{L}^{>\lambda}(\text{QBA}|\text{D})$.

The above theorem is the disturbing acceptance counterpart of Theorem 6.2 3). But at this moment we don't know whether the conclusions given Theorem 6.2 1) and 2) hold for disturbing acceptance or not.

6.4. Relationship between QBA|NDs and QBA|Ds

In this subsection, we consider the relationship between various semantics of QBA|NDs and that of QBA|Ds.

Theorem 6.4. 1. *For any* $\lambda \in (0,1)$ *and* $\mu \in [0,1)$, *it holds that* $\mathbb{L}^{>\lambda}(\text{QBA}|\text{ND}) \not\subseteq \mathbb{L}^{>\mu}(\text{QBA}|\text{D})$.

2. *For any* $\lambda \in [0,1)$, *we have:* $\mathbb{L}^{=1}(\text{QBA}|\text{ND}) \not\subseteq \mathbb{L}^{>\lambda}(\text{QBA}|\text{D})$.
3. *For any* $\lambda \in [0,1)$, *we have:* $\mathbb{L}^{=1}(\text{QBA}|\text{D}) \not\subseteq \mathbb{L}^{>\lambda}(\text{QBA}|\text{ND})$.

Part 1) of the above theorem shows that the threshold semantics of quantum Büchi automata with non-disturbing acceptance is not included in that with disturbing acceptance. But it is still an unsolved problem whether the reverse inclusion is true. Furthermore, parts 2) and 3) shows that the almost sure semantics of quantum Büchi automata with non-disturbing (resp. disturbing) acceptance is not included in the threshold semantics with disturbing (resp. non-disturbing) acceptance.

7. Relationship between Quantum Büchi Automata and Classical ω-Automata

The aim of this section is to clarify the relationship between classical ω-automata and quantum Büchi automata with the two different acceptance conditions QBA|NDs and QBA|Ds.

7.1. Relationship between QBA|NDs and Classical ω-Automata

The following theorem shows the relationship between the languages accepted by QBA|NDs and classical ω-regular languages and ω-context free languages.

Theorem 7.1. 1. *For any* $\lambda \in [0,1)$, *we have:* $\omega\text{-RL} \not\subseteq \mathbb{L}^{>\lambda}(\text{QBA}|\text{ND})$.

2. *For any* $\lambda \in [0,1)$, *we have:* $\omega\text{-CFL} \setminus \omega\text{-RL} \not\subseteq \mathbb{L}^{>\lambda}(\text{QBA}|\text{ND})$.
3. *For any* $\lambda \in (0,1)$, *it holds that* $\mathbb{L}^{>\lambda}(\text{QBA}|\text{ND}) \not\subseteq \omega\text{-RL}$.
4. $\mathbb{L}^{=1}(\text{QBA}|\text{ND}) \not\subseteq \omega\text{-RL}$.
5. $\mathbb{L}^{=1}(\text{QBA}|\text{ND}) \not\subseteq \omega\text{-CFL}$.

Parts 1) and 2) of the above theorem indicates that both ω-regular and context-free languages are not included in the threshold semantics of quantum Büchi automata with non-disturbing acceptance. Parts 3) asserts that the threshold semantics of quantum Büchi automata with non-disturbing acceptance is not included in ω-regular languages, and the almost sure semantics is not included in either ω-regular or context-free languages.

7.2. Relationship between QBA|Ds and Classical ω-Automata

The following theorem describes the relationship between the languages accepted by QBA|Ds and classical ω-regular languages and ω-context free languages.

Theorem 7.2. 1. *For any $\lambda \in [0, 1)$, we have:* $\omega\text{-RL} \not\subseteq \mathbb{L}^{>\lambda}(\text{QBA|D})$.

2. *For any $\lambda \in [0, 1)$, we have:* $\omega\text{-CFL} \setminus \omega\text{-RL} \not\subseteq \mathbb{L}^{>\lambda}(\text{QBA|D})$.
3. $\mathbb{L}^{=1}(\text{QBA|D}) \not\subseteq \omega\text{-RL}$.

The above theorem is the disturbing acceptance counterpart of Theorem 7.1 1), 2) and 4), but it is still unknown whether the conclusions given in Theorem 7.1 3) and 5) are valid for disturbing acceptance.

8. Closure Properties

The aim of this section is to investigate the closure properties of the languages accepted by quantum Büchi automata under the Boolean operations.

8.1. Closure Properties of QBA|NDs

In this section, we consider the closure properties of the languages accepted by QBA|NDs under the threshold semantics with respect to Boolean operations: union, intersection and complement.

Theorem 8.1. 1. $\mathbb{L}^{>0}(\text{QBA|ND})$ *is closed under union:*

- *if $L_1, L_2 \in \mathbb{L}^{>0}(\text{QBA|ND})$, then $L_1 \cup L_2 \in \mathbb{L}^{>0}(\text{QBA|ND})$.*

2. *For $\lambda \in (0, 1)$, $\mathbb{L}^{>\lambda}(\text{QBA|ND})$ is closed under union in the limit:*
   - *if $L_1, L_2 \in \mathbb{L}^{>\lambda}(\text{QBA|ND})$, then there is a sequence of ω-languages $\left\{L^{(k)} \in \mathbb{L}^{>\lambda}(\text{QBA|ND}) : k \in \mathbb{N}\right\}$ such that $\lim_{k\to\infty} L^{(k)} = L_1 \cup L_2$.*
3. $\mathbb{L}^{>\lambda}(\text{QBA|ND})$ *is not closed in the limit for $\lambda \in (0, 1)$.*
4. $\mathbb{L}^{>\lambda}(\text{QBA|ND})$ *is not closed under complementation for $\lambda \in [0, 1)$.*
5. $\mathbb{L}^{>\lambda}(\text{QBA|ND})$ *is not closed under intersection for $\lambda \in [0, 1)$.*

For threshold value $\lambda = 0$, Theorem 8.1.1 shows that $\mathbb{L}^{>\lambda}(\text{QBA|ND})$ is closed under union. However, for $\lambda \in (0, 1)$, we only have a weaker conclusion, namely the closure of $\mathbb{L}^{>\lambda}(\text{QBA|ND})$ under union in the limit given in Theorem 8.1.2, and still don't know whether it is closed under union.

8.2. Closure Properties of QBA|Ds

The following theorem shows that the languages accepted by QBA|Ds under the threshold semantics are not closed under intersection and complement.

Theorem 8.2.
1. $\mathbb{L}^{>\lambda}(\text{QBA}|\text{D})$ *is not closed under intersection for* $\lambda \in [0, 1)$.
2. $\mathbb{L}^{>\lambda}(\text{QBA}|\text{D})$ *is not closed under complementation for* $\lambda \in [0, 1)$.

But we still don't know wether $\mathbb{L}^{>\lambda}(\text{QBA}|\text{D})$ is closed under union.

9. Decision Problems

In this section, we consider the decision problems about the emptiness of the languages accepted by QBA|NDs and QBA|Ds.

9.1. Emptiness Problem for Threshold Semantics

We first show that the emptiness of the languages non-disturbingly or disturbingly accepted by quantum Büchi automata is equivalent to the emptiness of the languages accepted by quantum finite automata in the measure-once scenario.

Lemma 9.1. *For any quantum automaton* $\mathcal{A}$ *and* $\lambda \in [0, 1)$*, the following statements are equivalent:*

1. $\mathcal{L}^{>\lambda}(\mathcal{A}|\text{MO}) \neq \emptyset$.
2. $\mathcal{L}^{>\lambda}(\mathcal{A}|\text{ND}) \neq \emptyset$.
3. $\mathcal{L}^{>\lambda}(\mathcal{A}|\text{D}) \neq \emptyset$.

It was proved in [38] that given a QFA $\mathcal{A}$ and a threshold $\lambda \in [0, 1)$, whether there exists a word $w \in \Sigma^*$ such that $f_{\mathcal{A}}^{\text{MO}}(w) > \lambda$ is decidable. This fact together with Lemma 9.1 immediately yields:

Theorem 9.2 (Decidability of the emptiness problem)**.** *For any QBA* $\mathcal{A}$ *and any* $\lambda \in [0, 1)$,

1. *whether* $\mathcal{L}^{>\lambda}(\mathcal{A}|\text{ND}) = \emptyset$ *is decidable.*
2. *whether* $\mathcal{L}^{>\lambda}(\mathcal{A}|\text{D}) = \emptyset$ *is decidable.*

9.2. Emptiness Problem for Non-strict Threshold Semantics

In this subsection, we deal with the emptiness problem under the non-strict threshold semantics. The proof techniques for this case are quite different from that used for the (strict) threshold semantics considered in the last subsection.

Lemma 9.3. *Let $\mathcal{A}$ be a quantum automaton, and $\lambda \in (0,1]$. Then $\mathcal{L}^{\geq\lambda}(\mathcal{A}|\text{ND}) \neq \emptyset$ if and only if there is a $U \in \overline{\mathcal{U}}$ such that $f(U) \geq \lambda$, where $\overline{\mathcal{U}}$ is the closure of the semigroup $\mathcal{U} = \{U_w : w \in \Sigma^*\}$ and $f(U) = \|P_F U\,|s_0\rangle\|^2$.*

Then we can reduce the emptiness problem under the non-strict threshold semantics into a first-order formula, which can be decided by the Tarski-Seidenberg elimination method [39].

Theorem 9.4 (Decidability of the emptiness problem for non-strict thresholds)**.** *For any quantum automaton $\mathcal{A}$, and any $\lambda \in (0,1]$, whether $\mathcal{L}^{\geq\lambda}(\mathcal{A}|\text{ND}) \neq \emptyset$ is decidable.*

The universality problems both for strict and non-strict threshold semantics are left open, e.g. whether $\mathcal{L}^{>\lambda}(\mathcal{A}|\text{ND}) = \Sigma^\omega$.

9.3. Emptiness Problem under Intersection

In this subsection, we consider the emptiness of the intersection of languages accepted by two different quantum Büchi automata under the threshold semantics. The following lemma gives a necessary and sufficient condition for this emptiness in terms of the languages accepted by the corresponding quantum finite automata.

Lemma 9.5. *Let $\mathcal{A}$ and $\mathcal{B}$ be two quantum automata, and $\lambda \in [0,1)$. Then $\mathcal{L}^{>\lambda}(\mathcal{A}|\text{ND}) \cap \mathcal{L}^{>\lambda}(\mathcal{B}|\text{ND}) \neq \emptyset$ if and only if there are two finite words $u, v \in \Sigma^*$ such that $u \in \mathcal{L}^{>\lambda}(\mathcal{A}|\text{MO})$ and $uv \in \mathcal{L}^{>\lambda}(\mathcal{B}|\text{MO})$.*

Using a similar technique in [38], we can reduce the emptiness problem into a first-order formula, which can also be decided by the Tarski-Seidenberg elimination method [39].

Theorem 9.6 (Decidability of the emptiness problem under intersection)**.** *For any two quantum automata $\mathcal{A}$ and $\mathcal{B}$, and any $\lambda \in [0,1)$, whether $\mathcal{L}^{>\lambda}(\mathcal{A}|\text{ND}) \cap \mathcal{L}^{>\lambda}(\mathcal{B}|\text{ND}) \neq \emptyset$ is decidable.*

10. Conclusion

In this paper, we defined the notion of quantum Büchi automata with two different accepting conditions: non-disturbing and disturbing, and studied the ω-languages accepted by quantum Büchi automata under the probable, almost sure and threshold semantics. In particular, we clarified the relationship between the ω-languages accepted by classical Büchi automata and quantum ones, and we proved several closure properties of the languages accepted by quantum Büchi automata and decidability of the emptiness problems for them. But quite a few basic problems are still unsolved, and they were already pointed out in the previous sections wherever appropriate. Certainly, finding solutions to these problems are interesting topics for further research. On the other hand, we are even more interested in discovering applications of quantum Büchi automata in the fields like model-checking quantum systems and analysis and verification of quantum programs. Another interesting topic for future research is to extend the results obtained in this paper to timed quantum automata, which, we believe, will be very useful in not only quantum computing but more general quantum technology, including real-time and embedded quantum systems.

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Appendix A. Proofs of the Results in Section 4

We first present a simple technical lemma for the comparison between the accepting probability $f_{\mathcal{A}}^{\mathrm{ND}}(w)$ and a threshold λ. It will be used in all of the proofs presented in the Appendices.

Lemma Appendix A.1. *Suppose that* $\mathcal{A} = (\mathcal{H}, |s_0\rangle, \Sigma, \{U_\sigma : \sigma \in \Sigma\}, F)$ *is a quantum automaton, and* $\lambda \in [0,1]$ *is a real number. For any* $w \in \Sigma^\omega$, *let* $|s_n\rangle$ *be the non-disturbing run of* $\mathcal{A}$ *over* w. *Then*

1. *The following three statements are equivalent:*
   (a) $f_{\mathcal{A}}^{\mathrm{ND}}(w) > \lambda$.
   (b) $\exists \varepsilon > 0, \exists |\psi\rangle \in F, \exists \{n_i\}, \forall i \in \mathbb{N}^+ : |\langle \psi \mid s_{n_i}\rangle|^2 > \lambda + \varepsilon$.
   (c) $\exists |\psi\rangle \in F, \exists \{n_i\} : \lim_{i\to\infty} |\langle \psi \mid s_{n_i}\rangle|^2 > \lambda$.
2. $f_{\mathcal{A}}^{\mathrm{ND}}(w) < \lambda$ *if and only if:* $\exists \varepsilon > 0, \forall |\psi\rangle \in F, \forall \{n_i\}, \exists i \in \mathbb{N}^+. |\langle \psi \mid s_{n_i}\rangle|^2 < \lambda - \varepsilon$.

Proof. Immediate from the definition of $f_{\mathcal{A}}^{\mathrm{ND}}(w)$. □

Lemma Appendix A.2. *Suppose that* $\mathcal{A} = (\mathcal{H}, |s_0\rangle, \Sigma, \{U_\sigma : \sigma \in \Sigma\}, F)$ *is a quantum automaton, and* $\lambda \in [0,1]$ *is a real number. Then*

1. $f_{\mathcal{A}}^{\mathrm{D}}(w) > \lambda$ *if and only if:* $\exists \varepsilon > 0, \exists |\psi\rangle \in F, \exists \{n_i\}, \forall i \in \mathbb{N}^+. |\langle \psi \mid s_{n_i}\rangle|^2 > \lambda + \varepsilon$. *where* $|s_n\rangle$ *is the disturbing run of* $\mathcal{A}$ *over* w *under* M_ψ *with* $\{n_i\}$.

2. $f_{\mathcal{A}}^{\mathrm{D}}(w) < \lambda$ *if and only if:* $\exists \varepsilon > 0, \forall |\psi\rangle \in F, \forall \{n_i\}, \forall i \in \mathbb{N}^+. |\langle \psi \mid s_{n_i}\rangle|^2 > \lambda + \varepsilon$, *where* $|s_n\rangle$ *is the disturbing run of* $\mathcal{A}$ *over* w *under* M_ψ *with* $\{n_i\}$.

Proof. Immediate from the definition of $f_{\mathcal{A}}^{\mathrm{D}}(w)$. □

Appendix A.1. Proof of Proposition 4.1

By Definition 4.1, we immediately have:

Lemma Appendix A.3. *Suppose that $\mathcal{A} = (\mathcal{H}, |s_0\rangle, \Sigma, \{U_\sigma : \sigma \in \Sigma\}, F)$ is a quantum automaton, and $\lambda \in [0,1]$ is a real number. For any $w \in \Sigma^\omega$, let $|s_n\rangle$ be the non-disturbing run of $\mathcal{A}$ over w. The following statements are equivalent*

1. $f_{\mathcal{A}}^{\mathrm{IR}}(w) > \lambda$.
2. $\exists \varepsilon > 0, \exists \{n_i\}, \forall i \in \mathbb{N}^+ : \| P_F |s_{n_i}\rangle \|^2 > \lambda + \varepsilon$.
3. $\exists \{n_i\}. \lim_{i\to\infty} \| P_F |s_{n_i}\rangle \|^2 > \lambda$.

Now we start to prove Proposition 4.1. It is sufficient to show that $\forall \lambda > 0. f_{\mathcal{A}}^{\mathrm{ND}}(w) > \lambda \Longleftrightarrow f_{\mathcal{A}}^{\mathrm{IR}}(w) > \lambda$,

"$\Longrightarrow$". If $f_{\mathcal{A}}^{\mathrm{ND}}(w) > \lambda$, by Lemma Appendix A.1, we obtain: $\exists \varepsilon > 0, \exists |\psi\rangle \in F, \exists \{n_i\}, \forall i \in \mathbb{N}^+. |\langle \psi \mid s_{n_i}\rangle|^2 > \lambda + \varepsilon$. Let $|v_k\rangle$ be the orthogonal basis of F and $|\psi\rangle = \sum_k a_k |v_k\rangle$ with $\sum_k |a_k|^2 = 1$. Note that

$$\begin{aligned} |\langle \psi \mid s_{n_i}\rangle|^2 &= \left| \sum_k a_k^* \langle v_k \mid s_{n_i}\rangle \right|^2 \leq \sum_k |a_k|^2 \sum_k |\langle v_k \mid s_{n_i}\rangle|^2 \\ &= \sum_k |\langle v_k \mid s_{n_i}\rangle|^2 = \| P_F |s_{n_i}\rangle \|^2 . \end{aligned}$$

Then $\| P_F |s_{n_i}\rangle \|^2 \geq |\langle \psi \mid s_{n_i}\rangle|^2 > \lambda + \varepsilon$. By Lemma Appendix A.3, it holds that $f_{\mathcal{A}}^{\mathrm{IR}}(w) > \lambda$.

"$\Longleftarrow$". If $f_{\mathcal{A}}^{\mathrm{IR}}(w) > \lambda$, by Lemma Appendix A.3, we have: $\exists \varepsilon > 0, \exists \{n_i\}, \forall i \in \mathbb{N}^+. \| P_F |s_{n_i}\rangle \|^2 > \lambda + \varepsilon$. Since the set of states is compact, we assert that $\exists |\hat{s}\rangle \in \mathcal{H}, \exists \{m_i\}. \lim_{i\to\infty} |s_{n_{m_i}}\rangle = |\hat{s}\rangle$. Let $|\psi\rangle = \frac{P_F |\hat{s}\rangle}{\| P_F |\hat{s}\rangle \|} \in F$. Note that

$$\begin{aligned} \lim_{i\to\infty} \left| \langle \psi \mid s_{n_{m_i}}\rangle \right|^2 &= \lim_{i\to\infty} \frac{\left| \langle \hat{s} | P_F | s_{n_{m_i}}\rangle \right|^2}{\| P_F |\hat{s}\rangle \|^2} = \frac{|\langle \hat{s} | P_F |\hat{s}\rangle|^2}{\| P_F |\hat{s}\rangle \|^2} \\ &= \| P_F |\hat{s}\rangle \|^2 = \lim_{i\to\infty} \left\| P_F |s_{n_{m_i}}\rangle \right\|^2 \geq \lambda + \varepsilon > \lambda. \end{aligned}$$

By Lemma Appendix A.1, we obtain $f_{\mathcal{A}}^{\mathrm{ND}}(w) > \lambda$.

Appendix A.2. Proof of Proposition 4.2

By the equivalence of a) and c) in Lemma Appendix A.1 (2), we see that $f_{\mathcal{A}}^{\mathrm{IR}}(w) = \limsup_{n\to\infty} \| P_F |s_n\rangle \|^2 = \limsup_{n\to\infty} f_{\mathcal{A}}^{\mathrm{MO}}(w_n)$. Then using Proposition 4.1, we obtain: $f_{\mathcal{A}}^{\mathrm{ND}}(w) = f_{\mathcal{A}}^{\mathrm{IR}}(w) = \limsup_{n\to\infty} f_{\mathcal{A}}^{\mathrm{MO}}(w_n)$.

Appendix A.3. Proof of Corollary 4.3

If $f_{\mathcal{A}}^{\mathrm{MO}}(w) = f_{\mathcal{B}}^{\mathrm{MO}}(w)$ for any $w \in \Sigma^*$, then Proposition 4.2 yields:

$$f_{\mathcal{A}}^{\mathrm{ND}}(w) = \limsup_{n\to\infty} f_{\mathcal{A}}^{\mathrm{MO}}(w_n) = \limsup_{n\to\infty} f_{\mathcal{B}}^{\mathrm{MO}}(w_n) = f_{\mathcal{B}}^{\mathrm{ND}}(w).$$

Appendix A.4. Proof of Proposition 4.4

1. By Proposition 4.2, we obtain:

$$\begin{aligned}&f_{a\mathcal{A}\oplus b\mathcal{B}}^{\mathrm{ND}}(w) = \limsup_{n\to\infty} f_{a\mathcal{A}\oplus b\mathcal{B}}^{\mathrm{MO}}(w_n) = \limsup_{n\to\infty} \left(|a|^2 f_{\mathcal{A}}^{\mathrm{MO}}(w_n) + |b|^2 f_{\mathcal{B}}^{\mathrm{MO}}(w_n)\right)\\ &\leq \limsup_{n\to\infty} |a|^2 f_{\mathcal{A}}^{\mathrm{MO}}(w_n) + \limsup_{n\to\infty} |b|^2 f_{\mathcal{B}}^{\mathrm{MO}}(w_n) = |a|^2 f_{\mathcal{A}}^{\mathrm{ND}}(w) + |b|^2 f_{\mathcal{B}}^{\mathrm{ND}}(w).\end{aligned}$$

On the other hand, we have:

$$\begin{aligned}&\limsup_{n\to\infty} \left(|a|^2 f_{\mathcal{A}}^{\mathrm{MO}}(w_n) + |b|^2 f_{\mathcal{B}}^{\mathrm{MO}}(w_n)\right)\\ &\geq \max\left\{\limsup_{n\to\infty} |a|^2 f_{\mathcal{A}}^{\mathrm{MO}}(w_n), \limsup_{n\to\infty} |b|^2 f_{\mathcal{B}}^{\mathrm{MO}}(w_n)\right\} = \max\left\{|a|^2 f_{\mathcal{A}}^{\mathrm{ND}}(w), |b|^2 f_{\mathcal{B}}^{\mathrm{ND}}(w)\right\}\\ &\geq \frac{1}{2}\left(|a|^2 f_{\mathcal{A}}^{\mathrm{ND}}(w) + |b|^2 f_{\mathcal{B}}^{\mathrm{ND}}(w)\right) \geq \frac{1}{2} f_{a\mathcal{A}\oplus b\mathcal{B}}^{\mathrm{ND}}(w).\end{aligned}$$

Hence, it follows that

$$|a|^2 f_{\mathcal{A}}^{\mathrm{ND}}(w) + |b|^2 f_{\mathcal{B}}^{\mathrm{ND}}(w) \geq f_{a\mathcal{A}\oplus b\mathcal{B}}^{\mathrm{ND}}(w) \geq \max\{|a|^2 f_{\mathcal{A}}^{\mathrm{ND}}(w), |b|^2 f_{\mathcal{B}}^{\mathrm{ND}}(w)\} \geq \frac{1}{2} f_{a\mathcal{A}\oplus b\mathcal{B}}^{\mathrm{ND}}(w).$$

2. By Proposition 4.2, we have:

$$\begin{aligned}f_{\mathcal{A}\otimes\mathcal{B}}^{\mathrm{ND}}(w) &= \limsup_{n\to\infty} f_{\mathcal{A}\otimes\mathcal{B}}^{\mathrm{MO}}(w_n) = \limsup_{n\to\infty} f_{\mathcal{A}}^{\mathrm{MO}}(w_n) f_{\mathcal{B}}^{\mathrm{MO}}(w_n)\\ &\leq \limsup_{n\to\infty} f_{\mathcal{A}}^{\mathrm{MO}}(w_n) \limsup_{n\to\infty} f_{\mathcal{B}}^{\mathrm{MO}}(w_n) = f_{\mathcal{A}}^{\mathrm{ND}}(w) f_{\mathcal{B}}^{\mathrm{ND}}(w).\end{aligned}$$

3. By Proposition 4.2, we obtain:

$$\begin{aligned}f_{\mathcal{A}}^{\mathrm{ND}}(w) + f_{\mathcal{A}^\perp}^{\mathrm{ND}}(w) &= \limsup_{n\to\infty} f_{\mathcal{A}}^{\mathrm{MO}}(w_n) + \limsup_{n\to\infty} f_{\mathcal{A}^\perp}^{\mathrm{MO}}(w_n)\\ &= \limsup_{n\to\infty} f_{\mathcal{A}}^{\mathrm{MO}}(w_n) + \limsup_{n\to\infty} \left(1 - f_{\mathcal{A}}^{\mathrm{MO}}(w_n)\right)\\ &= 1 + \limsup_{n\to\infty} f_{\mathcal{A}}^{\mathrm{MO}}(w_n) - \liminf_{n\to\infty} f_{\mathcal{A}}^{\mathrm{MO}}(w_n) \geq 1,\end{aligned}$$

and the equality holds if and only if $\limsup_{n\to\infty} f_{\mathcal{A}}^{\mathrm{MO}}(w_n) = \liminf_{n\to\infty} f_{\mathcal{A}}^{\mathrm{MO}}(w_n)$, which implies the existence of $\lim_{n\to\infty} f_{\mathcal{A}}^{\mathrm{MO}}(w_n)$.

Appendix A.5. Proof of Corollary 4.5

1. By Proposition 4.2, we obtain:

$$f^{\mathrm{ND}}_{a\mathcal{A}\oplus b\mathcal{A}}(w) = \limsup_{n\to\infty} f^{\mathrm{MO}}_{a\mathcal{A}\oplus b\mathcal{A}}(w_n) = \limsup_{n\to\infty} f^{\mathrm{MO}}_{\mathcal{A}}(w_n) = f^{\mathrm{ND}}_{\mathcal{A}}(w).$$

2. By Proposition 4.2, we obtain:

$$\begin{aligned} f^{\mathrm{ND}}_{\mathcal{A}^{\otimes k}}(w) &= \limsup_{n\to\infty} f^{\mathrm{MO}}_{\mathcal{A}^{\otimes k}}(w_n) = \limsup_{n\to\infty} \left(f^{\mathrm{MO}}_{\mathcal{A}}(w_n)\right)^k \\ &= \left(\limsup_{n\to\infty} f^{\mathrm{MO}}_{\mathcal{A}}(w_n)\right)^k = \left(f^{\mathrm{ND}}_{\mathcal{A}}(w)\right)^k. \end{aligned}$$

Appendix A.6. Proof of Proposition 4.6

1. For any $\lambda \in [0,1]$, we construct a quantum automaton $\mathcal{C}_\lambda = (\mathcal{H}, |s_0\rangle, \Sigma, \{U_\sigma : \sigma \in \Sigma\}, F)$, where:

- $\mathcal{H} = \mathrm{span}\{|0\rangle, |1\rangle\}$,
- $F = \mathrm{span}\{|0\rangle\}$,
- $|s_0\rangle = \sqrt{\lambda}\,|0\rangle + \sqrt{1-\lambda}\,|1\rangle$, and
- $U_\sigma = I$ for all $\sigma \in \Sigma$.

It can be easily verified that $f^{\mathrm{ND}}_{\mathcal{C}_\lambda}(w) = \lambda$.

2. Let $\mathcal{B} = \mathcal{C}_\lambda \otimes \mathcal{A}$. Then

$$\begin{aligned} f^{\mathrm{ND}}_{\mathcal{B}}(w) &= \limsup_{n\to\infty} f^{\mathrm{MO}}_{\mathcal{C}_\lambda\otimes\mathcal{A}}(w_n) = \limsup_{n\to\infty} f^{\mathrm{MO}}_{\mathcal{C}_\lambda}(w_n) f^{\mathrm{MO}}_{\mathcal{A}}(w_n) \\ &= \limsup_{n\to\infty} \lambda f^{\mathrm{MO}}_{\mathcal{A}}(w_n) = \lambda f^{\mathrm{ND}}_{\mathcal{A}}(w). \end{aligned}$$

3. Let $\mathcal{B} = \sqrt{\lambda}\mathcal{A} \oplus \sqrt{1-\lambda}\mathcal{C}_1$. Then we have:

$$\begin{aligned} f^{\mathrm{ND}}_{\mathcal{B}}(w) &= \limsup_{n\to\infty} f^{\mathrm{MO}}_{\sqrt{\lambda}\mathcal{A}\oplus\sqrt{1-\lambda}\mathcal{C}_1}(w_n) \\ &= \limsup_{n\to\infty} \left(\lambda f^{\mathrm{MO}}_{\mathcal{A}}(w_n) + (1-\lambda)\right) = \lambda f^{\mathrm{ND}}_{\mathcal{A}}(w) + (1-\lambda) \end{aligned}$$

Appendix A.7. Proof of Proposition 4.7

1. For any $\lambda \in [0,1]$, we construct a quantum automaton $\mathcal{A} = (\mathcal{H}, |s_0\rangle, \Sigma, \{U_\sigma : \sigma \in \Sigma\}, F)$, where:

- $\mathcal{H} = \text{span}\{|0\rangle, |1\rangle\}$,
- $F = \text{span}\{|0\rangle\}$,
- $|s_0\rangle = \sqrt{\lambda}\,|0\rangle + \sqrt{1-\lambda}\,|1\rangle$, and
- $U_\sigma = I$ for all $\sigma \in \Sigma$.

Note that by definition, the only possible measurement for checking Büchi acceptance is: $M_0 = \{|0\rangle\langle 0|, |1\rangle\langle 1|\}$. Then for any $w \in \Sigma^\omega$, and any check point sequence $\{n_i\}$, let $|s_n\rangle$ be the disturbing run of $\mathcal{A}$ over w under M_0 with $\{n_i\}$. At each checkpoint n_i, we have $|s_{n_1}\rangle = |s_0\rangle$ and $|s_{n_i}\rangle = |0\rangle$ for all $i \geq 2$. As a result, it holds that $\inf_{i=1}^{\infty} |\langle \psi \mid s_{n_i}\rangle|^2 = |\langle \psi \mid s_{n_1}\rangle|^2 = \lambda$, and then $f^{\mathrm{D}}_{\mathcal{A}}(w) = \lambda$.

2. Let $\mathcal{A} = (\mathcal{H}^{\mathcal{A}}, |s_0^{\mathcal{A}}\rangle, \Sigma, \{U_\sigma^{\mathcal{A}} : \sigma \in \Sigma\}, F^{\mathcal{A}})$ and $\mathcal{B} = (\mathcal{H}^{\mathcal{B}}, |s_0^{\mathcal{B}}\rangle, \Sigma, \{U_\sigma^{\mathcal{B}} : \sigma \in \Sigma\}, F^{\mathcal{B}})$. For any $w \in \Sigma^\omega$, and any $\lambda^{\mathcal{A}} < f^{\mathrm{D}}_{\mathcal{A}}(w)$, by Lemma Appendix A.2, we have: $\exists \varepsilon^{\mathcal{A}} > 0, \exists\, |\psi^{\mathcal{A}}\rangle \in F^{\mathcal{A}}, \exists \{n_i\}, \forall i \in \mathbb{N}^+.\ \left|\langle \psi^{\mathcal{A}} \mid s^{\mathcal{A}}_{n_i}\rangle\right|^2 > \lambda^{\mathcal{A}} + \varepsilon^{\mathcal{A}}$, where $|s^{\mathcal{A}}_n\rangle$ is the disturbing run of $\mathcal{A}$ over w under $M_{\psi^{\mathcal{A}}}$ with $\{n_i\}$. Let $|s_n\rangle$ be the disturbing run of $a\mathcal{A} \oplus b\mathcal{B}$ over w under $M_{\psi^{\mathcal{A}}}$ with $\{n_i\}$. Then when $i = 1$, it holds that $\left|\langle \psi^{\mathcal{A}} \mid s_{n_i}\rangle\right|^2 = \left|a\,\langle \psi^{\mathcal{A}} \mid s^{\mathcal{A}}_{n_i}\rangle\right|^2 > |a|^2 (\lambda^{\mathcal{A}} + \varepsilon^{\mathcal{A}})$, and when $i \geq 2$, $\left|\langle \psi^{\mathcal{A}} \mid s_{n_i}\rangle\right|^2 = \left|\langle \psi^{\mathcal{A}} \mid s^{\mathcal{A}}_{n_i}\rangle\right|^2 > \lambda^{\mathcal{A}} + \varepsilon^{\mathcal{A}} \geq |a|^2 (\lambda^{\mathcal{A}} + \varepsilon^{\mathcal{A}})$. By Lemma Appendix A.2, we obtain: $f^{\mathrm{D}}_{a\mathcal{A}\oplus b\mathcal{B}}(w) > |a|^2 \lambda$. Since λ can arbitrarily tend to $f^{\mathrm{D}}_{\mathcal{A}}(w)$, it holds that $f^{\mathrm{D}}_{a\mathcal{A}\oplus b\mathcal{B}}(w) \geq |a|^2 f^{\mathrm{D}}_{\mathcal{A}}(w)$. In the same way, we can prove that $f^{\mathrm{D}}_{a\mathcal{A}\oplus b\mathcal{B}}(w) \geq |b|^2 f^{\mathrm{D}}_{\mathcal{B}}(w)$. As a result, it follows that $f^{\mathrm{D}}_{a\mathcal{A}\oplus b\mathcal{B}}(w) \geq \max\{|a|^2 f^{\mathrm{D}}_{\mathcal{A}}(w), |b|^2 f^{\mathrm{D}}_{\mathcal{B}}(w)\}$.

Appendix B. Proofs of the Results in Section 5

Appendix B.1. Proof of Theorem 5.1

The proof of Theorem 5.1 is based on the following two lemmas.

Lemma Appendix B.1. *For any n real numbers $\alpha_1, \alpha_2, \ldots, \alpha_n$, and for any $0 < \varepsilon < 1$, there are infinitely many positive integers k such that*

$$\min\{\{k\alpha_j\}, \{-k\alpha_j\}\} < \varepsilon$$

for all $1 \le j \le n$, where $\{x\}$ is the non-negative fractional part of x. Moreover, the smallest k can be bounded by $k \le (c\varepsilon)^{-n}$, where c is a constant.

Proof. Let $M = \lceil 1/\varepsilon \rceil$ and $a_j = j/M$ for $0 \le j \le M$. Then the set $\{[a_{j-1}, a_j) : 1 \le j \le M\}$ forms a partition of $[0, 1)$. Now that there are M parts in the partition, we define a function: $h(x) = j$ if and only if $\{x\} \in [a_{j-1}, a_j)$. We further define:

$$g(k) = (h(k\alpha_1), h(k\alpha_2), \ldots, h(k\alpha_n)) \in \{1, 2, \ldots, M\}^n.$$

By the pigeonhole principle, there are two integers $1 \le p < q \le M^n + 1$ such that $g(p) = g(q)$, i.e. $h(p\alpha_j) = h(q\alpha_j)$ for all $1 \le j \le n$. Then $|\{p\alpha_j\} - \{q\alpha_j\}| < \frac{1}{M}$. Let $k = q - p \le M^n$. Note that

$$|\{p\alpha_j\} - \{q\alpha_j\}| = \min\{\{(p-q)\alpha_j\}, \{(q-p)\alpha_j\}\} = \min\{\{k\alpha_j\}, \{-k\alpha_j\}\}.$$

Then we have: $\min\{\{k\alpha_j\}, \{-k\alpha_j\}\} < \frac{1}{M} \le \varepsilon$. On the other hand, there is a constant c such that $\lceil \frac{1}{\varepsilon} \rceil \le (c\varepsilon)^{-1}$, e.g. $c = 1/2$. Hence, $k \le M^n \le (c\varepsilon)^{-n}$. Moreover, since $g(k)$ can take only a finite number of different values, we can find an infinite sequence $q_1, q_2, \ldots$ such that $g(q_1) = g(q_i)$ for all $i \in \mathbb{N}$. We choose $k_i = g(q_{i+1}) - g(q_1)$ for all $i \in \mathbb{N}^+$. Then we can verify that each k_i satisfies the condition $\min\{\{k\alpha_j\}, \{-k\alpha_j\}\} < \varepsilon$. □

Lemma Appendix B.2. *Let D be an n-dimensional unitary diagonal matrix. Then for any $0 < \delta < 1$, there are infinitely many positive integers k such that*

$$D^k = I + \delta J$$

where J is a diagonal matrix with $\sum_{i=1}^{n} |J_{ii}|^2 < 1$. Moreover, the smallest k can be bounded by $k \le (c\delta)^{-n}$, where c is a constant.

Proof. Assume that $D = \mathrm{diag}(\exp(\mathrm{i}\theta_1), \exp(\mathrm{i}\theta_2), \ldots, \exp(\mathrm{i}\theta_n))$, where $\theta_j = 2\alpha_j\pi$ and α_j is a real number for all $1 \le j \le n$. By Lemma Appendix B.1, there are infinitely many positive integers k such that

$$\min\{\{k\alpha_j\}, \{-k\alpha_j\}\} < \frac{\delta}{2n\pi},$$

where the smallest k can be bounded by $k \le (c\delta)^{-n}$ for some constant c. Let $J = (I - D^k)/\delta$, then

$$J = \frac{1}{\delta}\,\mathrm{diag}(1 - \exp(\mathrm{i}k\theta_1), 1 - \exp(\mathrm{i}k\theta_2), \ldots, 1 - \exp(\mathrm{i}k\theta_n)).$$

Note that for any $1 \le j \le n$,

$$\begin{aligned} |1-\exp(\mathrm{i}k\theta_j)| &= \sqrt{2-2\cos(k\theta_j)} = \sqrt{2-2\cos(2k\alpha_j\pi)} \\ &= \sqrt{2-2\cos(2\{k\alpha_j\}\pi)} \le \sqrt{2\cdot\frac{1}{2}(2\pi\min\{\{k\alpha_j\},\{-k\alpha_j\}\})^2} \\ &= 2\pi\min\{\{k\alpha_j\},\{-k\alpha_j\}\} < 2\pi\cdot\frac{\delta}{2n\pi} = \frac{\delta}{n}. \end{aligned}$$

Thus, $|J_{ii}| < 1/n \le 1$ for all $1 \le i \le n$, and $\sum_{i=1}^{n} |J_{ii}|^2 \le \sum_{i=1}^{n} |J_{ii}| < \sum_{i=1}^{n} \frac{1}{n} = 1$. □

Now we start to prove Theorem 5.1. Let $\mathcal{A} = (\mathcal{H}, |s_0\rangle, \Sigma, \{U_\sigma : \sigma \in \Sigma\}, F)$ with $\mathcal{H}$ being n-dimensional. For any $w \in \Sigma^+$, by the Spectral Decomposition Theorem, there is a unitary matrix V and a diagonal matrix D such that $U_w = V^\dagger D V$. Then by Lemma Appendix B.2, chosen some $\delta > 0$, there is a positive integer $k \le (c\delta)^{-n}$ for some constant c such that $D^k = I + \delta J$, where J is a diagonal matrix with $\sum_i |J_{ii}|^2 < 1$. For any $u \in \Sigma^*$ and $v \in \Sigma^\omega$, we split the proof into two cases.

Case 1. If $\lambda < f_{\mathcal{A}}^{\mathrm{ND}}(uv)$, then by Lemma Appendix A.1, $\exists \varepsilon > 0, \exists |\psi\rangle \in F, \exists \{n_i\}, \forall i \in \mathbb{N}^+.\ |\langle\psi \mid s_{n_i}\rangle|^2 > \lambda + \varepsilon$, where $|s_n\rangle$ is the non-disturbing run of $\mathcal{A}$ over uv. For convenience, we may assume that $n_1 > |u|$ (Note that u is a finite word). Then $|s_{n_i}\rangle = U_{v_{n_i-|u|}} U_u |s_0\rangle$. Now we consider the word uw^kv. Let $m_i = n_i + k|w|$ and $|t_n\rangle$ be the non-disturbing run of $\mathcal{A}$ over uw^kv. Then

$$\begin{aligned} |\langle\psi \mid t_{m_i}\rangle|^2 &= \left|\langle\psi| U_{v_{m_i-|u|-k|w|}} U_w^k U_u |s_0\rangle\right|^2 = \left|\langle\psi| U_{v_{n_i-|u|}} V^\dagger D^k V U_u |s_0\rangle\right|^2 \\ &= \left|\langle\psi| U_{v_{n_i-|u|}} V^\dagger (I+\delta J) V U_u |s_0\rangle\right|^2 \\ &= \left|\langle\psi| U_{v_{n_i-|u|}} U_u |s_0\rangle + \delta \langle\psi| U_{v_{n_i-|u|}} V^\dagger J V U_u |s_0\rangle\right|^2 \\ &= \left|\langle\psi \mid s_{n_i}\rangle + \delta \langle\psi| U_{v_{n_i-|u|}} V^\dagger J V U_u |s_0\rangle\right|^2 \\ &\ge \left(|\langle\psi \mid s_{n_i}\rangle| - \delta\left|\langle\psi| U_{v_{n_i-|u|}} V^\dagger J V U_u |s_0\rangle\right|\right)^2 \\ &\ge |\langle\psi \mid s_{n_i}\rangle|^2 - 2\delta > \lambda + \varepsilon_0 - 2\delta \end{aligned}$$

If we choose $2\delta < \varepsilon$, then $|\langle\psi \mid t_{m_i}\rangle|^2 > \lambda + \varepsilon_0 - \varepsilon$. By Lemma Appendix A.1, it holds that $f_{\mathcal{A}}^{\mathrm{ND}}(uw^kv) > \lambda - \varepsilon$. Since λ can arbitrarily tend to $f_{\mathcal{A}}^{\mathrm{ND}}(uv)$, we obtain:

$$f_{\mathcal{A}}^{\mathrm{ND}}(uw^k v) \geq f_{\mathcal{A}}^{\mathrm{ND}}(uv) - \varepsilon. \tag{B.1}$$

Case 2. If $\lambda > f_{\mathcal{A}}^{\mathrm{ND}}(uv)$, then by Lemma Appendix A.1, there is a $\varepsilon_1 > 0$ such that for any state $|\psi\rangle \in F$, and any checkpoints $\{n_i\}$, there is a $i \in \mathbb{N}^+$ such that $|\langle\psi \mid s_{n_i}\rangle|^2 < \lambda - \varepsilon_1$, where $|s_n\rangle$ is the non-disturbing run of $\mathcal{A}$ over uv. For convenience, we may assume that $n_1 > |u|$ (Note that u is a finite word). Then $|s_{n_i}\rangle = U_{v_{n_i - |u|}} U_u |s_0\rangle$. Now we consider the word $uw^k v$. Let $m_i = n_i + k\,|w|$ and $|t_n\rangle$ be the non-disturbing run of $\mathcal{A}$ over $uw^k v$. Then

$$\begin{aligned} |\langle\psi \mid t_{m_i}\rangle|^2 &= \left|\langle\psi \mid s_{n_i}\rangle + \delta\,\langle\psi|\, U_{v_{n_i-|u|}} V^\dagger J V U_u \,|s_0\rangle\right|^2 \\ &\leq \left(|\langle\psi \mid s_{n_i}\rangle| + \delta\left|\langle\psi|\, U_{v_{n_i-|u|}} V^\dagger J V U_u \,|s_0\rangle\right|\right)^2 \\ &\leq |\langle\psi \mid s_{n_i}\rangle|^2 + 3\delta < \lambda - \varepsilon_1 + 3\delta \end{aligned}$$

If we choose $3\delta < \varepsilon$, then $|\langle\psi \mid t_{m_i}\rangle|^2 < \lambda - \varepsilon_1 + \varepsilon$. By Lemma Appendix A.1, it holds that $f_{\mathcal{A}}^{\mathrm{ND}}(uw^k v) < \lambda + \varepsilon$. Since λ can arbitrarily tend to $f_{\mathcal{A}}^{\mathrm{ND}}(uv)$, we have:

$$f_{\mathcal{A}}^{\mathrm{ND}}(uw^k v) \leq f_{\mathcal{A}}^{\mathrm{ND}}(uv) + \varepsilon. \tag{B.2}$$

Combining Eqs. (B.1) and (B.2), we obtain: $\left|f_{\mathcal{A}}^{\mathrm{ND}}(uv) - f_{\mathcal{A}}^{\mathrm{ND}}(uw^k v)\right| \leq \varepsilon$.

Finally, we consider the relationship between k and δ. In both Case 1 and Case 2, it suffices to take $\delta < \varepsilon/3$. As a result, it holds that $k \leq (c\delta)^{-n} \leq (c\varepsilon/3)^{-n}$, and we complete the proof.

Appendix B.2. Proof of Theorem 5.2

1. By Lemma 5.1, for any $w \in \Sigma^+$ and any $\varepsilon > 0$, there are infinitely many ks such that for any $u \in \Sigma^*$ and $v \in \Sigma^\omega$, $\left|f_{\mathcal{A}}^{\mathrm{ND}}(uv) - f_{\mathcal{A}}^{\mathrm{ND}}(uw^k v)\right| < \varepsilon$. If $f_{\mathcal{A}}^{\mathrm{ND}}(uv) > \lambda$, then there is a $\varepsilon_0 > 0$ such that $f_{\mathcal{A}}^{\mathrm{ND}}(uv) > \lambda + \varepsilon_0$. If we choose $\varepsilon = \varepsilon_0/2$, then $f_{\mathcal{A}}^{\mathrm{ND}}(uw^k v) > f_{\mathcal{A}}^{\mathrm{ND}}(uv) - \varepsilon > \lambda + \varepsilon_0/2 > \lambda$. As a result, we have $uw^k v \in \mathcal{L}^{>\lambda}(\mathcal{A}|\mathrm{ND})$.

2. We need the following simple lemma:

Lemma Appendix B.3. *Suppose $\mathcal{A}$ is a quantum automaton. For any $w \in \Sigma^*$ and $v \in \Sigma^+$, $f_{\mathcal{A}}^{\mathrm{ND}}(uv^\omega) \geq f_{\mathcal{A}}^{\mathrm{MO}}(u)$.*

Proof. It holds that

$$f_{\mathcal{A}}^{\mathrm{ND}}(uv^\omega) = \limsup_{n\to\infty} f_{\mathcal{A}}^{\mathrm{MO}}\left((uv^\omega)_n\right) \geq \limsup_{n\to\infty} f_{\mathcal{A}}^{\mathrm{MO}}(uv^n).$$

Let $U_v = V^\dagger D V$, where V is a unitary matrix and D is a diagonal matrix. By Lemma Appendix B.2, for any $0 < \delta < 1$, there are infinitely many integers k such that $D^k = I + \delta J$, where J is a diagonal matrix with $\sum_i |J_{ii}|^2 < 1$. Note that

$$\begin{aligned} f_{\mathcal{A}}^{\mathrm{MO}}(uv^k) &= \left\| P_F U_v^k \left| s_{|u|} \right\rangle \right\|^2 = \left\| P_F V^\dagger D^k V \left| s_{|u|} \right\rangle \right\|^2 \\ &= \left\| P_F V^\dagger (I + \delta J) V \left| s_{|u|} \right\rangle \right\|^2 = \left\| P_F \left| s_{|u|} \right\rangle + \delta P_F V^\dagger J V \left| s_{|u|} \right\rangle \right\|^2 \\ &\geq \left(\left\| P_F \left| s_{|u|} \right\rangle \right\| - \delta \left\| P_F V^\dagger J V \left| s_{|u|} \right\rangle \right\| \right)^2 \\ &\geq \left\| P_F \left| s_{|u|} \right\rangle \right\|^2 - 2\delta = f_{\mathcal{A}}^{\mathrm{MO}}(u) - 2\delta. \end{aligned}$$

Putting this result into limit supreme, we obtain: $\limsup_{n\to\infty} f_{\mathcal{A}}^{\mathrm{MO}}(uv^n) \geq f_{\mathcal{A}}^{\mathrm{MO}}(u) - 2\delta$. Since δ can be arbitrarily small, let $\delta \to 0$, it holds that $\limsup_{n\to\infty} f_{\mathcal{A}}^{\mathrm{MO}}(uv^n) \geq f_{\mathcal{A}}^{\mathrm{MO}}(u)$, which means $f_{\mathcal{A}}^{\mathrm{ND}}(uv^\omega) \geq f_{\mathcal{A}}^{\mathrm{MO}}(u)$. □

Now we are ready to prove Theorem 5.2.2. If $w \in \mathcal{L}^{>\lambda}(\mathcal{A}|\mathrm{ND})$, i.e. $f_{\mathcal{A}}^{\mathrm{ND}}(w) > \lambda$, then by Lemma Appendix A.3, we have: $\exists \varepsilon > 0, \exists \{n_i\}, \forall i \in \mathbb{N}^+$. $\| P_F |s_{n_i}\rangle \|^2 > \lambda + \varepsilon$. We choose the prefixes $x_i = w_{n_i}$, and thus $f_{\mathcal{A}}^{\mathrm{MO}}(x_i) > \lambda$. Then for any $y \in \Sigma^+$, by Lemma Appendix B.3, we have: $f_{\mathcal{A}}^{\mathrm{ND}}(x_i y^\omega) \geq f_{\mathcal{A}}^{\mathrm{MO}}(x_i) > \lambda$, and thus $x_i y^\omega \in \mathcal{L}^{>\lambda}(\mathcal{A}|\mathrm{ND})$.

Appendix B.3. Proof of Theorem 5.3

Let quantum automaton $\mathcal{A} = (\mathcal{H}, |s_0\rangle, \Sigma, \{U_\sigma : \sigma \in \Sigma\}, F)$ be n-dimensional. For any $w \in \Sigma^+$, by the Spectral Decomposition Theorem, there are a unitary matrix V and a diagonal matrix D such that $U_w = V^\dagger D V$. By Lemma Appendix B.2, chosing some $\delta > 0$, there is a positive integer $k \leq (c\delta)^{-n}$ for some constant c such that $D^k = I + \delta J$, where J is a diagonal matrix with $\sum_i |J_{ii}|^2 < 1$.

For any $u \in \Sigma^*$ and $v \in \Sigma^\omega$, and any $\lambda < f_{\mathcal{A}}^{\mathrm{D}}(uv)$. By Lemma Appendix A.2, there are a $\varepsilon_0 > 0$, a state $|\psi\rangle \in F$ and a sequence $\{n_i\}$ such that for all $i \in \mathbb{N}$, we have $|\langle \psi \mid s_{n_i} \rangle|^2 > \lambda + \varepsilon_0$, where $|s_n\rangle$ is the disturbing run of $\mathcal{A}$ over uv under M_ψ with $\{n_i\}$. Let n_m be the smallest element in $\{n_i\}$ such that $n_m > |u|$. Here we construct another sequence $\{n'_i\}$, which is $n'_i = n_i$ if $i < m$ and $n'_i = n_i + k|w|$ otherwise. Let $|s'_n\rangle$ be the disturbing run of $\mathcal{A}$ over $uw^k v$ under M_ψ with $\{n'_i\}$. For any $i \neq m$, we have $\left| \left\langle \psi \mid s'_{n'_i} \right\rangle \right|^2 = |\langle \psi \mid s_{n_i} \rangle|^2 > \lambda + \varepsilon_0$. Let $|\phi\rangle$ be the initial state starting from

$\sigma_{n_{m-1}}$, i.e. $|\phi\rangle = |s_0\rangle$ if $m = 1$ and $|\phi\rangle = |\psi\rangle$ otherwise. Note that

$$
\begin{aligned}
&\left|\left\langle \psi \mid s'_{n'_m}\right\rangle\right|^2 = \left|\langle\psi| U_{\sigma_{n_m}} \ldots U_{\sigma_{|u|+1}} U_w^k U_{\sigma_{|u|}} \ldots U_{\sigma_{n_{m-1}+1}} |\phi\rangle\right|^2 \\
&= \left|\langle\psi| U_{\sigma_{n_m}} \ldots U_{\sigma_{|u|+1}} V^\dagger D^k V U_{\sigma_{|u|}} \ldots U_{\sigma_{n_{m-1}+1}} |\phi\rangle\right|^2 \\
&= \left| \langle\psi| U_{\sigma_{n_m}} \ldots U_{\sigma_{|u|+1}} V^\dagger \quad (I + \delta J) V U_{\sigma_{|u|}} \ldots U_{\sigma_{n_{m-1}+1}} |\phi\rangle \right|^2 \\
&= \left| \langle\psi| U_{\sigma_{n_m}} \ldots U_{\sigma_{n_{m-1}+1}} |\phi\rangle + \delta \langle\psi| U_{\sigma_{n_m}} \ldots U_{\sigma_{|u|+1}} V^\dagger J V U_{\sigma_{|u|}} \ldots U_{\sigma_{n_{m-1}+1}} |\phi\rangle \right|^2 \\
&= \left| \langle\psi \mid s_{n_m}\rangle + \delta \langle\psi| U_{\sigma_{n_m}} \ldots U_{\sigma_{|u|+1}} V^\dagger J V U_{\sigma_{|u|}} \ldots U_{\sigma_{n_{m-1}+1}} |\phi\rangle \right|^2 \\
&\geq \left(|\langle\psi \mid s_{n_m}\rangle| - \delta \left|\langle\psi| U_{\sigma_{n_m}} \ldots U_{\sigma_{|u|+1}} V^\dagger J V U_{\sigma_{|u|}} \ldots U_{\sigma_{n_{m-1}+1}} |\phi\rangle\right| \right)^2 \\
&\geq |\langle\psi \mid s_{n_m}\rangle|^2 - 2\delta > \lambda + \varepsilon_0 - 2\delta.
\end{aligned}
$$

If we choose $2\delta < \varepsilon$, then $\left|\left\langle \psi \mid s'_{n'_m}\right\rangle\right|^2 > \lambda + \varepsilon_0 - \varepsilon$. By Lemma Appendix A.2, we obtain: $f_{\mathcal{A}}^{\mathrm{D}}(uw^k v) > \lambda - \varepsilon$. Since λ can arbitrarily tend to $f_{\mathcal{A}}^{\mathrm{D}}(uv)$, it holds that $f_{\mathcal{A}}^{\mathrm{D}}(uw^k v) \geq f_{\mathcal{A}}^{\mathrm{D}}(uv) - \varepsilon$. From the above, it is enough to choose $\delta < \varepsilon/2$, and then $k \leq (c\delta)^{-n} \leq (c\varepsilon/2)^{-n}$, and we complete the proof.

Appendix B.4. Proof of Theorem 5.4

1. By Theorem 5.3, for any $w \in \Sigma^+$ and any $\varepsilon > 0$, there are infinitely many ks such that for any $u \in \Sigma^*$ and $v \in \Sigma^\omega$, $f_{\mathcal{A}}^{\mathrm{D}}(uw^k v) \geq f_{\mathcal{A}}^{\mathrm{D}}(uv) - \varepsilon$. If $f_{\mathcal{A}}^{\mathrm{D}}(uv) > \lambda$, then there is a $\varepsilon_0 > 0$ such that $f_{\mathcal{A}}^{\mathrm{D}}(uv) > \lambda + \varepsilon_0$. If we further choose $\varepsilon = \varepsilon_0/2$, then $f_{\mathcal{A}}^{\mathrm{D}}(uw^k v) \geq f_{\mathcal{A}}^{\mathrm{D}}(uv) - \varepsilon > \lambda + \varepsilon_0/2 > \lambda$. As a result, we have $uw^k v \in \mathcal{L}^{>\lambda}(\mathcal{A}|\mathrm{D})$.

2. Let $\mathcal{A} = (\mathcal{H}, |s_0\rangle, \Sigma, \{U_\sigma : \sigma \in \Sigma\}, F)$ and $w \in \mathcal{L}^{>\lambda}(\mathcal{A}|\mathrm{D})$. By Theorem 6.1 (see Appendix Appendix C), w can be written as $w = u_1 u_2 \ldots$, where $u_1 \in \mathcal{L}^{\lambda+\varepsilon}(\mathcal{A}_{s_0,\psi}|\mathrm{MO})$ and $u_i \in \mathcal{L}^{\lambda+\varepsilon}(\mathcal{A}_{\psi,\psi}|\mathrm{MO}) \setminus \{\epsilon\}$ for all $i \geq 2$, for some $\varepsilon > 0$ and some state $|\psi\rangle \in F$.

Now for any $n \geq 1$, let $x = u_1 u_2 \ldots u_n$ be a prefix of w. For any $y \in \Sigma^+$, by the Spectral Decomposition Theorem, it holds that $U_y = V^\dagger D V$ for some unitary matrix V and some diagonal matrix D. By Lemma Appendix B.2, chosen $\delta > 0$, there is a positive integer k such that $D^k = I + \delta J$, where J is a diagonal matrix with $\sum_i |J_{ii}|^2 < 1$. Note that

$$
\begin{aligned}
f_{\mathcal{A}}^{\mathrm{MO}}(y^k) &= \left|\langle\psi| U_y^k |\psi\rangle\right|^2 = \left|\langle\psi| V^\dagger D^k V |\psi\rangle\right|^2 = \left|\langle\psi| V^\dagger (I + \delta J) V |\psi\rangle\right|^2 \\
&= \left|1 + \delta \langle\psi| V^\dagger J V |\psi\rangle\right|^2 \geq \left(1 - \delta \left|\langle\psi| V^\dagger J V |\psi\rangle\right|\right)^2 \geq 1 - 2\delta
\end{aligned}
$$

If we choose $\delta < (1-\lambda-\varepsilon)/2$, then $f_{\mathcal{A}}^{\mathrm{MO}}(y^k) > \lambda+\varepsilon$, i.e. $y^k \in \mathcal{L}^{>\lambda+\varepsilon}(\mathcal{A}|\mathrm{MO}) \setminus \{\epsilon\}$. Therefore,

$$xy^\omega = u_1u_2\dots u_n(y^k)^\omega \in \mathcal{L}^{>\lambda+\varepsilon}(\mathcal{A}_{s_0,\psi}|\mathrm{MO})\left(\mathcal{L}^{>\lambda+\varepsilon}(\mathcal{A}_{\psi,\psi}|\mathrm{MO}) \setminus \{\epsilon\}\right)^\omega.$$

By Proposition 6.1, we obtain $xy^\omega \in \mathcal{L}^{>\lambda}(\mathcal{A}|\mathrm{D})$.

By the way, the above proof can also be used to obtain a lemma similar to Lemma Appendix B.3:

3. We first observe:

Lemma Appendix B.4. *Suppose $\mathcal{A}$ is a quantum automaton. For any $w \in \Sigma^*$ and $v \in \Sigma^+$, we have: $f_{\mathcal{A}}^{\mathrm{D}}(uv^\omega) \geq f_{\mathcal{A}}^{\mathrm{MO}}(u)$.*

Now we can prove Theorem 5.4. Let $\mathcal{A} = (\mathcal{H}, |s_0\rangle, \Sigma, \{U_\sigma : \sigma \in \Sigma\}, F)$ be a quantum automaton, $w \in \Sigma^\omega$ and $\lambda \in [0, 1)$. By Theorem 6.1 (see its proof in Appendix Appendix C), $w \in \mathcal{L}^{>\lambda}(\mathcal{A}|\mathrm{D})$ if and only if there is a $\varepsilon > 0$, a state $|\psi\rangle \in F$, such that $w \in \mathcal{L}^{>\lambda+\varepsilon}(\mathcal{A}_{s_0,\psi}|\mathrm{MO})\left(\mathcal{L}^{>\lambda+\varepsilon}(\mathcal{A}_{\psi,\psi}|\mathrm{MO}) \setminus \{\epsilon\}\right)^\omega$, where $\mathcal{A}_{u,v} = (\mathcal{H}, |u\rangle, \Sigma, \{U_\sigma : \sigma \in \Sigma\}, F_v)$ and $F_v = \mathrm{span}\{|v\rangle\}$. Thus $w = u_1u_2u_3\dots$, where $u_1 \in \mathcal{L}^{>\lambda+\varepsilon}(\mathcal{A}_{s_0,\psi}|\mathrm{MO})$, and $u_2, u_3, \dots \in \left(\mathcal{L}^{>\lambda+\varepsilon}(\mathcal{A}_{\psi,\psi}|\mathrm{MO}) \setminus \{\epsilon\}\right)^\omega$.

Now for any $n \in \mathbb{N}^+$, we choose:

- $x = u_1u_2\dots u_{n+1}$,
- $y = u_{n+2}$ and
- $z = u_{n+3}u_{n+4}\dots$.

It can be verified that the split $w = xyz$ satisfied the requirement that for any $k \in \mathbb{N}$, $xy^kz \in \mathcal{L}^{>\lambda}(\mathcal{A}|\mathrm{D})$.

Appendix C. Proofs of the Results in Section 6

Appendix C.1. Proof of Proposition 6.1

Let $w = \sigma_1\sigma_2\dots \in \Sigma^\omega$. The proof is divided into the following two parts:

"$\Longrightarrow$". Assume that $w \in \mathcal{L}^{>\lambda}(\mathcal{A}|\mathrm{D})$. Then by Lemma Appendix A.2, there are a $\varepsilon > 0$, a state $|\psi\rangle \in F$ and a sequence $\{n_i\}$ such that for all $i \in \mathbb{N}$, we have $|\langle\psi \mid s_{n_i}\rangle|^2 > \lambda + \varepsilon$, where $|s_n\rangle$ is the disturbing run of $\mathcal{A}$ over w under M_ψ with $\{n_i\}$. Here, we construct a sequence of finite words, $u_i = \sigma_{n_{i-1}+1}\dots\sigma_{n_i-1}\sigma_{n_i}$ for all $i \in \mathbb{N}^+$, where $n_0 = 0$ is taken for convenience. Note that $w = u_1u_2u_3\dots$ where $u_i \neq \epsilon$ for all $i \geq 2$ (note that the sequence

$\{n_i\}$ is strictly increasing, i.e. $n_i < n_{i+1}$), $f^{\mathrm{MO}}_{\mathcal{A}_{s_0,\psi}}(u_1) = |\langle\psi| U_{u_1} |s_0\rangle|^2 = |\langle\psi \mid s_{n_1}\rangle|^2 > \lambda + \varepsilon$, and $f^{\mathrm{MO}}_{\mathcal{A}_{\psi,\psi}}(u_i) = |\langle\psi| U_{u_i} |\psi\rangle|^2 = |\langle\psi \mid s_{n_i}\rangle|^2 > \lambda + \varepsilon$ for all $i \geq 2$. Thus $u_1 \in \mathcal{L}^{>\lambda+\varepsilon}(\mathcal{A}_{s_0,\psi}|\mathrm{MO})$ and $u_i \in \mathcal{L}^{>\lambda+\varepsilon}(\mathcal{A}_{\psi,\psi}|\mathrm{MO}) \setminus \{\epsilon\}$ for all $i \geq 2$. This leads to $w \in \mathcal{L}^{>\lambda+\varepsilon}(\mathcal{A}_{s_0,\psi}|\mathrm{MO}) \left(\mathcal{L}^{>\lambda+\varepsilon}(\mathcal{A}_{\psi,\psi}|\mathrm{MO}) \setminus \{\epsilon\}\right)^\omega$.

"$\Longleftarrow$". Now we assume that there are a real number $\varepsilon > 0$ and a state $|\psi\rangle \in F$ such that $w \in \mathcal{L}^{>\lambda+\varepsilon}(\mathcal{A}_{s_0,\psi}|\mathrm{MO}) \left(\mathcal{L}^{>\lambda+\varepsilon}(\mathcal{A}_{\psi,\psi}|\mathrm{MO}) \setminus \{\epsilon\}\right)^\omega$. Then w can be written $w = u_1 u_2 \ldots$ as the concatenation of infinitely many finite words, where $u_1 \in \mathcal{L}^{>\lambda+\varepsilon}(\mathcal{A}_{s_0,\psi}|\mathrm{MO})$ and $u_i \in \mathcal{L}^{>\lambda+\varepsilon}(\mathcal{A}_{\psi,\psi}|\mathrm{MO}) \setminus \{\epsilon\}$ for all $i \geq 2$. Here, we construct a sequence $\{n_i\}$ with $n_1 = |u_1|$ and $n_i = n_{i-1} + |u_i|$ for all $i \geq 2$. Let $|s_n\rangle$ be the disturbing run of $\mathcal{A}$ over w under M_ψ with $\{n_i\}$. Note that $|\langle\psi \mid s_{n_1}\rangle|^2 = |\langle\psi| U_{u_1} |s_0\rangle|^2 = f^{\mathrm{MO}}_{\mathcal{A}_{s_0,\psi}}(u_1) > \lambda + \varepsilon$, and $|\langle\psi \mid s_{n_i}\rangle|^2 = |\langle\psi| U_{u_i} |\psi\rangle|^2 = f^{\mathrm{MO}}_{\mathcal{A}_{\psi,\psi}}(u_i) > \lambda + \varepsilon$ for all $i \geq 2$. Then by Lemma Appendix A.2, we obtain $f^{\mathrm{D}}_{\mathcal{A}}(w) > \lambda$, i.e. $w \in \mathcal{L}^{>\lambda}(\mathcal{A}|\mathrm{D})$.

Appendix C.2. Proof of Theorem 6.2

We first prove part 1. Suppose $0 < \lambda < \mu < 1$. We use the abbreviations $\mathbb{L}^{>\mu} = \mathbb{L}^{>\mu}(\mathrm{QBA}|\mathrm{ND})$ and $\mathbb{L}^{>\lambda} = \mathbb{L}^{>\lambda}(\mathrm{QBA}|\mathrm{ND})$.

Claim 1: $\mathbb{L}^{>\mu} \subseteq \mathbb{L}^{>\lambda}$.

Note that $0 < \frac{\lambda}{\mu} < 1$. For any quantum automaton $\mathcal{A}$, by Proposition 4.6, there is a quantum automaton $\mathcal{B}$ such that for any $w \in \Sigma^\omega$, $f^{\mathrm{ND}}_{\mathcal{B}}(w) = \frac{\lambda}{\mu} f^{\mathrm{ND}}_{\mathcal{A}}(w)$. Then $f^{\mathrm{ND}}_{\mathcal{A}}(w) > \mu$ if and only if $f^{\mathrm{ND}}_{\mathcal{B}}(w) > \lambda$, which means $\mathcal{L}^{>\mu}(\mathcal{A}|\mathrm{ND}) = \mathcal{L}^{>\lambda}(\mathcal{B}|\mathrm{ND})$. Thus, $\mathbb{L}^{>\mu} \subseteq \mathbb{L}^{>\lambda}$.

Claim 2: $\mathbb{L}^{>\lambda} \subseteq \mathbb{L}^{>\mu}$.

Note that $0 < \frac{1-\mu}{1-\lambda} < 1$. For any quantum automaton $\mathcal{A}$, by Proposition 4.6, there is a quantum automaton $\mathcal{B}$ such that for any $w \in \Sigma^\omega$, $f^{\mathrm{ND}}_{\mathcal{B}}(w) = \frac{1-\mu}{1-\lambda} f^{\mathrm{ND}}_{\mathcal{A}}(w) + \frac{\mu-\lambda}{1-\lambda}$. Then $f^{\mathrm{ND}}_{\mathcal{A}}(w) > \lambda$ if and only if $f^{\mathrm{ND}}_{\mathcal{B}}(w) > \mu$, which means $\mathcal{L}^{>\lambda}(\mathcal{A}|\mathrm{ND}) = \mathcal{L}^{>\mu}(\mathcal{B}|\mathrm{ND})$. Thus, $\mathbb{L}^{>\lambda} \subseteq \mathbb{L}^{>\mu}$.

The above two claims together yield that for any $\mu, \lambda \in (0,1)$, $\mathbb{L}^{>\mu} = \mathbb{L}^{>\lambda}$.

Note that "$\mathbb{L}^{>\lambda} \subseteq \mathbb{L}^{>\mu}$" holds even if $\lambda = 0$. Thus for any $\lambda \in (0,1)$, we have: $\mathbb{L}^{>0} \subseteq \mathbb{L}^{>\mu}$.

The proof of part 2 is similar to that of (1).

Finally, we prove part 3. Let $\mathcal{A} = (\mathcal{H}, |s_0\rangle, \Sigma, \{U_\sigma : \sigma \in \Sigma\}, F)$, where:

- $\mathcal{H} = \mathrm{span}\{|0\rangle, |1\rangle\}$,
- $|s_0\rangle = |0\rangle$,

- $\Sigma = \{a, b\}$,
- $F = \text{span}\{|0\rangle\}$, and
- $U_a = R_x(\sqrt{2}\pi)$ and $U_b = R_x(-\sqrt{2}\pi)$.

Moreover, let $L = \mathcal{L}^{=1}(\mathcal{A}|\text{ND})$. We now use Theorem 5.2 (2) to show that $L \notin \mathbb{L}^{>\lambda}(\text{QBA}|\text{ND})$ for any $\lambda \in [0, 1)$. We choose $w = a^\omega \in L$, for any prefix x of a^ω, say $x = a^n$ for some $n > 0$, and we choose $y = ab \in \Sigma^+$. Note that $xy^\omega = a^n(ab)^\omega \notin L$, and thus $L \notin \mathbb{L}^{>\lambda}(\text{QBA}|\text{ND})$. As a result, we have: $\mathbb{L}^{=1}(\text{QBA}|\text{ND}) \not\subseteq \mathbb{L}^{>\lambda}(\text{QBA}|\text{ND})$ for any $\lambda \in [0, 1)$.

It remains to show that $a^\omega \in L$ and $a^n(ab)^\omega \notin L$ for all $n > 0$.

Claim 1: $a^\omega \in L$.

By Lemma Appendix B.3, we obtain: $f^{\text{ND}}_{\mathcal{A}}(a^\omega) \geq f^{\text{MO}}_{\mathcal{A}}(\epsilon) = 1$, and thus $f^{\text{ND}}_{\mathcal{A}}(a^\omega) = 1$, i.e. $a^\omega \in L$.

Claim 2: $a^n(ab)^\omega \notin L$.

For any $n > 0$, note that $f^{\text{MO}}_{\mathcal{A}}\left((a^n(ab)^\omega)_{n+2k}\right) = \cos^2\left(\frac{\sqrt{2}n\pi}{2}\right)$, and

$$f^{\text{MO}}_{\mathcal{A}}\left((a^n(ab)^\omega)_{n+2k+1}\right) = \cos^2\left(\frac{\sqrt{2}(n+1)\pi}{2}\right).$$

By Proposition 4.2,

$$f^{\text{ND}}_{\mathcal{A}}(a^n(ab)^\omega) = \max\left\{\cos^2\left(\frac{\sqrt{2}n\pi}{2}\right), \cos^2\left(\frac{\sqrt{2}(n+1)\pi}{2}\right)\right\} < 1.$$

Thus, $a^n(ab)^\omega \notin L$.

Appendix C.3. Proof of Theorem 6.3

Let $\mathcal{A} = (\mathcal{H}, |s_0\rangle, \Sigma, \{U_\sigma : \sigma \in \Sigma\}, F)$, where:

- $\mathcal{H} = \text{span}\{|0\rangle, |1\rangle\}$,
- $|s_0\rangle = |0\rangle$,
- $\Sigma = \{a, b\}$,
- $F = \text{span}\{|0\rangle\}$ and
- $U_a = R_x(\sqrt{2}\pi), U_b = R_x(-\sqrt{2}\pi)$.

It can be verified that $f^{\mathrm{D}}_{\mathcal{A}}(a^\omega) = 1$, i.e. $a^\omega \in \mathcal{L}^{=1}(\mathcal{A}|\mathrm{D})$. For any non-empty prefix of a^ω, say $x = a^n$ for some $n \in \mathbb{N}^+$, we choose $y = ab$, then it can be verified that $xy^\omega = a^n(ab)^\omega \notin \mathcal{L}^{=1}(\mathcal{A}|\mathrm{D})$. By Theorem 5.4 (2), we have: $\mathbb{L}^{=1}(\mathrm{QBA}|\mathrm{D}) \not\subseteq \mathbb{L}^{>\lambda}(\mathrm{QBA}|\mathrm{D})$ for any $\lambda \in [0, 1)$.

The same as above, Theorem 6.4 (3) can be proved: $\mathbb{L}^{=1}(\mathrm{QBA}|\mathrm{D}) \not\subseteq \mathbb{L}^{>\lambda}(\mathrm{QBA}|\mathrm{ND})$ for any $\lambda \in [0, 1)$.

Appendix C.4. Proof of Theorem 6.4

1. Let $\mathcal{A} = (\mathcal{H}, |s_0\rangle, \Sigma, \{U_\sigma : \sigma \in \Sigma\}, F)$, where:

- $\mathcal{H} = \mathrm{span}\{|0\rangle, |1\rangle\}$,
- $|s_0\rangle = |0\rangle$,
- $\Sigma = \{a, b\}$,
- $F = \mathrm{span}\{|0\rangle\}$, and
- $U_a = R_x(\alpha\pi), U_b = R_x(-\beta\pi)$ with $\beta > \alpha > 0$ and α and β are irrational numbers to be determined.

For simplicity, we use $d(w)$ to denote the number of occurrences of symbol d in a finite word w; e.g. $a(aaab) = 3$. We choose an infinite word $w = \sigma_1\sigma_2\dots$ with $\sigma_1 = a$, and

$$\sigma_{n+1} = \begin{cases} b, & \alpha a(w_n) - \beta b(w_n) > \beta \\ a, & \text{otherwise} \end{cases}.$$

It can be shown that $f^{\mathrm{ND}}_{\mathcal{A}}(w) = 1$ and for any prefix x of w, $\alpha + \beta > \alpha a(x) - \beta b(x) > 0$. We further choose some $\lambda \in (0, 1)$ to be determined, then $w \in \mathcal{L}^{>\lambda}(\mathcal{A}|\mathrm{ND})$. For any split $w = xyz$, where $|y| \geq 1$, since x and xy are prefixes of w, we have: $\alpha + \beta > \alpha a(x) - \beta b(x) > 0$ and $\alpha + \beta > \alpha a(xy) - \beta b(xy) > 0$, which lead to $\alpha + \beta > \alpha a(y) - \beta b(y) > -(\alpha + \beta)$. Note that $|y| \geq 1$, then $\alpha a(y) - \beta b(y) \neq 0$ and it's irrational itself. Choose some $\theta \in (0, 2)$ to be determined and some $\varepsilon > 0$ small enough, and then there exists $k \in \mathbb{N}$ such that $|(k(\alpha a(y) - \beta b(y)) \bmod 2) - \theta| < \varepsilon$, i.e. $k(\alpha a(y) - \beta b(y)) \in (2m + \theta - \varepsilon, 2m + \theta + \varepsilon)$ for some $m \in \mathbb{N}$.

Now we consider word $w' = xy^{k+1}z$, and let r be a prefix of z. Note that

$$\alpha a(xy^{k+1}r) - \beta b(xy^{k+1}r) = \alpha a(xyr) - \beta b(xyr) + k(\alpha a(y) - \beta b(y)).$$

Since xyr is a prefix of w, it holds that $\alpha a(xyr) - \beta b(xyr) \in (0, \alpha + \beta)$. As a result, we obtain:

$$\alpha a(xy^{k+1}r) - \beta b(xy^{k+1}r) \in (2m + \theta - \varepsilon, 2m + \theta + \varepsilon + \alpha + \beta).$$

We can choose $\alpha, \beta, \theta, \varepsilon$ such that $0 < \theta - \varepsilon < \theta + \varepsilon + \alpha + \beta < 1/2$. Then

$$f_{\mathcal{A}}^{\mathrm{ND}}(w') \leq \sup_{\varphi \in (2m+\theta-\varepsilon, 2m+\theta+\varepsilon+\alpha+\beta)} \cos^2 \frac{\varphi\pi}{2} = \cos^2 \frac{(\theta - \varepsilon)\pi}{2}$$

Now we set $\lambda = \cos^2 \frac{(\theta-\varepsilon)\pi}{2}$, and hence $f_{\mathcal{A}}^{\mathrm{ND}}(w') \leq \lambda$, i.e. $w' \notin \mathcal{L}^{>\lambda}(\mathcal{A}|\mathrm{ND})$. By Theorem 5.2, it holds that $\mathcal{L}^{>\lambda}(\mathcal{A}|\mathrm{ND}) \notin \mathbb{L}^{>\mu}(\mathrm{QBA}|\mathrm{D})$ for any $\mu \in [0, 1)$, and therefore $\mathbb{L}^{>\lambda}(\mathrm{QBA}|\mathrm{ND}) \not\subseteq \mathbb{L}^{>\mu}(\mathrm{QBA}|\mathrm{D})$.

2. Similar to the proof of Theorem 6.2 (3).

3. Similar to the proof of Theorem 6.3.

Appendix D. Proofs of the Results in Section 7

Appendix D.1. Proof of Theorem 7.1

1. Consider an ω-regular language $L = (a + b)^* a^\omega$. Suppose there is a quantum automaton $\mathcal{A}$ such that $\mathcal{L}^{>\lambda}(\mathcal{A}|\mathrm{ND}) = L$. We choose $a^\omega \in L$, i.e. $a^\omega \in \mathcal{L}^{>\lambda}(\mathcal{A}|\mathrm{ND})$. By Theorem 5.2 (2), we see that there are infinitely many ks such that $a^k y^\omega \in \mathcal{L}^{>\lambda}(\mathcal{A}|\mathrm{ND})$ for any $y \in \Sigma^+$. We just choose $y = b$ and get that $a^k b^\omega \in \mathcal{L}^{>\lambda}(\mathcal{A}|\mathrm{ND})$, i.e. $a^k b^\omega \in L$, which is a contradiction. Therefore, there is an ω-regular language which is not in $\mathbb{L}^{>\lambda}(\mathrm{QBA}|\mathrm{ND})$ for any $\lambda \in [0, 1)$.

2. Consider an ω-context-free but not ω-regular language $L = \{a^n b^n (a + b)^\omega : n \geq 1\}$. Suppose there is a quantum automaton $\mathcal{A}$ such that $\mathcal{L}^{>\lambda}(\mathcal{A}|\mathrm{ND}) = L$. Let $n \geq 2$. Then $a^n b^n a^\omega \in L = \mathcal{L}^{>\lambda}(\mathcal{A}|\mathrm{ND})$. By Theorem 5.2 (1), we can choose

- $w = a$,
- $u = a^n b$, and
- $v = b^{n-1} a^\omega$ (note that $uv = a^n b^n a^\omega \in L$),

and then there are infinitely many ks such that $uw^k v = a^n b a^k b^{n-1} a^\omega \in L$, which is a contradiction. As a result, there is an ω-context-free but not ω-regular language which is not in $\mathbb{L}^{>\lambda}(\mathrm{QBA}|\mathrm{ND})$ for any $\lambda \in [0, 1)$.

3. In order to prove that an ω-language is not ω-regular, we recall the following useful pumping lemma for ω-RL from [40].

Theorem Appendix D.1 (A pumping lemma for ω-RL, [40])**.** *Let $L \subseteq \Sigma^\omega$ be an ω-regular language. There exists an integer n_0 such that for any word $w \in L$, and for any integer $n \geq n_0$, w can be written as $w = xyz$, where $x \in \Sigma^*, y \in \Sigma^+$, and $z \in \Sigma^\omega$, such that*

1. $|x| = n$,
2. $|y| \leq n_0$,
3. *For all $k \in \mathbb{N}$, $xy^k z \in L$.*

Now we start to prove Theorem 7.1.3. Let $\mathcal{A} = (\mathcal{H}, |s_0\rangle, \Sigma, \{U_\sigma : \sigma \in \Sigma\}, F)$, where:

- $\mathcal{H} = \text{span}\{|0\rangle, |1\rangle\}$,
- $|s_0\rangle = |0\rangle$,
- $\Sigma = \{a, b\}$,
- $F = \text{span}\{|0\rangle\}$, and
- $U_a = R_x(\sqrt{2}\pi)$ and $U_b = R_x(-\sqrt{2}\pi)$.

Now let $\lambda = 0.9$ and $L = \mathcal{L}^{>\lambda}(\mathcal{A}|\text{ND})$. We use Theorem Appendix D.1 to show that L is not ω-regular. For any positive integer n_0, we choose the infinite word $w = a^{2n_0} b^{2n_0} (ab)^\omega \in L$ and the position $n = n_0$. Then for any split $w = xyz$, where $|x| = n_0$ and $1 \leq |y| \leq n_0$, we choose some k, and then have $xy^k z = a^{2n_0 + (k-1)|y|} b^{2n_0} (ab)^\omega \notin L$. By Theorem Appendix D.1, we conclude that L is not ω-regular.

It remains to prove that for any $n \in \mathbb{N}$, $a^{2n} b^{2n} (ab)^\omega \in L$, and there is a k such that $a^{2n+km} b^{2n} (ab)^\omega \notin L$ for any $1 \leq m \leq n$.

Claim 1. $a^{2n} b^{2n} (ab)^\omega \in L$.

For any $n \in \mathbb{N}$, note that the state of the non-disturbing run of $\mathcal{A}$ over $a^{2n} b^{2n} (ab)^\omega$ at the checkpoint $n_i = 4n + 2i$ is $|0\rangle \in F$, and thus $f_{\mathcal{A}}^{\text{ND}}(a^{2n} b^{2n} (ab)^\omega) = 1 > 0.9 = \lambda$, i.e. $a^{2n} b^{2n} (ab)^\omega \in L$.

Claim 2. $\forall n, \forall 1 \leq m \leq n, \exists k, a^{2n+km} b^{2n} (ab)^\omega \notin L$.

For any $n \in \mathbb{N}$ and $1 \leq m \leq n$, note that the state of the non-disturbing run of $\mathcal{A}$ over $a^{2n+km} b^{2n} (ab)^\omega$ at the checkpoint $4n + km + 2l$ and $4n + km +$

$2l+1$ are $|s_{4n+km+2l}\rangle = R_x(\sqrt{2}km\pi)\,|0\rangle$ and $|s_{4n+km+2l+1}\rangle = R_x(\sqrt{2}(km+1)\pi)\,|0\rangle$, respectively, for all $l \in \mathbb{N}$. Thus

$$\begin{aligned} f_{\mathcal{A}}^{\mathrm{ND}}(a^{2n+km}b^{2n}(ab)^{\omega}) &= \max\{|\langle 0 \mid s_{4n+km+2l}\rangle|^2, |\langle 0 \mid s_{4n+km+2l+1}\rangle|^2\} \\ &= \max\left\{\cos^2\left(\frac{km\pi}{\sqrt{2}}\right), \cos^2\left(\frac{(km+1)\pi}{\sqrt{2}}\right)\right\} \end{aligned}$$

Note that $\cos^2(0.4) = 0.848353\cdots < 0.9 = \lambda$ and $\cos^2(0.4+\frac{\pi}{\sqrt{2}}) = 0.752979\cdots < 0.9 = \lambda$. If we choose k such that $\frac{km\pi}{\sqrt{2}} \bmod 2\pi$ is close enough to 0.4, then $f_{\mathcal{A}}^{\mathrm{ND}}(a^{2n+km}b^{2n}(ab)^{\omega}) < \lambda$, i.e. $a^{2n+km}b^{2n}(ab)^{\omega} \notin L$. As a result, we have $\mathbb{L}^{>\lambda}(\mathrm{QBA}|\mathrm{ND}) \not\subseteq \omega$-RL.

4. Let's consider the language $L = \mathcal{L}^{=1}(\mathcal{A}|\mathrm{ND})$, where $\mathcal{A}$ is the quantum automaton introduced in the proof of Theorem 7.1 (3). Here we use Theorem Appendix D.1 to show that L is not ω-regular. For any positive integer n_0, we choose the infinite word $w = a^{2n_0}b^{2n_0}(ab)^{\omega} \in L$, and the position $n = n_0$. Then for any split $w = xyz$, where $|x| = n_0$ and $1 \leq |y| \leq n_0$, we choose $k = 2$, we have: $xy^kz = a^{2n_0+|y|}b^{2n_0}(ab)^{\omega} \notin L$. By Theorem Appendix D.1, we conclude that L is not ω-regular.

It remains to prove that for any $n \in \mathbb{N}$, $a^{2n}b^{2n}(ab)^{\omega} \in L$, and $a^{2n+m}b^{2n}(ab)^{\omega} \notin L$ for any $1 \leq m \leq n$.

Claim 1. $a^{2n}b^{2n}(ab)^{\omega} \in L$.

For any $n \in \mathbb{N}$, let $|s_n\rangle$ be the non-disturbing run of $\mathcal{A}$ over $w = a^{2n}b^{2n}(ab)^{\omega}$. For any $\varepsilon > 0$, we choose $|\psi\rangle = |0\rangle \in F$ and $n_i = 4n + 2i$, then for all $i \in \mathbb{N}$, $|\langle 0 \mid s_{n_i}\rangle|^2 = 1 > 1 - \varepsilon$. By Lemma Appendix A.1, we obtain: $f_{\mathcal{A}}^{\mathrm{ND}}(a^{2n}b^{2n}(ab)^{\omega}) = 1$, i.e. $a^{2n}b^{2n}(ab)^{\omega} \in L$.

Claim 2. $a^{2n+m}b^{2n}(ab)^{\omega} \notin L$.

For any $1 \leq m \leq n$, let $|s_n\rangle$ be the non-disturbing run of $\mathcal{A}$ over $w = a^{2n+m}b^{2n}(ab)^{\omega}$. Note that $|s_{4n+m+2k}\rangle = R_x(\sqrt{2}m\pi)\,|0\rangle$ and $|s_{4n+m+2k+1}\rangle = R_x(\sqrt{2}(m+1)\pi)\,|0\rangle$ for all $k \in \mathbb{N}$. We choose

$$M = \max\left\{\cos^2\left(\frac{\sqrt{2}m\pi}{2}\right), \cos^2\left(\frac{\sqrt{2}(m+1)\pi}{2}\right)\right\},$$

for any checkpoints $\{n_i\}$. Furthermore, we choose a checkpoint n_i such that $n_i > 4n+m$, then $|\langle 0 \mid s_{n_i}\rangle|^2 \leq M$. By Lemma Appendix A.1, $f_{\mathcal{A}}^{\mathrm{ND}}(a^{2n+m}b^{2n}(ab)^{\omega}) < 1$, i.e. $a^{2n+m}b^{2n}(ab)^{\omega} \notin L$. Hence, $L = \mathcal{L}^{=1}(\mathcal{A}|\mathrm{ND})$ is an counterexample to show that $\mathbb{L}^{=1}(\mathrm{QBA}|\mathrm{ND}) \not\subseteq \omega$-RL.

5. In order to prove that a ω-language is not ω-context-free, we need a pumping lemma for ω-CFL, which, to the best of our knowledge, seems not appear in the previous literature. First, we recall the following pumping lemma for CFL from [41].

Theorem Appendix D.2 (Pumping lemma for CFL, [41])**.** *If $L \subseteq \Sigma^*$ is a context-free language, then there exists an integer $n \geq 1$ such that each word $z \in L$ with $|z| \geq n$ can be written as $z = uvwxy$, where $u, v, w, x, y \in \Sigma^*$, such that*

1. $|vwx| \leq n$,
2. $|vx| \geq 1$, *and*
3. *For all $k \in \mathbb{N}$, $uv^kwx^ky \in L$.*

The above pumping lemma for CFL can be generalised to the case of ω-CFL.

Theorem Appendix D.3 (A pumping lemma for ω-CFL)**.** *Let $L \subseteq \Sigma^\omega$ be an ω-context-free language. Then there exists an positive integer n such that each $z \in L$ can be written as $z = uvwxy$, where $u, v, w, x \in \Sigma^*$ and $y \in \Sigma^\omega$, such that*

1. $|vwx| \leq n$,
2. $|vx| \geq 1$, *and*
3. *For all $k \in \mathbb{N}$, $uv^kwx^ky \in L$.*

Proof. For any ω-CFL L, there are some m pairs of CFLs U_i and V_i such that $L = \bigcup_{i=1}^m U_i V_i^\omega$. Let $z = \sigma_1\sigma_2 \cdots \in \Sigma^\omega$ be an infinite word. If $z \in L$, then $z \in U_i V_i^\omega$ for some $1 \leq i \leq m$. Furthermore, z can be written as $z = z_0 z_1 z_2 \ldots$, where $z_0 \in U_i$ and $z_j \in V_i \setminus \{\epsilon\}$ for all $j \geq 1$. Note that each V_i has its own "pumping length", say n_i for all $1 \leq i \leq m$. That is, by Theorem Appendix D.2, for any $1 \leq i \leq m$, there is an integer $n_i \geq 0$ such that each $z \in V_i$ with $|z| \geq n_i$ can be written as $z = uvwxy$ such that

1. $|vwx| \leq n_i$,
2. $|vx| \geq 1$, and

3. For all $k \in \mathbb{N}$, $uv^k wx^k y \in V_i$.

Now let $n_0 = \max\limits_{1 \le i \le m} \{n_i\}$ and $N_0 = 3n_0$. We choose N_0 to be the "pumping length" of L.

Case 1. There is a z_j such that $|z_j| \ge n_0$ for some $j \ge 1$. We apply the pumping lemma just on z_j. That is, z_j can be written as $z_j = abcde$, where $a, b, c, d, e \in \Sigma^*$, such that

- $|bcd| \le n_i \le n_0$, $|bd| \ge 1$, and
- $ab^k cd^k e \in V_i$ for all $k \in \mathbb{N}$.

Here we choose

- $u = z_0 z_1 \dots z_{j-1} a$,
- $v = b$,
- $w = c$,
- $x = d$ and
- $y = e z_{j+1} z_{j+2} \dots$,

then

- $|vwx| = |bcd| \le n_0 < N_0$,
- $|vx| = |bd| \ge 1$, and
- for any $k \in \mathbb{N}$, we have $uv^k wx^k y = z_0 z_1 \dots z_{j-1} ab^k cd^k e z_{j+1} z_{j+2} \cdots \in U_i V_i^\omega \subseteq L$.

Case 2. $|z_j| < n_0$ for all $j \ge 1$. We choose

- $u = z_0$,
- $v = z_1$,
- $w = z_2$,
- $x = z_3$,
- $y = z_4 z_5 \dots$,

then

- $|vwx| = |z_1 z_2 z_3| < 3n_0 = N_0$,
- $|vx| = |z_1| + |z_3| \geq 1$, and for all $k \in \mathbb{N}$, $uv^k wx^k y = z_0 z_1^k z_2 z_3^k z_4 \cdots \in U_i V_i^\omega \subseteq L$.

Hence, we complete the proof. □

Now we are ready to prove Theorem 7.1.5. Consider such a quantum automaton $\mathcal{A} = (H, |s_0\rangle, \Sigma, \{U_\sigma : \sigma \in \Sigma\}, F)$, where:

- $\mathcal{H} = \text{span}\{|0\rangle, |1\rangle\}$,
- $|s_0\rangle = |0\rangle$,
- $\Sigma = \{a, b, c\}$,
- $F = \text{span}\{|0\rangle\}$, and
- $U_a = R_x((\sqrt{2} + \sqrt{3})\pi)$, $U_b = R_x(-\sqrt{2}\pi)$ and $U_c = R_x(-\sqrt{3}\pi)$.

Let $L = \mathcal{L}^{=1}(\mathcal{A}|\text{ND})$. We use Theorem Appendix D.3 to show that L is not ω-context-free. For any positive integer n, we choose $z = (a^n b^n c^n)^\omega \in L$, for any split $z = uvwxy$ where $u, v, w, x \in \Sigma^*$ and $y \in \Sigma^\omega$ with $|vwx| \leq n$ and $|vx| \geq 1$, we choose some k such that $uv^k wx^k y \notin L$. Hence, $L \notin \omega$-CFL.

It remains to show that $(a^n b^n c^n)^\omega \in L$ and there exists a k such that $uv^k wx^k y \notin L$.

Claim 1. $(a^n b^n c^n)^\omega \in L$. For any $(a^n b^n c^n)^\omega$ with $n \geq 0$, we choose the sequence of checkpoints $n_i = 3ni$. Note that $|s_{n_i}\rangle = |0\rangle$ for all $i \in \mathbb{N}$, where $|s_i\rangle$ is the non-disturbing run of $(a^n b^n c^n)^\omega$. Thus, $|\langle 0 \mid s_{n_i}\rangle|^2 = 1$, and $f_{\mathcal{A}}^{\text{ND}}((a^n b^n c^n)^\omega) = 1$, i.e. $(a^n b^n c^n)^\omega \in L$.

Claim 2. $\exists k, uv^k wx^k y \notin L$. For any split $z = uvwxy$ where $|vwx| \leq n$ and $|vx| \geq 1$, note that vwx cannot contain all the letters in the alphabet Σ, which are a, b and c. There are several possible cases, among which here we only consider one, and the other cases can be proved in the same way. We choose the case where vx doesn't contain b and does contain a. For convenience, we use $d(w)$ to denote the number of occurrences symbol d in word w, e.g. $a(aaab) = 3$. Then the case we are considering can be interpreted as $a(vx) \geq 1$ and $b(vx) = 0$. Note that for any prefix p of z, we have $a(p) - b(p) \geq 0$ and the equality holds when $|p| \bmod 3n = 0$.

We further note that there is a word $q \in \Sigma^*$, which is a prefix of y, such that $|uvwxq| \bmod 3n = 0$, and thus $z = uvwxq(a^n b^n c^n)^\omega = uvwxqz$. Then $a(uv^k wx^k q) - b(uv^k wx^k q) = a(uvwxq) - b(uvwxq) + (k-1)a(vx) = (k-1)a(vx)$. On the other hand, for any $z \in \Sigma^\omega$, the non-disturbing run of $\mathcal{A}$ over z is $|s_i\rangle = R_x(\sqrt{2}(a(z_i) - b(z_i))\pi + \sqrt{3}(a(z_i) - c(z_i)))\,|0\rangle$ for all $i \in \mathbb{N}$.

Now we consider the non-disturbing run $|s_n\rangle$ of $\mathcal{A}$ over $uv^k wx^k y$, and the checkpoint $n_i = \left|uv^k wx^k q\right| + i$ for any $i \in \mathbb{N}^+$. Note that

$$\begin{aligned}
|s_{n_i}\rangle &= U_{z_i} R_x\Big(\sqrt{2}(a(uv^k wx^k q) - b(uv^k wx^k q))\pi + \sqrt{3}(a(uv^k wx^k q) - c(uv^k wx^k q))\Big)\,|0\rangle \\
&= U_{z_i} R_x\Big(\sqrt{2}((k-1)a(vx))\pi + \sqrt{3}(a(uv^k wx^k q) - c(uv^k wx^k q))\Big)\,|0\rangle \\
&= R_x\Big(\sqrt{2}((k-1)a(vx) + a(z_i) - b(z_i))\pi \\
&\quad + \sqrt{3}(a(uv^k wx^k q) - c(uv^k wx^k q) + a(z_i) - c(z_i))\Big)\,|0\rangle
\end{aligned}$$

We further note that if we choose $k = 2$, then $(k-1)a(vx) + a(z_i) - b(z_i) \geq (k-1)a(vx) = a(vx) \geq 1$. Therefore, $|s_{n_i}\rangle \neq |0\rangle$ for any $i \in \mathbb{N}^+$. Moreover, note that $|s_{n_i}\rangle$ has a cycle of length $3n$, i.e. $|s_{n_i}\rangle = \left|s_{n_{i+3n}}\right\rangle$. If we choose $M = \max_{0 \leq i < 3n}\{|\langle 0 \mid s_{n_i}\rangle|^2\} < 1$, then $|\langle 0 \mid s_{n_i}\rangle|^2 \leq M < 1$ for all $i \in \mathbb{N}$. By Lemma Appendix A.1, we have: $f^{\mathrm{ND}}_{\mathcal{A}}(uv^2wx^2y) < 1$, i.e. $uv^2wx^2y \notin L$. Hence, there is an ω-language in $\mathbb{L}^{=1}(\mathrm{QBA|ND})$ which is not ω-context-free, i.e. $\mathbb{L}^{=1}(\mathrm{QBA|ND}) \not\subseteq \omega$-CFL.

Appendix D.2. Proof of Theorem 7.2

1. Similar to the proof of Theorem 7.1 (1).
2. Similar to the proof of Theorem 7.1 (2).
3. Let $\mathcal{A} = (\mathcal{H}, |s_0\rangle, \Sigma, \{U_\sigma : \sigma \in \Sigma\}, F)$, where:

- $\mathcal{H} = \mathrm{span}\{|0\rangle, |1\rangle\}$,
- $|s_0\rangle = |0\rangle$,
- $\Sigma = \{a, b\}$,
- $F = \mathrm{span}\{|0\rangle\}$ and
- $U_a = R_x(\sqrt{2}\pi), U_b = R_x(-\sqrt{2}\pi)$.

It can be verified that $f^{\mathrm{D}}_{\mathcal{A}}(a^n b^n (ab)^\omega) = 1$, i.e. $a^n b^n (ab)^\omega \in \mathcal{L}^{=1}(\mathcal{A}|\mathrm{D})$, for any $n \in \mathbb{N}$. With this property, we will show that $L = \mathcal{L}^{=1}(\mathcal{A}|\mathrm{D})$ is not ω-regular. For any $n_0 \in \mathbb{N}^+$, we choose $n = n_0$ and $w = a^{2n} b^{2n} (ab)^\omega$, then $w \in L$. For any split $w = xyz$ where $|x| = n$ and $1 \leq |y| \leq n$, then $x = a^n$ and $y = a^{|y|}$. As a result, it holds that $z = a^{n-|y|} b^{2n} (ab)^\omega$. We now choose $k = 2$, then $w' = xy^k z = xy^2 z = a^{2n+|y|} b^{2n} (ab)^\omega$. Let the non-disturbing run of $\mathcal{A}$ over w' be $\left| {s'}^{\mathrm{ND}}_n \right\rangle$. Then $\left| {s'}^{\mathrm{ND}}_n \right\rangle = |0\rangle$ if and only if $n = 0$, and $\left| {s'}^{\mathrm{ND}}_n \right\rangle$ can range over finitely many different states. For any checkpoints $\{n_i\}$, if $n_1 \neq 0$, let the disturbing run of $\mathcal{A}$ over w under M_0 with $\{n_i\}$ be $|s'_n\rangle$, then $|\langle 0 \mid {s'}_{n_1} \rangle|^2 = \left| \left\langle 0 \mid {s'}^{\mathrm{ND}}_{n_1} \right\rangle \right|^2 < 1$. And if $n_1 = 0$, then $|\langle 0 \mid {s'}_{n_2} \rangle|^2 = \left| \left\langle 0 \mid {s'}^{\mathrm{ND}}_{n_2} \right\rangle \right|^2 < 1$. Thus $f^{\mathrm{D}}_{\mathcal{A}}(w') < 1$, i.e. $w' = xy^2 z \notin L$. Finally by Theorem Appendix D.1, L is not ω-regular, and thus $\mathbb{L}^{=1}(\mathrm{QBA}|\mathrm{D}) \not\subseteq \omega\text{-RL}$.

Appendix E. Proofs of the Results in Section 8

Appendix E.1. Proof of Theorem 8.1

1. Suppose $\mathcal{A}_1$ and $\mathcal{A}_2$ are the QBAs of L_1 and L_2, respectively. That is, $L_1 = \mathcal{L}^{>0}(\mathcal{A}_1|\mathrm{ND})$ and $L_2 = \mathcal{L}^{>0}(\mathcal{A}_2|\mathrm{ND})$. Let $\mathcal{A}_{12} = \frac{1}{\sqrt{2}}\mathcal{A}_1 \oplus \frac{1}{\sqrt{2}}\mathcal{A}_2$. We will prove that $\mathcal{L}^{>0}(\mathcal{A}_{12}|\mathrm{ND}) = L_1 \cup L_2$.

Claim 1. $\mathcal{L}^{>0}(\mathcal{A}_{12}|\mathrm{ND}) \subseteq L_1 \cup L_2$.

For any $w \in \mathcal{L}^{>0}(\mathcal{A}_{12}|\mathrm{ND})$, i.e. $f^{\mathrm{ND}}_{\mathcal{A}_{12}}(w) > 0$. By Lemma 4.4, $\max\{f^{\mathrm{ND}}_{\mathcal{A}_1}(w), f^{\mathrm{ND}}_{\mathcal{A}_2}(w)\} \geq f^{\mathrm{ND}}_{\mathcal{A}_{12}}(w) > 0$. Then $f^{\mathrm{ND}}_{\mathcal{A}_1}(w) > 0$ or $f^{\mathrm{ND}}_{\mathcal{A}_2}(w) > 0$, which implies $w \in L_1 \cup L_2$. Thus, $\mathcal{L}^{>0}(\mathcal{A}_{12}|\mathrm{ND}) \subseteq L_1 \cup L_2$.

Claim 2. $L_1 \cup L_2 \subseteq \mathcal{L}^{>0}(\mathcal{A}_{12}|\mathrm{ND})$.

For any $w \in L_1 \cup L_2$, i.e. $\max\{f^{\mathrm{ND}}_{\mathcal{A}_1}(w), f^{\mathrm{ND}}_{\mathcal{A}_2}(w)\} > 0$. By Lemma 4.4, $f^{\mathrm{ND}}_{\mathcal{A}_{12}}(w) \geq \frac{1}{2}\max\{f^{\mathrm{ND}}_{\mathcal{A}_1}(w), f^{\mathrm{ND}}_{\mathcal{A}_2}(w)\} > 0$. Then $w \in \mathcal{L}^{>0}(\mathcal{A}_{12}|\mathrm{ND})$. Thus, $L_1 \cup L_2 \subseteq \mathcal{L}^{>0}(\mathcal{A}_{12}|\mathrm{ND})$.

2. Let $\mathcal{A}$ and $\mathcal{B}$ be quantum automata such that $L_A = \mathcal{L}^{>\lambda}(\mathcal{A}|\mathrm{ND})$ and $L_B = \mathcal{L}^{>\lambda}(\mathcal{B}|\mathrm{ND})$. Furthermore, we put: $\mathcal{M}_k = \frac{1}{\sqrt{2}}\left(\mathcal{A}^{\otimes k} \oplus \mathcal{B}^{\otimes k}\right)$, and $L_k = \mathcal{L}^{>\lambda^k}(\mathcal{M}_k|\mathrm{ND}) \in \mathbb{L}^{>\lambda}(\mathrm{QBA}|\mathrm{ND})$.

Claim 1. For any $k, m \in \mathbb{N}$, if $k < m$, then $L_k \subseteq L_m$.

Let $w \in L_k$, then $f^{\mathrm{ND}}_{\mathcal{M}_k}(w) = \limsup_{n\to\infty} f^{\mathrm{MO}}_{\mathcal{M}_k}(w_n) > \lambda^k$. Note that

$$\begin{aligned}\left(f^{\mathrm{MO}}_{\mathcal{M}_k}(w_n)\right)^{1/k} &= \left(\frac{1}{2}\left((f^{\mathrm{MO}}_{\mathcal{A}}(w_n))^k + (f^{\mathrm{MO}}_{\mathcal{B}}(w_n))^k\right)\right)^{1/k} \\ &\le \left(\frac{1}{2}\left((f^{\mathrm{MO}}_{\mathcal{A}}(w_n))^m + (f^{\mathrm{MO}}_{\mathcal{B}}(w_n))^m\right)\right)^{1/m} = \left(f^{\mathrm{MO}}_{\mathcal{M}_m}(w_n)\right)^{1/m}.\end{aligned}$$

Thus,

$$\limsup_{n\to\infty}\left(f^{\mathrm{MO}}_{\mathcal{M}_m}(w_n)\right)^{1/m} \ge \limsup_{n\to\infty}\left(f^{\mathrm{MO}}_{\mathcal{M}_k}(w_n)\right)^{1/k} = \left(\limsup_{n\to\infty} f^{\mathrm{MO}}_{\mathcal{M}_k}(w_n)\right)^{1/k} > \lambda,$$

i.e. $\limsup_{n\to\infty} f^{\mathrm{MO}}_{\mathcal{M}_m}(w_n) > \lambda^m$, which means $w \in L_m$, and thus $L_k \subseteq L_m$.

Claim 2. For any $k \in \mathbb{N}$, $L_k \subseteq L_A \cup L_B$.

Let $w \in L_k$, then $f^{\mathrm{ND}}_{\mathcal{M}_k}(w) = \limsup_{n\to\infty} f^{\mathrm{MO}}_{\mathcal{M}_k}(w_n) > \lambda^k$. On the other hand, if $w \notin L_A$ and $w \notin L_B$, then

$$\begin{aligned}\limsup_{n\to\infty} f^{\mathrm{MO}}_{\mathcal{M}_k}(w_n) &= \limsup_{n\to\infty}\frac{1}{2}\left((f^{\mathrm{MO}}_{\mathcal{A}}(w_n))^k + (f^{\mathrm{MO}}_{\mathcal{B}}(w_n))^k\right) \\ &\le \frac{1}{2}\left(\limsup_{n\to\infty}(f^{\mathrm{MO}}_{\mathcal{A}}(w_n))^k + \limsup_{n\to\infty}(f^{\mathrm{MO}}_{\mathcal{B}}(w_n))^k\right) \\ &= \frac{1}{2}\left((f^{\mathrm{ND}}_{\mathcal{A}}(w))^k + (f^{\mathrm{ND}}_{\mathcal{B}}(w))^k\right) < \lambda^k.\end{aligned}$$

There a contradiction arises here. Thus, $w \in L_A$ or $w \in L_B$, i.e. $w \in L_A \cup L_B$, which leads to $L_k \subseteq L_A \cup L_B$.

The above two claims imply that $\{L_k\}$ does have a limit, and $\lim_{k\to\infty} L_k \subseteq L_A \cup L_B$.

Claim 3. For any proper subset $L \subset L_A \cup L_B$, there is an infinite word $w \notin L$, and a k such that $w \in L_k$. Then for any proper subset $L \subset L_A \cup L_B$, there is an infinite word $w \in L_A \cup L_B$ but $w \notin L$. Without loss of generality, we may assume that $w \in L_A$, then there are checkpoints $\{n_i\}$ such that $f^{\mathrm{MO}}_{\mathcal{A}}(w_{n_i}) > \lambda$. Note that

$$\lim_{k\to\infty}\left(\frac{f^{\mathrm{MO}}_{\mathcal{A}}(w_{n_i})}{\lambda}\right)^k = +\infty.$$

Thus there is a k such that $\left(\frac{f_{\mathcal{A}}^{\mathrm{MO}}(w_{n_i})}{\lambda}\right)^k > 4$. Then $\frac{1}{2}\left((f_{\mathcal{A}}^{\mathrm{MO}}(w_{n_i}))^k + (f_{\mathcal{B}}^{\mathrm{MO}}(w_{n_i}))^k\right) > 2\lambda^k$, and consequently,

$$f_{\mathcal{M}_k}^{\mathrm{ND}}(w) = \limsup_{n\to\infty} \frac{1}{2}\left((f_{\mathcal{A}}^{\mathrm{MO}}(w_n))^k + (f_{\mathcal{B}}^{\mathrm{MO}}(w_n))^k\right) \geq 2\lambda^k > \lambda^k,$$

i.e. $w \in L_k$.

Claim 3 shows that $\lim_{k\to\infty} L_k = L_A \cup L_B$, and we complete the proof.

3. Let $\mathcal{A} = (\mathcal{H}, |s_0\rangle, \Sigma, \{U_\sigma : \sigma \in \Sigma\}, F)$, where:

- $\mathcal{H} = \mathrm{span}\{|0\rangle, |1\rangle\}$,
- $|s_0\rangle = |0\rangle$,
- $\Sigma = \{a, b\}$,
- $F = \mathrm{span}\{|0\rangle\}$, and
- $U_a = R_x(\frac{\sqrt{2}}{100}\pi)$ and $U_b = R_x(-\frac{\sqrt{2}}{100}\pi)$.

We further let $L_k = \mathcal{L}^{>1-\frac{1}{k+10}}(\mathcal{A}|\mathrm{ND}) \in \mathbb{L}^{>\lambda}(\mathrm{QBA}|\mathrm{ND})$. It can be verified that $\lim_{k\to\infty} L_k = \mathcal{L}^{=1}(\mathcal{A}|\mathrm{ND}) \notin \mathbb{L}^{>\lambda}(\mathrm{QBA}|\mathrm{ND})$.

4. Let $\mathcal{A} = (\mathcal{H}, |s_0\rangle, \Sigma, \{U_\sigma : \sigma \in \Sigma\}, F)$, where:

- $\mathcal{H} = \mathrm{span}\{|0\rangle, |1\rangle\}$,
- $|s_0\rangle = |0\rangle$,
- $\Sigma = \{a, b\}$,
- $F = \mathrm{span}\{|1\rangle\}$, and
- $U_a = I$ and $U_b = X$.

Note that $\mathcal{L}^{>0}(\mathcal{A}|\mathrm{ND}) \in \mathbb{L}^{>0}(\mathrm{QBA}|\mathrm{ND}) \subseteq \mathbb{L}^{>\lambda}(\mathrm{QBA}|\mathrm{ND})$. We choose $a^\omega \notin \mathcal{L}^{>0}(\mathcal{A}|\mathrm{ND})$, i.e. $a^\omega \in \overline{\mathcal{L}^{>0}(\mathcal{A}|\mathrm{ND})} = \Sigma^\omega \setminus \mathcal{L}^{>0}(\mathcal{A}|\mathrm{ND})$. For any prefixes x of a^ω, then $x = a^n$ for some $n \in \mathbb{N}$, we choose $y = b$, and then $xy^\omega = a^n b^\omega \in \mathcal{L}^{>0}(\mathcal{A}|\mathrm{ND})$, i.e. $a^n b^\omega \notin \overline{\mathcal{L}^{>0}(\mathcal{A}|\mathrm{ND})}$. By Theorem 5.2 (2), we obtain: $\overline{\mathcal{L}^{>0}(\mathcal{A}|\mathrm{ND})} \notin \mathbb{L}^{>\lambda}(\mathrm{QBA}|\mathrm{ND})$ for any $\lambda \in [0, 1)$. Therefore, we see that $\mathbb{L}^{>\lambda}(\mathrm{QBA}|\mathrm{ND})$ is not closed under complementation for any $\lambda \in [0, 1)$.

5. Let $\mathcal{A} = (\mathcal{H}, |s_0\rangle, \Sigma, \{U_\sigma : \sigma \in \Sigma\}, F)$, where:

- $\mathcal{H} = \text{span}\{|0\rangle, |1\rangle\}$,
- $|s_0\rangle = |0\rangle$,
- $\Sigma = \{a, b\}$,
- $U_a = X$, and $U_b = I$, and
- $F = \text{span}\{|0\rangle\}$.

It can be verified that $f_{\mathcal{A}}^{\text{ND}}(a^\omega) = f_{\mathcal{A}^\perp}^{\text{ND}}(a^\omega) = 1$. However, for any $n \in \mathbb{N}$, $f_{\mathcal{A}}^{\text{ND}}(a^n b^\omega) f_{\mathcal{A}^\perp}^{\text{ND}}(a^n b^\omega) = 0$, which together with Theorem 5.2 (2) implies that $\mathcal{L}^{>\lambda}(\mathcal{A}|\text{ND}) \cap \mathcal{L}^{>\lambda}(\mathcal{A}^\perp|\text{ND}) \notin \mathbb{L}^{>\lambda}(\mathcal{A}|X)$.

Appendix E.2. Proof of Theorem 8.2

1. Similar to the proof of Theorem 8.1 (5).
2. Consider $\mathcal{A} = (\mathcal{H}, |s_0\rangle, \Sigma, \{U_\sigma : \sigma \in \Sigma\}, F)$, where:

- $\mathcal{H} = \text{span}\{|0\rangle, |1\rangle\}$,
- $|s_0\rangle = |0\rangle$,
- $\Sigma = \{a, b\}$,
- $F = \text{span}\{|1\rangle\}$, and
- $U_a = I$, and $U_b = X$.

It can be verified that $f_{\mathcal{A}}^{\text{D}}(a^\omega) = 0$, but $f_{\mathcal{A}}^{\text{D}}(a^n b^\omega) = 1$ for any $n \in \mathbb{N}$. By Theorem 5.4 (2), $\mathbb{L}^{>\lambda}(\text{QBA}|\text{D})$ is not closed under complememtation for any $\lambda \in [0, 1)$.

Appendix F. Proofs of the Results in Section 9

Appendix F.1. Proof of Lemma 9.1

The proof consists of the following three parts:

"(1) $\Longrightarrow$ (2)". If $\mathcal{L}^{>\lambda}(\mathcal{A}|\text{MO}) \neq \emptyset$, then there is a finite word $u \in \Sigma^*$ such that $f_{\mathcal{A}}^{\text{MO}}(u) > \lambda$. By Lemma Appendix B.3, choosing an arbitrary $v \in \Sigma^+$, we have: $f_{\mathcal{A}}^{\text{ND}}(uv^\omega) \geq f_{\mathcal{A}}^{\text{MO}}(u) > \lambda$, i.e. $uv^\omega \in \mathcal{L}^{>\lambda}(\mathcal{A}|\text{ND})$ and thus $\mathcal{L}^{>\lambda}(\mathcal{A}|\text{ND}) \neq \emptyset$.

"(2) $\Longrightarrow$ (3)". If $\mathcal{L}^{>\lambda}(\mathcal{A}|\text{ND}) \neq \emptyset$, then there is an infinite word $w \in \Sigma^\omega$ such that $f_\mathcal{A}^{\text{ND}}(w) > \lambda$. By Proposition 4.2, $\limsup_{n\to\infty} f_\mathcal{A}^{\text{MO}}(w_n) > \lambda$. Then there is a checkpoint n such that $f_\mathcal{A}^{\text{MO}}(w_n) > \lambda$. By Lemma Appendix B.4, chosing an arbitrary $v \in \Sigma^+$, we have: $f_\mathcal{A}^{\text{D}}(uv^\omega) \geq f_\mathcal{A}^{\text{MO}}(u) > \lambda$, i.e. $uv^\omega \in \mathcal{L}^{>\lambda}(\mathcal{A}|\text{D})$ and thus $\mathcal{L}^{>\lambda}(\mathcal{A}|\text{D}) \neq \emptyset$.

"(3) $\Longrightarrow$ (1)". If $\mathcal{L}^{>\lambda}(\mathcal{A}|\text{D}) \neq \emptyset$, then there is an infinite word $w \in \Sigma^\omega$ such that $f_\mathcal{A}^{\text{D}}(w) > \lambda$. By Proposition 6.1, $w = u_0u_1u_2\ldots$, where

1. $u_0 \in \mathcal{L}^{>\lambda+\varepsilon}(\mathcal{A}_{s_0,\psi}|\text{MO})$;
2. $u_1, u_2, \cdots \in \mathcal{L}^{>\lambda+\varepsilon}(\mathcal{A}_{\psi,\psi}|\text{MO}) \setminus \{\epsilon\}$

for some real number $\varepsilon > 0$ and state $|\psi\rangle \in F$. Note that

$$f_\mathcal{A}^{\text{MO}}(u_0) = \|P_F U_{u_0} |s_0\rangle\|^2 = |\langle\psi| U_{u_0} |s_0\rangle|^2 > \lambda + \varepsilon.$$

Then $u_0 \in \mathcal{L}^{>\lambda}(\mathcal{A}|\text{MO})$ and thus $\mathcal{L}^{>\lambda}(\mathcal{A}|\text{MO}) \neq \emptyset$.

Appendix F.2. Proof of Lemma 9.5 and Theorem 9.6

We first prove Lemma 9.5.

Proof of Lemma 9.5. "$\Longrightarrow$". If $\mathcal{L}^{>\lambda}(\mathcal{A}|\text{ND}) \cap \mathcal{L}^{>\lambda}(\mathcal{B}|\text{ND}) \neq \emptyset$, i.e. there is an infinite word $w \in \Sigma^\omega$ such that $w \in \mathcal{L}^{>\lambda}(\mathcal{A}|\text{ND})$ and $w \in \mathcal{L}^{>\lambda}(\mathcal{B}|\text{ND})$. Then there are two sequences of checkpoints $\{n_i^\mathcal{A}\}$ and $\{n_i^\mathcal{B}\}$ such that for all $i \in \mathbb{N}$, $f_\mathcal{A}^{\text{MO}}(w_{n_i^\mathcal{A}}) > \lambda$, and $f_\mathcal{B}^{\text{MO}}(w_{n_i^\mathcal{B}}) > \lambda$. Without loss of generality, we may assume that $n_1^\mathcal{A} < n_1^\mathcal{B}$. Suppose $w = \sigma_1\sigma_2\ldots$. Let $u = \sigma_1\ldots\sigma_{n_1^\mathcal{A}}$ and $v = \sigma_{n_1^\mathcal{A}+1}\ldots\sigma_{n_1^\mathcal{B}}$. It can be easily verified that $f_\mathcal{A}^{\text{MO}}(u) > \lambda$ and $f_\mathcal{B}^{\text{MO}}(uv) > \lambda$.

"$\Longleftarrow$". If there are two finite words u and v such that $u \in \mathcal{L}^{>\lambda}(\mathcal{A}|\text{MO})$ and $uv \in \mathcal{L}^{>\lambda}(\mathcal{B}|\text{MO})$. It can be easily verified that $w = uv^\omega \in \mathcal{L}^{>\lambda}(\mathcal{A}|\text{ND}) \cap \mathcal{L}^{>\lambda}(\mathcal{B}|\text{ND})$. □

Now we prove Theorem 9.6. By Lemma 9.5, if $\mathcal{L}^{>\lambda}(\mathcal{A}|\text{ND}) \cap \mathcal{L}^{>\lambda}(\mathcal{B}|\text{ND}) = \emptyset$, then for any $u, v \in \Sigma^*$, $f_\mathcal{A}^{\text{MO}}(u) \leq \lambda$, or $f_\mathcal{B}^{\text{MO}}(uv) \leq \lambda$. Using a technique similar to that in [38], we can reduce the problem into a first-order formula, which can be decided by the Tarski-Seidenberg elimination method [39]. Suppose $\mathcal{A} = (\mathcal{H}^\mathcal{A}, \Sigma, |s_0^\mathcal{A}\rangle, \{U_\sigma^\mathcal{A} : \sigma \in \Sigma\}, F^\mathcal{A})$ and $\mathcal{B} = (\mathcal{H}^\mathcal{B}, \Sigma, |s_0^\mathcal{B}\rangle, \{U_\sigma^\mathcal{B} : \sigma \in \Sigma\}, F^\mathcal{B})$. Let $\mathcal{C} = \frac{1}{\sqrt{2}}\mathcal{A} \oplus \frac{1}{\sqrt{2}}\mathcal{B}$. For each $\sigma \in \Sigma$, we denote $U_\sigma = U_\sigma^\mathcal{A} \oplus U_\sigma^\mathcal{B}$. For convenience, let $\mathcal{C}_\mathcal{A}$ and $\mathcal{C}_\mathcal{B}$ be the analogs of $\mathcal{C}$ with accepting space $F^\mathcal{A}$ and $F^\mathcal{B}$, respectively, and the corresponding projections are $P^\mathcal{A}$ and $P^\mathcal{B}$. It can be verified that for any $w \in \Sigma^*$ $f_{\mathcal{C}_\mathcal{A}}^{\text{MO}}(w) = \frac{1}{2}f_\mathcal{A}^{\text{MO}}(w)$, and $f_{\mathcal{C}_\mathcal{B}}^{\text{MO}}(w) = \frac{1}{2}f_\mathcal{B}^{\text{MO}}(w)$.

On the other hand, $f^{\mathrm{MO}}_{\mathcal{C}_{\mathcal{A}}}(w) = \left\| P^{\mathcal{A}} U_w \left| s_0^{\mathcal{A}} \right\rangle \right\|^2$, and $f^{\mathrm{MO}}_{\mathcal{C}_{\mathcal{B}}}(w) = \left\| P^{\mathcal{B}} U_w \left| s_0^{\mathcal{B}} \right\rangle \right\|^2$. Let $\mathcal{U}$ be the semigroup generated by U_σ: $\mathcal{U} = \{U_w : w \in \Sigma^*\}$. And let $f_{\mathcal{A}}, f_{\mathcal{B}} : \mathbb{C}^{n\times n} \to \mathbb{R}$ be the functions defined by $f_{\mathcal{A}}(U) = \left\| P^{\mathcal{A}} U \left| s_0^{\mathcal{A}} \right\rangle \right\|^2$, and $f_{\mathcal{B}}(U) = \left\| P^{\mathcal{B}} U \left| s_0^{\mathcal{B}} \right\rangle \right\|^2$. Now the problem is to determine if $f_{\mathcal{A}}(U) \le \lambda/2$ or $f_{\mathcal{B}}(UV) \le \lambda/2$ for all $U, V \in \mathcal{U}$. The functions $f_{\mathcal{A}}$ and $f_{\mathcal{B}}$ are (continuous) polynomial maps and so the problem is equivalent to determine if $f_{\mathcal{A}}(U) \le \lambda/2$ or $f_{\mathcal{B}}(UV) \le \lambda/2$ for all $U, V \in \overline{\mathcal{U}}$, where $\overline{\mathcal{U}}$ is the closure of $\mathcal{U}$ in $\mathbb{C}^{n\times n}$.

Now we recall a property of $\overline{\mathcal{U}}$ from [38]:

Lemma Appendix F.1 ([38])**.** *Let U_σ for $\sigma \in \Sigma$ be unitary matrices of dimension n. Let $\overline{\mathcal{U}}$ be the closure of the semigroup $\mathcal{U} = \{U_w : w \in \Sigma^*\}$. Then $\overline{\mathcal{U}}$ is algebraic, and if U_σ have rational entries, we can effectively computed a sequence of polynomial $f_1, \dots f_k, \dots$ such that*

1. *If $U \in \overline{\mathcal{U}}$, $f_k(U) = 0$ for all k.*
2. *There exists some k such that $\overline{\mathcal{U}} = \{U : f_i(U) = 0, i = 1, 2, \dots, k\}$.*

After having computed $f_1, f_2, \dots, f_k$ (finitely many) introduced in Lemma Appendix F.1, the problem of determining whether $f_{\mathcal{A}}(U) \le \lambda/2$ or $f_{\mathcal{B}}(U) \le \lambda/2$ for all $U \in \overline{\mathcal{U}}$ can be written as

$$\forall U \forall V \left[\bigvee_{i=1}^{k} (f_i(U) = 0) \Longrightarrow \left(f_{\mathcal{A}}(U) \le \frac{\lambda}{2} \vee f_{\mathcal{B}}(UV) \le \frac{\lambda}{2} \right) \right],$$

which is a first-order formula and can be decided by the Tarski-Seidenberg elimination method [39]. Hence, for any two quantum automata $\mathcal{A}$ and $\mathcal{B}$, and any $\lambda \in [0, 1)$, whether $\mathcal{L}^{>\lambda}(\mathcal{A}|\mathrm{ND}) \cap \mathcal{L}^{>\lambda}(\mathcal{B}|\mathrm{ND}) \ne \emptyset$ is decidable.

Note that in Lemma Appendix F.1, matrices with rational entries are required. In fact, the restriction can be released to the case of algebraic number entries [38].

Appendix F.3. Proof of Lemma 9.3 and Theorem 9.4

Here we need a lemma in [7]:

Lemma Appendix F.2 (Pumping lemma for QFA|MO, [7])**.** *Let $\mathcal{A}$ be a quantum automaton. Then for any $w \in \Sigma^*$ and any $\varepsilon > 0$, there is a k such that*

$$\left| f^{\mathrm{MO}}_{\mathcal{A}}(uw^k v) - f^{\mathrm{MO}}_{\mathcal{A}}(uv) \right| \le \varepsilon$$

for any $u, v \in \Sigma^*$. Moreover, if $\mathcal{A}$ is n-dimensional, there is a constant c such that $k \leq (c\varepsilon)^{-n}$.

Now we are going to prove Lemma 9.3.

Proof. "$\Longrightarrow$". If $\mathcal{L}^{\geq\lambda}(\mathcal{A}|\text{ND}) \neq \emptyset$, then there is a $w \in \Sigma^\omega$ such that $f_{\mathcal{A}}^{\text{ND}}(w) \geq \lambda$, i.e. $\limsup_{n\to\infty} f_{\mathcal{A}}^{\text{MO}}(w_n) \geq \lambda$, which implies that there is a sequence $\{n_k\}$ such that $\lim_{k\to\infty} f_{\mathcal{A}}^{\text{MO}}(w_{n_k}) \geq \lambda$, i.e. $\lim_{k\to\infty} f(U_{n_k}) \geq \lambda$, where we use U_{n_k} short for $U_{w_{n_k}}$. Since $\overline{\mathcal{U}} \subseteq \mathbb{C}^{n\times n}$ is closed, there is a subsequence $\{n_{k_l}\}$, and a $U \in \overline{\mathcal{U}}$, such that $\lim_{l\to\infty} U_{n_{k_l}} = U$. Note that f is continuous, which leads to $f(U) = \lim_{l\to\infty} f(U_{n_{k_l}}) = \lim_{k\to\infty} f(U_{n_k}) \geq \lambda$.

"$\Longleftarrow$". Assume that there is $U \in \overline{\mathcal{U}}$ such that $f(U) \geq \lambda$. Then there must be a sequence $\{U_k\}$ such that $U_k \in \mathcal{U}$ for all k and $\lim_{k\to\infty} U_k = U$. Since $f(U)$ is continuous, we obtain: $\lim_{k\to\infty} f(U_k) = f(U) \geq \lambda$, that is, $\forall\varepsilon > 0, \exists k_0 \in \mathbb{N}^+, \forall k > k_0. |f(U_k) - f(U)| < \varepsilon$. In particular, let $\varepsilon = 1/n$, there is a k_n such that $f(U_{k_n}) > f(U) - \frac{1}{n} \geq \lambda - \frac{1}{n}$. Note that for any k, there is a finite word $u_k \in \Sigma^*$ such that $U_{u_k} = U_k$. Thus, $f_{\mathcal{A}}^{\text{MO}}(u_{k_n}) > \lambda - \frac{1}{n}$.

We now can construct an infinite word by induction as follows:

- Initially, set $v_1 = u_{k_1}$ with $f_{\mathcal{A}}^{\text{MO}}(v_1) > \lambda - 1$.
- Suppose we have chosen v_n for some $n \geq 1$ with $f_{\mathcal{A}}^{\text{MO}}(v_n) > \lambda - \frac{1}{n}$. By Lemma Appendix F.2, let $\varepsilon = f_{\mathcal{A}}^{\text{MO}}(u_{k_{n+1}}) - \lambda + \frac{1}{n+1} > 0$. Then there is $k \leq (c\varepsilon)^{-n}$ for some constant c such that

$$f_{\mathcal{A}}^{\text{MO}}(v_n^k u_{k_{n+1}}) \geq f_{\mathcal{A}}^{\text{MO}}(u_{k_{n+1}}) - \varepsilon > \lambda - \frac{1}{n+1}.$$

  We set $v_{n+1} = v_n^k u_{k_{n+1}}$. Then it holds that $f_{\mathcal{A}}^{\text{MO}}(v_{n+1}) > \lambda - \frac{1}{n+1}$.

Using the above construction, we obtain a sequence of finite words $\{v_n\}$, where $v_{n+1} = v_n u$ for some $u \in \Sigma^*$. Let $w = \lim_{n\to\infty} v_n \in \Sigma^\omega$. Thus, there must be a subsequence $\{v_{n_m}\}$ such that $\lim_{m\to\infty} f_{\mathcal{A}}^{\text{MO}}(v_{n_m})$ exists. Then:

$$f_{\mathcal{A}}^{\text{ND}}(w) = \limsup_{n\to\infty} f_{\mathcal{A}}^{\text{MO}}(w_n) \geq \limsup_{n\to\infty} f_{\mathcal{A}}^{\text{MO}}(v_n) \geq \lim_{m\to\infty} f_{\mathcal{A}}^{\text{MO}}(v_{n_m}) \geq \lambda.$$

Consequently, $f_{\mathcal{A}}^{\text{ND}}(w) \geq \lambda$, i.e. $w \in \mathcal{L}^{\geq\lambda}(\mathcal{A}|\text{ND})$, which implies that $\mathcal{L}^{\geq\lambda}(\mathcal{A}|\text{ND}) \neq \emptyset$. □

Finally, we can prove Theorem 9.4. The emptiness problem is equivalent to: whether there is a $U \in \overline{\mathcal{U}}$ such that $f(U) \geq \lambda$, which can be written as a first-order formula

$$\exists U \left[\bigvee_{i=1}^{k} (f_i(U) = 0) \Longrightarrow f(U) \geq \lambda \right],$$

where $f_1, \ldots, f_k$ can be computed as in Lemma Appendix F.1, and f is defined previously in this section. Moreover, the above first-order formula is decidable using the Tarski-Seidenberg elimination method [39].