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Stability of Switched Systems with Unstable Subsystems: A Sequence-based Average Dwell Time Approach

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Abstract This paper proposes a new sequence-based approach to resolve the stability problems found in switched systems with unstable subsystems. In existing approaches, the sequence information of switching subsystems is seldom exploited. By exploiting the sequence information, threshold values can be less restrictive and more appropriate for the situation. We study two cases in this paper: (a) all subsystems are unstable, and (b) part of the subsystems are unstable. Both continuous-time and discrete-time systems are studied, and a numerical example is given to show the advantage of our approach.

Keywords Sequence-based average dwell time · unstable subsystems · switched systems

1 Introduction

A switched system is a dynamic system that consists of a family of subsystems and a rule that dictates how the system as a whole switches between the active subsystem [4]. It has attracted extensive research interest due to its potentially wide applications in practice and theory [2, 11, 15, 18, 22, 24, 28].

Stability is one of the fundamental problems of switched systems, and has been extensively studied in the past two decades [4, 5, 12, 13, 16, 19]. The feedback control for a discrete-time integrator with unitary delay was studied in [7]. The H_∞ filtering problem for a class of nonlinear switched systems with stable and unstable subsystems was discussed in [25]. The problem of asymptotic stability of continuous-time positive switched linear systems under both arbitrary and restricted switching was studied in [14]. The stabilizability of controlled discrete-time switched linear systems was analyzed in [3].

Recently, switched systems with unstable subsystems have also been studied [8, 9, 11, 15, 17, 25]. The input-to-state stability of switched nonlinear input delay systems under asynchronous switching was studied in [11]. Paper [9] investigated the asymptotic stability of Markov switched systems. Some stability results of a discrete-time switched system with unstable subsystems were presented in [17]. Robust adaptive tracking control schemes for uncertain switched linear systems subject to disturbances were investigated in [15].

In order to resolve the stability problem of switched systems, some new concepts have been proposed. The concepts of dwell time, average dwell time (ADT), and mode-dependent average dwell time (MDADT) were firstly introduced in [1, 6, 20]. The concept of sequence was proposed in [23] to resolve the stability problems when all subsystems are stable. However, it has not been applied to switched systems with unstable subsystems.

Based on these concepts, a large number of methods have been studied for switched systems with unstable subsystems. The approaches based on dwell time have some advantages that can be computed and verified. They have been widely applied to switched systems with unstable subsystems [12, 13, 16, 19]. A new method

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for the computation of ADT was proposed in [18]. By developing a novel Lyapunov function approach and exploring the features of mode-dependent dwell time switching, some new stability conditions were established in [19]. Based on mode-dependence dwell time approaches, a sufficient condition ensuring the asymptotic stability of switched continuous-time systems with all modes unstable was presented in [12]. Based on the delay-dependent average dwell-time approach, sufficient conditions for stability were derived and formulated in [16]. By using MDADT techniques, some less conservative stability conditions were derived in paper [13]. However, the stability problems of switched systems have not been resolved thoroughly; current methods are too conservative in nature and result in unnecessary drops in performance. In order to reduce conservativeness, much effort is still needed.

Exploiting the sequence information is a new direction of reducing conservativeness. For the stable subsystems, using the sequence information can get a smaller dwell time threshold value which can lead to a reduction of constraints involving average dwell time for the stable subsystems. Superfluous constraints are often what cause a system to behave more conservative than necessary. In this paper we utilize these same techniques for unstable subsystems, using the sequence information to enlarge the dwell time interval. The goal of this is to relax average dwell time constraints and improve the overall response of the system.

In this paper, we study a novel approach for the stability analysis of switched systems with unstable subsystems. The main contributions of this paper are: (a) proposing a fast switching concept for the sequence-based method, and (b) proposing a novel approach based on switching sequences to analyze the stability of switched systems with unstable subsystems. By exploiting the sequence information, our new approach can reduce conservativeness, release some constraints and obtain better threshold values compared to existing approaches.

The rest of this paper is organized as follows: In section 2 we introduce some basic concepts of switched systems. Our main stability analysis results are presented in Section 3 for two cases: when all subsystems are unstable, and only part of the subsystems are unstable. In Section 4, we provide a comparative study between our approach and two existing ones.

Notation: For a switched system with m subsystems, its switching signal is represented as function $\sigma(t)(or\ \sigma(k)) : [0,+\infty) \to \mathcal{M} = \{1, 2, \cdots, m\}$. The symbol "$\times$" represents the multiplication operation or Cartesian product of sets. The symbol "$\to 0^+(1^+)$" means any value approaching 0 (1) from the right side. For any given matrix P, $P > 0$ claims this matrix is symmetric and positive definite (or if $P < 0$ it is considered negative definite). The superscript "T" denotes the matrix transpose. The class $\mathcal{K}_\infty$ function κ denotes that the function $\kappa : [0,\infty) \to [0,\infty)$, $\kappa(0) = 0$, is unbounded, strictly increasing and continuous. We use the time $t_1,\ t_2, \cdots$ to stand for the switching time of continuous-time subsystems, and $k_1,\ k_2, \cdots$ for the switching time of discrete-time subsystems. The flags t_i^- represents the time approaching subsystems switching time t_i from the left side. The mark $[p|q]$ denotes the situation that the p^{th} subsystem is instantly activated after the q^{th} subsystem.

2 Preliminaries

In this paper, both linear and nonlinear systems are considered. For nonlinear systems, we consider the following system models:
discrete-time switched systems

$$x(k+1) = f_{\sigma(k)}\left(x(k)\right),\ x(k_0) = x_0; \tag{1}$$

continuous-time switched systems

$$\dot{x}(t) = f_{\sigma(t)}\left(x(t)\right),\ x(t_0) = x_0. \tag{2}$$

The symbol $x(k)\ (or\ x(t)) \in \mathbb{R}^n$ denotes a n-dimension state vector. For the initial time k_0 or t_0, we set $x(k_0) = x_0$ and $x(t_0) = x_0$, where x_0 is known as the initial state. We use $\sigma(k)$ (or $\sigma(t)$) to denote the switching signal which is a piecewise continuous function. Its range is the finite set $\mathcal{M} = \{1, \ldots, m\}$ where m is the number of subsystems.

Correspondingly, the following linear switched system models are considered:
discrete-time switched systems

$$x(k+1) = A_{\sigma(k)}x(k),\ x(k_0) = x(0); \tag{3}$$

continuous-time switched systems

$$\dot{x}(t) = A_{\sigma(t)}x(t),\ x(t_0) = x(0). \tag{4}$$

In order to obtain a new solution for the stability problem of switched systems (1)-(4), we first introduce some concepts about switched systems.

Definition 1 *[23] For a given linear or nonlinear discrete-time switched system with a switching signal $\sigma(k)$ and any switching time $k_1,\ k_2$, where $k_2 > k_1 \geq 0$, let $N_{\sigma[p|q]}(k_2, k_1)$ represent the number of the sequence that the p^{th} subsystem is activated immediately after the q^{th} subsystem over the time interval $[k_1, k_2)$. The symbol*

$T_{p,[p|q]}(k_2,k_1)$ *(or* $T_{q,[p|q]}(k_2,k_1)$*) stands for the total running time of the* p^{th}*(or the* q^{th}*) subsystem under this circumstance. It is said that:*
1) $\sigma(k)$ *has a slow sequence-based average subsequent dwell time (SBASDT)* $\tau_{a(p,[p|q])}$ *if there exist two numbers,* $\tau_{a(p,[p|q])}$ *and* $N_{0(p,[p|q])}$*, where* $N_{0(p,[p|q])}$ *is called the sequence-based subsequent slowly switching chatter bound here, such that:*

$$N_{\sigma[p|q]}(k_2,k_1) \leq N_{0(p,[p|q])} + \frac{T_{p,[p|q]}(k_2,k_1)}{\tau_{a(p,[p|q])}}; \tag{5}$$

2) $\sigma(k)$ *has a slow sequence-based average preceding dwell time (SBAPDT)* $\tau_{a(q,[p|q])}$ *if there exist two numbers,* $\tau_{a(q,[p|q])}$ *and* $N_{0(q,[p|q])}$*, where* $N_{0(q,[p|q])}$ *is called the sequence-based preceding slowly switching chatter bound here, such that:*

$$N_{\sigma[p|q]}(k_2,k_1) \leq N_{0(q,[p|q])} + \frac{T_{q,[p|q]}(k_2,k_1)}{\tau_{a(q,[p|q])}}. \tag{6}$$

According to [23], Definition 1 is known for the sequence-based slowly switching. In [23], the methods based on this definition were applied to asynchronous switched systems where all subsystems are stable. Below, we introduce a new concept for fast switching.

Definition 2 *For a given linear or nonlinear discrete-time switched system with a switching signal* $\sigma(k)$ *and any switching time* k_1*,* k_2*, where* $k_2 > k_1 \geq 0$*, let* $N^c_{\sigma[p|q]}(k_2,k_1)$ *represent the number of the sequences that the* p^{th} *subsystem is activated immediately after the* q^{th} *subsystem over the time interval* $[k_1,k_2)$*. The symbol* $T^c_{p,[p|q]}(k_2,k_1)$ *(or* $T^c_{q,[p|q]}(k_2,k_1)$*) stands for the total running time of the* p^{th}*(or* q^{th}*) subsystem under this circumstance. It is said that*
(1) $\sigma(k)$ *has a fast SBASDT* $\tau^c_{a(p,[p|q])}$ *if there exist two numbers,* $\tau^c_{a(p,[p|q])}$ *and* $N^c_{0(p,[p|q])}$*, where* $N^c_{0(p,[p|q])}$ *is called the sequence-based subsequent fast switching chatter bound here, such that:*

$$N^c_{\sigma[p|q]}(k_2,k_1) \geq N^c_{0(p,[p|q])} + \frac{T^c_{p,[p|q]}(k_2,k_1)}{\tau^c_{a(p,[p|q])}}; \tag{7}$$

(2) $\sigma(k)$ *has a fast SBAPDT* $\tau^c_{a(q,[p|q])}$ *if there exist two numbers,* $\tau^c_{a(q,[p|q])}$ *and* $N^c_{0(q,[p|q])}$*, where* $N^c_{0(q,[p|q])}$ *is called the sequence-based preceding fast switching chatter bound here, such that:*

$$N^c_{\sigma[p|q]}(k_2,k_1) \geq N^c_{0(q,[p|q])} + \frac{T^c_{q,[p|q]}(k_2,k_1)}{\tau^c_{a(q,[p|q])}}. \tag{8}$$

Similar definitions can be given for continuous switched systems, but are not explicitly provided in this paper.

3 Stability Analysis

In this section, we consider two cases: (a) all modes are unstable; and (b) only part of the subsystems are unstable.

3.1 All Modes Are Unstable

In this subsection, we consider the case that all subsystems are unstable. We study discrete systems first and then continuous systems.

3.1.1 Discrete systems

For the discrete-time switched system (1), we can get the following Lemma 1 can be derived using the fast sequence-based average dwell time approach.

Lemma 1 *Consider the nonlinear discrete-time switched system (1) with the given constants* $\varsigma_p > 0$*,* $\varsigma_q > 0$*,* $0 < \mu_{[p|q]} < 1$*. Suppose that there exist* $\mathcal{C}^1$ *functions* $V_{\sigma(k)} : \mathbb{R}^n \to \mathbb{R}$*, and class* $\mathcal{K}_\infty$ *functions* $\kappa_{p1}, \kappa_{p2}, \kappa_{q1}, \kappa_{q2}$*, such that* $\forall(\sigma(k_i) = p, \sigma(k_i - 1) = q) \in \mathcal{M} \times \mathcal{M}$*,* $p \neq q$*,*

$$\begin{cases} \kappa_{p1}(\parallel x(k) \parallel) \leq V_p(x(k),k) \leq \kappa_{p2}(\parallel x(k) \parallel), \\ \kappa_{q1}(\parallel x(k) \parallel) \leq V_q(x(k),k) \leq \kappa_{q2}(\parallel x(k) \parallel), \end{cases} \tag{9}$$

$$\begin{cases} V_p(x(k+1),k+1) - V_p(x(k),k) \leq \varsigma_p V_p(x(k),k), \\ V_q(x(k+1),k+1) - V_q(x(k),k) \leq \varsigma_q V_q(x(k),k), \end{cases} \tag{10}$$

$$V_p(x(k_i), k_i) \leq \mu_{[p|q]} V_q(x(k_i), k_i), \tag{11}$$

then the system is globally uniformly asymptotically stable for any switching signals with fast SBASDT

$$\tau^c_{a(p,[p|q])} < -\frac{\ln \mu_{[p|q]}}{\ln(1+\varsigma_p)}, \tag{12}$$

or with fast SBAPDT

$$\tau^c_{a(q,[p|q])} < -\frac{\ln \mu_{[p|q]}}{\ln(1+\varsigma_q)}. \tag{13}$$

Proof: For any $k > 0$ and $\forall k \in [k_i, k_{i+1})$, $i \in Z_+$: Let $k_0 = 0$, the set $\mathcal{S}'' \triangleq \{(p,q) : p = \sigma(k_j), q = \sigma(k_j - 1), j = 1, 2, 3, \cdots, i\}$, $V_{\sigma(k)}(k)$ denote $V_{\sigma(k)}(x(k), k)$, and $\overline{\varsigma}$ stand for $1 + \varsigma$. According to (9)-(11), one can get

$$\begin{aligned} & V_{\sigma(k)}(k) \\ & \leq (1+\varsigma_{\sigma(k_i)})^{(k-k_i)} V_{\sigma(k_i)}(k_i) \\ & \leq \mu_{\sigma(k_i)} (1+\varsigma_{\sigma(k_i)})^{(k-k_i)} V_{\sigma(k_i-1)}(k_i) \\ & \leq \mu_{\sigma(k_i)} \overline{\varsigma}^{(k-k_i)}_{\sigma(k_i)} \overline{\varsigma}^{(k_i-k_{i-1})}_{\sigma(k_{i-1})} V_{\sigma(k_{i-1})}(k_{i-1}). \end{aligned} \tag{14}$$

It follows that

$$V_{\sigma(k)}(k) \leq \prod_{j=1}^{i} \mu_{\sigma(k_j)} \overline{\varsigma}^{(k-k_i)}_{\sigma(k_i)} \cdots \overline{\varsigma}^{(k_1-0)}_{\sigma(0)} V_{\sigma(0)}(0). \tag{15}$$

If $\sigma(k_j) = p$ and $\sigma(k_j - 1) = q$, then $\mu_{\sigma(k_j)}$ is represented by $\mu_{[p|q]}$ to reveal the switching sequence.

We sort all the elements in the set $\mathcal{S}''$, and represent the ordered elements as $[p|q]_{(g)}$ which means (p,q) is the g^{th} element of the set $\mathcal{S}''$. The amount of all the elements of $\mathcal{S}''$ is given by s''.

Let $N^c_{\sigma[p|q]_{(g)}}(k,0)$, $T^c_{p,[p|q]_{(g)}}(k,0)$ and $T^c_{q,[p|q]_{(g)}}(k,0)$ denote the activated numbers, the total subsequent dwell time, and the total proceeding dwell time of the g^{th} element in the time interval $[0,k)$ for fast switching, respectively.

Therefore, it follows that

$$V_{\sigma(k)}(k) \leq \left\{ \prod_{g=1}^{s''} \mu_{[p|q]_{(g)}}^{N^c_{\sigma[p|q]_{(k)}}(k,0)} \overline{\varsigma}_{p,[p|q]_{(g)}}^{T^c_{p,[p|q]_{(g)}}(k,0)} \right\} \overline{\varsigma}^{(k_1)}_{\sigma(0)} V_{\sigma(0)}(x(0)). \tag{16}$$

Next, we provide the proofs for the SBASDT and SBAPDT switching, separately.

(a) SBASDT switching:

According to (7) and $T^c_{p,[p|q]_{(g)}}(k,0) = T^c_{p,[p|q]_{(g)}}(k,k_1)$, one can obtain

$$\begin{aligned} & V_{\sigma(k)}(k) \\ & \leq \prod_{g=1}^{s''} \mu_{[p|q]_{(g)}}^{N^c_{0(p,[p|q]_{(g)})} + \frac{T^c_{p,[p|q]_{(g)}}(k,0)}{\tau^c_{a(p,[p|q]_{(g)})}}} \overline{\varsigma}_{p,[p|q]_{(g)}}^{T^c_{p,[p|q]_{(g)}}(k,k_1)} \overline{\varsigma}^{(k_1)}_{\sigma(0)} V_{\sigma(0)}(x(0)) \\ & = \prod_{g=1}^{s''} \mu_{[p|q]_{(g)}}^{N^c_{0(p,[p|q]_{(g)})}} \mu_{[p|q]_{(g)}}^{\frac{T^c_{p,[p|q]_{(g)}}(k,0)}{\tau^c_{a(p,[p|q]_{(g)})}}} \overline{\varsigma}_{p,[p|q]_{(g)}}^{T^c_{p,[p|q]_{(g)}}(k,k_1)} \overline{\varsigma}^{(k_1)}_{\sigma(0)} V_{\sigma(0)}(x(0)) \\ & = \prod_{g=1}^{s''} \mu_{[p|q]_{(g)}}^{N^c_{0(p,[p|q]_{(g)})}} \left(\mu_{[p|q]_{(g)}}^{\frac{1}{\tau^c_{a(p,[p|q]_{(g)})}}} \overline{\varsigma}_{p,[p|q]_{(g)}}\right)^{T^c_{p,[p|q]_{(g)}}(k,k_1)} \overline{\varsigma}^{(k_1)}_{\sigma(0)} V_{\sigma(0)}(x(0)). \end{aligned} \tag{17}$$

We make the following definitions:

$$\gamma_1 \triangleq \max_g \left\{ \left(\mu_{[p|q]_{(g)}}^{\frac{1}{\tau^c_{a(p,[p|q]_{(g)})}}} \overline{\varsigma}_{p,[p|q]_{(g)}}\right) \right\},$$

$$K_1 \triangleq \prod_{g=1}^{s''} \mu_{[p|q]_{(g)}}^{N^c_{0(p,[p|q]_{(g)})}} \gamma_1^{-k_1} \overline{\varsigma}^{k_1}_{\sigma(0)}.$$

According to the switching condition (12) and $\varsigma_p > 0$, we know that $0 < \gamma_1 < 1$.

Therefore, we have

$$V_{\sigma(k)}(k) \leq K_1 \gamma_1^{(k-k_0)} V_{\sigma(0)}(x(0)). \tag{18}$$

Therefore, the switched system is globally uniformly asymptotically stable.

(b) SBAPDT switching:
According to (8), (16) and $T^c_{q,[p|q]_{(g)}}(k,0) = T^c_{q,[p|q]_{(g)}}(k_i,0)$, one can obtain

$$\begin{aligned} & V_{\sigma(k)}(k) \\ \leq & \prod_{g=1}^{s''} \mu_{[p|q]_{(g)}}^{N^c_{0(q,[p|q]_{(g)})}+\frac{T^c_{q,[p|q]_{(g)}}(k,0)}{\tau^c_{a(q,[p|q]_{(g)})}}} \bar{\varsigma}_{q,[p|q]_{(g)}}^{T^c_{q,[p|q]_{(g)}}(k_i,0)} \bar{\varsigma}_{\sigma(k_i)}^{(k-k_i)} V_{\sigma(0)}(x(0)) \\ = & \prod_{g=1}^{s''} \mu_{[p|q]_{(g)}}^{N^c_{0(q,[p|q]_{(g)})}} \mu_{[p|q]_{(g)}}^{\frac{T^c_{q,[p|q]_{(g)}}(k,0)}{\tau^c_{a(q,[p|q]_{(g)})}}} \bar{\varsigma}_{q,[p|q]_{(g)}}^{T^c_{q,[p|q]_{(g)}}(k_i,0)} \bar{\varsigma}_{\sigma(k_i)}^{(k-k_i)} V_{\sigma(0)}(x(0)) \\ = & \prod_{g=1}^{s''} \mu_{[p|q]_{(g)}}^{N^c_{0(q,[p|q]_{(g)})}} (\mu_{[p|q]_{(g)}}^{\frac{1}{\tau^c_{a(q,[p|q]_{(g)})}}} \bar{\varsigma}_{q,[p|q]_{(g)}})^{T^c_{q,[p|q]_{(g)}}(k_i,0)} \bar{\varsigma}_{\sigma(k_i)}^{(k-k_i)} V_{\sigma(0)}(x(0)). \end{aligned} \tag{19}$$

We make the following definitions:

$$\gamma_2 \triangleq \max_g \left\{ (\mu_{[p|q]_{(g)}}^{\frac{1}{\tau^c_{a(q,[p|q]_{(g)})}}} \bar{\varsigma}_{q,[p|q]_{(g)}}) \right\}, \tag{20}$$

$$K_2 \triangleq \prod_{g=1}^{s''} \mu_{[p|q]_{(g)}}^{N^c_{0(q,[p|q]_{(g)})}} \gamma_2^{-(k-k_i)} \bar{\varsigma}_{\sigma(k_i)}^{(k-k_i)}. \tag{21}$$

From the switching condition (13) and $\varsigma_p > 0$, we can know that $0 < \gamma_2 < 1$, which leads to

$$V_{\sigma(k)}(k) \leq K_2 \gamma_2^{(k-k_0)} V_{\sigma(0)}(x(0)). \tag{22}$$

Therefore, the switched system is globally uniformly asymptotically stable.
Combining (a) and (b), this lemma is proved. □
If the considered system is linear, we have the following theorem.

Theorem 1 *Consider the system (3). Let* $\varsigma_p > 0$, $\varsigma_q > 0$, $0 < \mu_{[p|q]} < 1$, $\tau_{min,p} = \min\limits_{\sigma(k_i)=p}(k_i - k_{i-1})$, $\tau_{min,q} = \min\limits_{\sigma(k_i)=q}(k_i - i_{i-1})$ *be given constants. If there exists a set of positive definite matrices* $P_{p,j_p} > 0$, $P_{q,j_q} > 0$, $j_p = 0, 1, \cdots, \tau_{min,p}$, $j_q = 0, 1, \cdots, \tau_{min,q}$, *such that* $\forall(\sigma(k_i) = p, \sigma(k_i - 1) = q) \in \mathcal{M} \times \mathcal{M}$, $p \neq q$, $\forall j_p = 0, 1, \cdots, \tau_{min,p} - 1$, $\forall j_q = 0, 1, \cdots, \tau_{min,q} - 1$,

$$\begin{cases} \begin{bmatrix} (\varsigma_p + 1)P_{p,j_p+1}, & A_p^T P_{p,j_p} \\ * & P_{p,j_p} \end{bmatrix} > 0, \\ \begin{bmatrix} (\varsigma_p + 1)P_{p,\tau_{min,p}}, & A_p^T P_{p,\tau_{min,p}} \\ * & P_{p,\tau_{min,p}} \end{bmatrix} > 0, \end{cases} \tag{23}$$

$$\begin{cases} \begin{bmatrix} (\varsigma_q + 1)P_{q,j_q+1}, & A_q^T P_{q,j_q} \\ * & P_{q,j_q} \end{bmatrix} > 0, \\ \begin{bmatrix} (\varsigma_q + 1)P_{q,\tau_{min,q}}, & A_q^T P_{q,\tau_{min,q}} \\ * & P_{q,\tau_{min,q}} \end{bmatrix} > 0, \end{cases} \tag{24}$$

$$P_{p,0} \leq \mu_{[p|q]} P_{q,\tau_{min,p}}, \tag{25}$$

then the system is globally uniformly asymptotically stable for any switching signals with fast SBASDT satisfying (12) or with fast SBAPDT satisfying (13).

Proof: For the multiple Lyapunov functions for linear discrete-time switched systems, (3) can be rewritten as

$$V_{\sigma(k)}(k) = x^T(k) P_{\sigma(k)}(k) x(k), \sigma(k) \in \mathcal{M}. \tag{26}$$

For $\forall k \in [k_i, k_{i+1})$, the matrix $P_{\sigma(k)}(k)$ is defined as

$$P_{\sigma(k)}(k) = \begin{cases} P_{\sigma(t_i)}(k - k_i), & k \in [k_i, k_i + \tau_{min,\sigma(k_i)}) \\ P_{\sigma(t_i)}(\tau_{min,\sigma(k_i)}), & k \in [k_i + \tau_{min,\sigma(k_i)}, k_{i+1}) \end{cases}. \tag{27}$$

We rewrite $P_{\sigma(k_i)}(k - k_i)$ and $P_{\sigma(k_i)}(\tau_{min,\sigma(k_i)})$ as $P_{\sigma(k_i),k-k_i}$ and $P_{\sigma(k_i),\tau_{min,\sigma(k_i)}}$.
The condition (23) implies (9). The condition (24) means (10). Because of (25), we can get (11). □

3.1.2 Continuous systems

For the continuous-time systems, we can get Lemma 2 and Theorem 2 below. They are similar to Lemma 1 and Theorem 1.

Lemma 2 *Consider the system (2). Let $0 < \mu_{[p|q]} < 1, \alpha_p > 0, \alpha_q > 0$ be the given constants. Suppose that there exist $\mathbb{C}^1$ functions $V_{\sigma(t)} : \mathbb{R}^n \to \mathbb{R}$, and class $\mathcal{K}_\infty$ functions κ_{p1}, κ_{p2}, κ_{q1} and κ_{q2}, such that $\forall(\sigma(t_i) = p, \sigma(t_i^-) = q) \in \mathcal{M} \times \mathcal{M}$ and $p \neq q$,*

$$\begin{cases} \kappa_{p1}(\| x(t) \|) \leq V_p(x(t), t) \leq \kappa_{p2}(\| x(t) \|), \\ \kappa_{q1}(\| x(t) \|) \leq V_q(x(t), t) \leq \kappa_{q2}(\| x(t) \|), \end{cases} \tag{28}$$

$$\begin{cases} \dot{V}_p(x(t), t) \leq \alpha_p V_p(x(t), t), \\ \dot{V}_q(x(t), t) \leq \alpha_q V_q(x(t), t), \end{cases} \tag{29}$$

and

$$V_p(x(t_i), t_i) \leq \mu_{[p|q]} V_q(x(t_i), t_i), \tag{30}$$

then the system is globally uniformly asymptotically stable for any switching signals with fast SBASDT

$$\tau_{a(p,[p|q])} < \tau^*_{a(p,[p|q])} = -ln\mu_{[p|q]}/\alpha_p, \tag{31}$$

or with fast SBAPDT

$$\tau_{a(q,[p|q])} < \tau^*_{a(q,[p|q])} = -ln\mu_{[p|q]}/\alpha_q. \tag{32}$$

Theorem 2 *Consider the system (4). Let $0 < \mu_{[p|q]} < 1$, $\alpha_p > 0$, $\alpha_q > 0$, $\tau_{min,p} = \min\limits_{\sigma(t_i)=p}(t_i - t_{i-1})$, $\tau_{min,q} = \min\limits_{\sigma(t_i)=q}(t_i - t_{i-1})$ and l_p, l_q be the given constants. If there exists a set of positive definite matrices $P_{p,j_p} > 0$, $P_{q,j_q} > 0$, $j_p = 0,\ 1, \cdots,\ l_p$, $j_q = 0,\ 1, \cdots,\ l_q$, such that $\forall(\sigma(t_i) = p, \sigma(t_i^-) = q) \in \mathcal{M} \times \mathcal{M}$, $p \neq q$, $\forall j_p = 0,\ 1, \cdots,\ l_p - 1$, $\forall j_q = 0,\ 1, \cdots,\ l_q - 1$,*

$$\begin{cases} A_p^T P_{p,j_p} + P_{p,j_p} A_p + \frac{l_p(P_{p,j_p+1} - P_{p,j_p})}{\tau_{min,p}} \leq \alpha_p P_{p,j_p}, \\ A_p^T P_{p,j_p+1} + P_{p,j_p+1} A_p + \frac{l_p(P_{p,j_p+1} - P_{p,j_p})}{\tau_{min,p}} \leq \alpha_p P_{p,j_p+1}, \\ A_p^T P_{p,l_p} + P_{p,l_p} A_p \leq \alpha_p P_{p,l_p}, \end{cases} \tag{33}$$

$$\begin{cases} A_q^T P_{q,j_q} + P_{q,j_q} A_q + \frac{l_q(P_{q,j_q+1} - P_{q,j_q})}{\tau_{min,q}} \leq \alpha_q P_{q,j_q}, \\ A_q^T P_{q,j_q+1} + P_{q,j_q+1} A_q + \frac{l_q(P_{q,j_q+1} - P_{q,j_q})}{\tau_{min,q}} \leq \alpha_q P_{q,j_q+1}, \\ A_q^T P_{q,l_q} + P_{q,l_q} A_q \leq \alpha_q P_{q,l_q}, \end{cases} \tag{34}$$

$$P_{p,0} \leq \mu_{[p|q]} P_{q,l_p}, \tag{35}$$

then the system is globally uniformly asymptotically stable for any switching signals with SBASDT satisfying (31) or with SBAPDT satisfying (32).

Remark 1 *For the sequence-based approach, it just requires $0 < \mu_{[p|q]} < 1$, when $\sigma(t_i) = p, \sigma(t_i^-) = q$, $i \in Z_+$. If some sequences do not appear, there is no constraint on their μ values. For example, Fig.1 shows the periodically switched systems. For this kind of switched systems, we do not need to check whether $\mu_{[3|1]}$,$\mu_{[2|3]}$ $\mu_{[1|2]}$ are less than 1 or not.*

3.2 Part of the Subsystems Are Unstable

In this subsection, we consider the switched systems with both unstable and stable subsystems. In this case, it means $\mathcal{M} = \mathcal{S} \cup \mathcal{U}$, where $\mathcal{S}$ and $\mathcal{U}$ denote the set of stable and unstable subsystems, respectively.

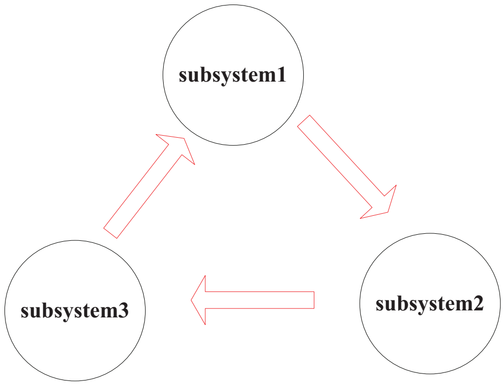

Fig. 1: A periodically switched system.

3.2.1 Discrete systems

For a discrete-time switched system (1), if the unstable subsystems are fast switching and the stable systems are slowly switching, we can get Lemmas 3,4 and Theorems 3,4 below.

Lemma 3 *Consider the system (1), and let ς_p, ς_q, $\mu_{[p|q]}$, be given constants. Suppose that there exist $\mathcal{C}^1$ functions $V_{\sigma(k)}:\mathbb{R}^n\rightarrow\mathbb{R}$, and class $\mathcal{K}_\infty$ functions κ_{q1},κ_{q2}, κ_{p1},κ_{p2}, such that $\forall(\sigma(k_i)=p,\sigma(k_i-1)=q)\in\mathcal{M}\times\mathcal{M}$, $p\neq q$,*

$$\begin{cases}\kappa_{p1}(\parallel x(k)\parallel)\leq V_p(x(k),k)\leq\kappa_{p2}(\parallel x(k)\parallel),\\ \kappa_{q1}(\parallel x(k)\parallel)\leq V_q(x(k),k)\leq\kappa_{q2}(\parallel x(k)\parallel),\end{cases}\tag{36}$$

$$\begin{cases}V_p(x(k+1),k+1)-V_p(x(k),k)\leq\varsigma_pV_p(x(k),k),\\ V_q(x(k+1),k+1)-V_q(x(k),k)\leq\varsigma_qV_q(x(k),k),\end{cases}\tag{37}$$

$$V_p(x(k_i),k_i)\leq\mu_{[p|q]}V_q(x(k_i),k_i).\tag{38}$$

Then the system is globally uniformly asymptotically stable for any switching signals with SBASDT

$$\begin{cases}\tau_{a(p,[p|q])}>\tau^*_{a(p,[p|q])}\triangleq-\frac{\ln\mu_{[p|q]}}{\ln(1+\varsigma_p)},(-1<\varsigma_p<0,\mu_{[p|q]}>1,p\in\mathcal{S}),\\ \tau^c_{a(p,[p|q])}<\tau^*_{a(p,[p|q])}\triangleq-\frac{\ln\mu_{[p|q]}}{\ln(1+\varsigma_p)},(\varsigma_p>0,0<\mu_{[p|q]}\leq1,p\in\mathcal{U}),\end{cases}\tag{39}$$

or with SBAPDT

$$\begin{cases}\tau_{a(q,[p|q])}>\tau^*_{a(q,[p|q])}\triangleq-\frac{\ln\mu_{[p|q]}}{\ln(1+\varsigma_q)},(-1<\varsigma_q<0,\mu_{[p|q]}>1,q\in\mathcal{S}).\\ \tau^c_{a(q,[p|q])}<\tau^*_{a(q,[p|q])}\triangleq-\frac{\ln\mu_{[p|q]}}{\ln(1+\varsigma_q)},(\varsigma_q>0,0<\mu_{[p|q]}\leq1,q\in\mathcal{U}).\end{cases}\tag{40}$$

Proof: For any $k>0$ and $\forall k\in[k_i,k_{i+1})$, $i\in Z_+$: Let $k_0=0$, the set $\mathcal{S}''=\{(p,q):p=\sigma(k_j),q=\sigma(k_j-1),j=1,2,3,\cdots,i\}$, $V_{\sigma(k)}(k)$ denote $V_{\sigma(k)}(x(k),k)$, and $\bar{\varsigma}$ stand for $1+\varsigma$. According to (36)-(38), one can get

$$\begin{aligned}&V_{\sigma(k)}(k)\\ &\leq(1+\varsigma_{\sigma(k_i)})^{(k-k_i)}V_{\sigma(k_i)}(k_i)\\ &\leq\mu_{\sigma(k_i)}(1+\varsigma_{\sigma(k_i)})^{(k-k_i)}V_{\sigma(k_i-1)}(k_i)\\ &\leq\mu_{\sigma(k_i)}\bar{\varsigma}_{\sigma(k_i)}^{(k-k_i)}\bar{\varsigma}_{\sigma(k_{i-1})}^{(k_i-k_{i-1})}V_{\sigma(k_{i-1})}(k_{i-1}).\end{aligned}\tag{41}$$

Finally, we can get

$$V_{\sigma(k)}(k)\leq\prod_{j=1}^{i}\mu_{\sigma(k_j)}\bar{\varsigma}_{\sigma(k_i)}^{(k-k_i)}\cdots\bar{\varsigma}_{\sigma(0)}^{(k_1-0)}V_{\sigma(0)}(0).\tag{42}$$

The symbols $N^c_{\sigma[p|q]_{(g)}}(k,0)$ ($N_{\sigma[p|q]_{(g)}}(k,0)$), $T^c_{p,[p|q]_{(g)}}(k,0)$ ($T_{p,[p|q]_{(g)}}(k,0)$) and $T^c_{q,[p|q]_{(g)}}(k,0)$ ($T_{q,[p|q]_{(g)}}(k,0)$) denote the activated numbers, total subsequent dwell time, and the total preceding dwell time of the g^{th} element in the time interval $[0,k)$ for fast (slowly) switching, respectively.

Next, we provide proof for the SBASDT and SBAPDT switching separately.
(a) The SBASDT switching:
According to (5) and (7), one can obtain

$$\begin{aligned}
&V_{\sigma(k)}(k)\\
&\leq \prod_{g=1,p\in\mathcal{S}}^{g=s''} \mu_{[p|q]_{(g)}}^{N_{0(p,[p|q]_{(g)})}+\frac{T_{p,[p|q]_{(g)}}(k,k_1)}{\tau_{a(p,[p|q]_{(g)})}}} \overline{\varsigma}_{p,[p|q]_{(g)}}^{T_{p,[p|q]_{(g)}}(k,k_1)}\\
&\times \prod_{g=1,p\in\mathcal{U}}^{g=s''} \mu_{[p|q]_{(g)}}^{N^c_{0(p,[p|q]_{(g)})}+\frac{T^c_{p,[p|q]_{(g)}}(k,k_1)}{\tau^c_{a(p,[p|q]_{(g)})}}} \overline{\varsigma}_{p,[p|q]_{(g)}}^{T^c_{p,[p|q]_{(g)}}(k,k_1)} \overline{\varsigma}_{\sigma(0)}^{(k_1)} V_{\sigma(0)}(x(0))\\
&= \prod_{g=1,p\in\mathcal{S}}^{g=s''} \mu_{[p|q]_{(g)}}^{N_{0(p,[p|q]_{(g)})}} \left(\mu_{[p|q]_{(g)}}^{\frac{1}{\tau_{a(p,[p|q]_{(g)})}}} \overline{\varsigma}_{p,[p|q]_{(g)}}\right)^{T_{p,[p|q]_{(g)}}(k,k_1)}\\
&\times \prod_{g=1,p\in\mathcal{U}}^{g=s''} \mu_{[p|q]_{(g)}}^{N^c_{0(p,[p|q]_{(g)})}} \left(\mu_{[p|q]_{(g)}}^{\frac{1}{\tau^c_{a(p,[p|q]_{(g)})}}} \overline{\varsigma}_{p,[p|q]_{(g)}}\right)^{T^c_{p,[p|q]_{(g)}}(k,k_1)} \overline{\varsigma}_{\sigma(0)}^{(k_1)} V_{\sigma(0)}(x(0)).
\end{aligned} \tag{43}$$

We give the following definitions:

$$\gamma_3 \triangleq \max_g \left\{ \mu_{[p|q]_{(g)}}^{\frac{1}{\tau_{a(p,[p|q]_{(g)})}}} \overline{\varsigma}_{p,[p|q]_{(g)}}, \mu_{[p|q]_{(g)}}^{\frac{1}{\tau^c_{a(p,[p|q]_{(g)})}}} \overline{\varsigma}_{p,[p|q]_{(g)}} \right\},$$

$$K_3 \triangleq \left\{ \prod_{g=1,p\in\mathcal{S}}^{g=s''} \mu_{[p|q]_{(g)}}^{N_{0(p,[p|q]_{(g)})}} \right\} \left\{ \prod_{g=1,p\in\mathcal{U}}^{g=s''} \mu_{[p|q]_{(g)}}^{N^c_{0(p,[p|q]_{(g)})}} \right\} \gamma_3^{-k_1} \overline{\varsigma}_{\sigma(0)}^{k_1}.$$

According to the switching condition (39), we know that $0 < \gamma_3 < 1$. It follows that

$$V_{\sigma(k)}(k) \leq K_3 \gamma_3^{(k-k_0)} V_{\sigma(0)}(x(0)). \tag{44}$$

Therefore, the switched system is globally uniformly asymptotically stable.

(b) The SBASDT switching:
According to (6) and (8), one can obtain

$$\begin{aligned}
&V_{\sigma(k)}(k)\\
&\leq \prod_{g=1,q\in\mathcal{S}}^{g=s''} \mu_{[p|q]_{(g)}}^{N_{0(q,[p|q]_{(g)})}+\frac{T_{q,[p|q]_{(g)}}(k_i,0)}{\tau_{a(q,[p|q]_{(g)})}}} \overline{\varsigma}_{q,[p|q]_{(g)}}^{T_{q,[p|q]_{(g)}}(k_i,0)}\\
&\times \prod_{g=1,q\in\mathcal{U}}^{g=s''} \mu_{[p|q]_{(g)}}^{N^c_{0(q,[p|q]_{(g)})}+\frac{T^c_{q,[p|q]_{(g)}}(k_i,0)}{\tau^c_{a(q,[p|q]_{(g)})}}} \overline{\varsigma}_{q,[p|q]_{(g)}}^{T^c_{q,[p|q]_{(g)}}(k_i,0)} \overline{\varsigma}_{\sigma(k-k_i)}^{(k-k_i)} V_{\sigma(0)}(x(0))\\
&= \prod_{g=1,q\in\mathcal{S}}^{g=s''} \mu_{[p|q]_{(g)}}^{N_{0(q,[p|q]_{(g)})}} \left(\mu_{[p|q]_{(g)}}^{\frac{1}{\tau_{a(q,[p|q]_{(g)})}}} \overline{\varsigma}_{q,[p|q]_{(g)}}\right)^{T_{q,[p|q]_{(g)}}(k_i,0)}\\
&\times \prod_{g=1,q\in\mathcal{U}}^{g=s''} \mu_{[p|q]_{(g)}}^{N^c_{0(q,[p|q]_{(g)})}} \left(\mu_{[p|q]_{(g)}}^{\frac{1}{\tau^c_{a(q,[p|q]_{(g)})}}} \overline{\varsigma}_{q,[p|q]_{(g)}}\right)^{T^c_{q,[p|q]_{(g)}}(k_i,0)} \overline{\varsigma}_{\sigma(k-k_i)}^{(k-k_i)} V_{\sigma(0)}(x(0)).
\end{aligned} \tag{45}$$

We give the following definitions:

$$\gamma_4 \triangleq \max_g \left\{ \mu_{[p|q]_{(g)}}^{\frac{1}{\tau_{a(q,[p|q]_{(g)})}}} \overline{\varsigma}_{q,[p|q]_{(g)}}, \mu_{[p|q]_{(g)}}^{\frac{1}{\tau^c_{a(q,[p|q]_{(g)})}}} \overline{\varsigma}_{q,[p|q]_{(g)}} \right\}$$

$$K_4 \triangleq \left\{ \prod_{g=1,q\in\mathcal{S}}^{g=s''} \mu_{[p|q]_{(g)}}^{N_{0(q,[p|q]_{(g)})}} \right\} \left\{ \prod_{g=1,q\in\mathcal{U}}^{g=s''} \mu_{[p|q]_{(g)}}^{N^c_{0(q,[p|q]_{(g)})}} \right\} \gamma_4^{-(k-k_i)} \overline{\varsigma}_{\sigma(k_i)}^{(k-k_i)}$$

According to the switching condition (40), we can know that $0 < \gamma_4 < 1$. It follows that

$$V_{\sigma(k)}(k) \leq K_4 \gamma_4^{(k-k_0)} V_{\sigma(0)}(x(0)). \tag{46}$$

Therefore, the switched system is globally uniformly asymptotically stable.
Combining (a) and (b), this lemma is proved.□

If the considered system is linear, we have Theorem 3.

Theorem 3 *Consider the system (3). Let ς_p, ς_q, $\mu_{[p|q]}$, $\tau_{min,p} = \min\limits_{\sigma(k_i)=p}(k_i - k_{i-1})$, $\tau_{min,q} = \min\limits_{\sigma(k_i)=q}(k_i - i_{i-1})$ be given constants. For $\forall(\sigma(k_i) = p, \sigma(k_i - 1) = q) \in \mathcal{M} \times \mathcal{M}$, $p \neq q$, assume there exists a set of positive definite matrices satisfying:*
(a) if $p \in \mathcal{U}$, then $P_{p,\tau_{min,p}} > 0$, $P_{p,j_p} > 0$, $j_p = 0,\ 1, \cdots,\ \tau_{min,p} - 1$, and

$$\begin{cases} \begin{bmatrix} (\varsigma_p + 1)P_{p,j_p+1}, & A_p^T P_{p,j_p} \\ * & P_{p,j_p} \end{bmatrix} > 0 \\ \begin{bmatrix} (\varsigma_p + 1)P_{p,\tau_{min,p}}, & A_p^T P_{p,\tau_{min,p}} \\ * & P_{p,\tau_{min,p}} \end{bmatrix} > 0 \end{cases}; \tag{47}$$

(b) if $p \in \mathcal{S}$, then $P_p > 0$ and

$$\begin{bmatrix} (\varsigma_p + 1)P_p & A_p^T P_p \\ * & P_p \end{bmatrix} > 0; \tag{48}$$

(c) if $q \in \mathcal{U}$, then $P_{q,\tau_{min,q}} > 0$, $P_{q,j_q} > 0$, $j_q = 0,\ 1, \cdots,\ \tau_{min,q} - 1$, and

$$\begin{cases} \begin{bmatrix} (\varsigma_q + 1)P_{q,j_q+1}, & A_q^T P_{q,j_q} \\ * & P_{q,j_q} \end{bmatrix} > 0 \\ \begin{bmatrix} (\varsigma_q + 1)P_{q,\tau_{min,q}}, & A_q^T P_{q,\tau_{min,q}} \\ * & P_{q,\tau_{min,q}} \end{bmatrix} > 0 \end{cases}; \tag{49}$$

(d) if $q \in \mathcal{S}$, then $P_q > 0$ and

$$\begin{bmatrix} (\varsigma_q + 1)P_q & A_q^T P_q \\ * & P_q \end{bmatrix} > 0, \tag{50}$$

and

$$P_{p,0} \leq \mu_{[p|q]} P_{q,\tau_{min,p}}. \tag{51}$$

Then for any switching signals with SBASDT satisfying (39) or with SBAPDT satisfying (40), the system (3) is globally uniformly asymptotically stable.

Proof: The proof of this theorem is largely similar to that of Theorem 1. The only additional requirement is:

$$\begin{cases} P_p = P_{p,j_p} > 0,\ j_p = 0,\ 1, \cdots,\ \tau_{min,p},\ p \in \mathcal{S} \\ P_q = P_{q,j_q} > 0,\ j_q = 0,\ 1, \cdots,\ \tau_{min,q},\ q \in \mathcal{S} \end{cases}.$$

It means that a constant P_p (or P_q) replaces all P_{p,j_p} (or P_{q,j_q}) if $p \in \mathcal{S}$ ($q \in \mathcal{S}$). □

For the continuous-time systems, we can get Lemma 4 and Theorem 4 below, and the proofs are similar to those for discrete-time systems.

Lemma 4 *Consider the system (2). Let $\mu_{[p|q]}$, α_p and α_q be given constants. Suppose that there exist class $\mathcal{K}_\infty$ functions κ_{p1}, κ_{p2}, κ_{q1}, and κ_{q2} and $\mathbb{C}^1$ functions $V_{\sigma(t)}(x(t)) : \mathbb{R}^n \to \mathbb{R}$, such that $\forall(\sigma(t_i) = p, \sigma(t_i^-) = q) \in \mathcal{M} \times \mathcal{M}$ and $p \neq q$,*

$$\begin{cases} \kappa_{p1}(\| x(t) \|) \leq V_p(x(t), t) \leq \kappa_{p2}(\| x(t) \|), \\ \kappa_{q1}(\| x(t) \|) \leq V_q(x(t), t) \leq \kappa_{q2}(\| x(t) \|), \end{cases} \tag{52}$$

$$\begin{cases} \dot{V}_p(x(t), t) \leq \alpha_p V_p(x(t), t), \\ \dot{V}_q(x(t), t) \leq \alpha_q V_q(x(t), t), \end{cases} \tag{53}$$

$$V_p(x(t_i), t_i) \leq \mu_{[p|q]} V_q(x(t_i), t_i). \tag{54}$$

Then for any switching signals with SBASDT

$$\begin{cases} \tau_{a(p,[p|q])} > \tau^*_{a(p,[p|q])} \triangleq -ln\mu_{[p|q]}/\alpha_p, (\alpha_p < 0, \mu_{[p|q]} > 1, p \in \mathcal{S}), \\ \tau^c_{a(p,[p|q])} < \tau^*_{a(p,[p|q])} \triangleq -ln\mu_{[p|q]}/\alpha_p, (\alpha_p > 0, 0 < \mu_{[p|q]} < 1, p \in \mathcal{U}), \end{cases} \tag{55}$$

or with SBAPDT

$$\begin{cases} \tau_{a(q,[p|q])} > \tau^*_{a(q,[p|q])} \triangleq -ln\mu_{[p|q]}/\alpha_q, (\alpha_q < 0, \mu_{[p|q]} > 1, q \in \mathcal{S}), \\ \tau^c_{a(q,[p|q])} < \tau^*_{a(q,[p|q])} \triangleq -ln\mu_{[p|q]}/\alpha_q, (\alpha_q > 0, 0 < \mu_{[p|q]} < 1, q \in \mathcal{U}), \end{cases} \tag{56}$$

the system is globally uniformly asymptotically stable.

Theorem 4 *Consider the system (4). Let* $\mu_{[p|q]}$, α_p, α_q, $\tau_{min,p} = \min\limits_{\sigma(t_i)=p}(t_i - t_{i-1})$, $\tau_{min,q} = \min\limits_{\sigma(t_i)=q}(t_i - t_{i-1})$ *and* l_p, l_q *be the given constants. For* $\forall(\sigma(t_i) = p, \sigma(t_i^-) = q) \in \mathcal{M} \times \mathcal{M}$, $p \neq q$, *assume that there exists a set of positive definite matrices satisfying:*
(a) if $p \in \mathcal{U}$ *then* $P_{p,l_p} > 0$, $P_{p,j_p} > 0$, $j_p = 0, 1, \cdots, l_p - 1$, *and*

$$\begin{cases} A_p^T P_{p,j_p} + P_{p,j_p} A_p + \frac{l_p(P_{p,j_p+1} - P_{p,j_p})}{\tau_{min,p}} \leq \alpha_p P_{p,j_p} \\ A_p^T P_{p,j_p+1} + P_{p,j_p+1} A_p + \frac{l_p(P_{p,j_p+1} - P_{p,j_p})}{\tau_{min,p}} \leq \alpha_p P_{p,j_p+1} \\ A_p^T P_{p,l_p} + P_{p,l_p} A_p \leq \alpha_p P_{p,l_p} \end{cases}; \tag{57}$$

(b) if $p \in \mathcal{S}$ *then* $P_p > 0$, *and*

$$A_p^T P_p + P_p A_p \leq \alpha_p P_p; \tag{58}$$

(c) if $q \in \mathcal{U}$ *then* $P_{q,l_q} > 0$, $P_{q,j_q} > 0$, $j_q = 0, 1, \cdots, l_q - 1$, *and*

$$\begin{cases} A_q^T P_{q,j_q} + P_{q,j_q} A_q + \frac{l_q(P_{q,j_q+1} - P_{q,j_q})}{\tau_{min,q}} \leq \alpha_q P_{q,j_q} \\ A_q^T P_{q,j_q+1} + P_{q,j_q+1} A_q + \frac{l_q(P_{q,j_q+1} - P_{q,j_q})}{\tau_{min,q}} \leq \alpha_q P_{q,j_q+1} \\ A_q^T P_{q,l_q} + P_{q,l_q} A_q \leq \alpha_q P_{q,l_q} \end{cases}; \tag{59}$$

(d) if $p \in \mathcal{S}$ *then* $P_p > 0$, *and*

$$A_q^T P_q + P_q A_q \leq \alpha_q P_q;$$

and

$$P_{p,0} \leq \mu_{[p|q]} P_{q,l_p}. \tag{60}$$

Then the system (4) is globally uniformly asymptotically stable for any switching signals with SBASDT satisfying (55) or with SBAPDT satisfying (56).

Remark 2 *In most existing methods that solve the stability problems of switched systems with both stable and unstable systems, the precondition is either there is a common Lyapunov function when unstable subsystems switch to unstable subsystems [19] or that unstable subsystems must be followed by a stable systems [17]. According to Theorem 3 and 4, both preconditions are not required any more.*

The preceding theorems in this subsections require $\mu_{[p|q]} < 1$ when the subsequent or preceding subsystem is unstable. This is a relatively strong requirement in some cases. In the following, we relax this requirement.

Lemma 5 *Consider the system (1). Let* ς_p, ς_q, $\mu_{[p|q]} > 1$ *be given constants. Suppose that there exist class* $\mathcal{K}_\infty$ *functions* $\kappa_{p1}, \kappa_{p2}, \kappa_{q1}, \kappa_{q2}$ *and* $\mathcal{C}^1$ *functions* $V_{\sigma(k)} : \mathbb{R}^n \to \mathbb{R}$, *such that* $\forall(\sigma(k_i) = p, \sigma(k_i - 1) = q) \in \mathcal{M} \times \mathcal{M}$, $p \neq q$,

$$\begin{cases} \kappa_{p1}(\| x(k) \|) \leq V_p(k) \leq \kappa_{p2}(\| x(k) \|), \\ \kappa_{q1}(\| x(k) \|) \leq V_q(k) \leq \kappa_{q2}(\| x(k) \|), \end{cases} \tag{61}$$

$$\begin{cases} V_p(k+1) - V_p(k) \leq \varsigma_p V_p(k), \\ V_q(k+1) - V_q(k) \leq \varsigma_q V_q(k), \end{cases} \tag{62}$$

$$V_p(k_i) \leq \mu_{[p|q]} V_q(k_i). \tag{63}$$

Then for any switching signals with SBASDT,

$$\begin{cases} \tau_{a(p,[p|q])} > -\frac{\ln \mu_{[p|q]}}{\ln(1+\varsigma_p)}, (-1 < \varsigma_p < 0, p \in \mathcal{S}), \\ \tau_{a(p,[p|q])} > \tau^*_{a(p,[p|q])}, (\forall \tau^*_{a(p,[p|q])} > 0, \varsigma_p > 0, p \in \mathcal{U}), \\ \frac{T^-}{T^+} > \frac{\ln \gamma_s^+ - \ln \gamma}{\ln \gamma - \ln \gamma_s^-}, (0 < \gamma_s^- < \gamma < 1), \end{cases} \tag{64}$$

or with SBAPDT

$$\begin{cases} \tau_{a(q,[p|q])} > -\frac{\ln \mu_{[p|q]}}{\ln(1+\varsigma_q)}, \ (-1 < \varsigma_q < 0, \ q \in \mathcal{S}), \\ \tau_{a(q,[p|q])} > \tau^*_{a(q,[p|q])}, (\forall \tau^*_{a(q,[p|q])} > 0, \varsigma_q > 0, q \in \mathcal{U}), \\ \frac{T^-}{T^+} > \frac{\ln \gamma_p^+ - \ln \gamma}{\ln \gamma - \ln \gamma_p^-}, (0 < \gamma_p^- < \gamma < 1), \end{cases} \tag{65}$$

the system is globally uniformly asymptotically stable with marginal γ, *where* T^- *and* T^+ *stand for the total running time of stable and unstable subsystems. The definitions of* γ_s^+, γ_p^+, γ_s^-, γ_p^- *are given in (69)-(70) and (76)-(77).*

Proof: For any $k > 0$ and $\forall k \in [k_i, k_{i+1})$, $i \in Z_+$: Let $k_0 = 0$, the set $\mathcal{S}'' \triangleq \{(p,q) : p = \sigma(k_j), q = \sigma(k_j - 1), j = 1,2,3,\cdots,i\}$, $V_{\sigma(k)}(k)$ denote $V_{\sigma(k)}(x(k))$, and $\bar{\varsigma}$ stand for $1+\varsigma$. According to (61)-(63), one can get the system (42).

Let $N_{\sigma[p|q]_{(g)}}(k,0)$, $T_{q,[p|q]_{(g)}}(k,0)$ and $T_{p,[p|q]_{(g)}}(k,0)$ denote the activated numbers, total proceeding dwell time and total subsequent dwell time of the g^{th} element in the time interval $[0,k)$ respectively.

Next, we provide proof for the SBASDT and SBAPDT switching separately.

(a) The SBASDT switching:

According to (5), one can obtain

$$\begin{aligned}
&V_{\sigma(k)}(k)\\
\leq &\prod_{g=1,p\in\mathcal{S}}^{g=s''} \mu_{[p|q]_{(g)}}^{N_{0(p,[p|q]_{(g)})}+\frac{T_{p,[p|q]_{(g)}}(k,k_1)}{\tau_{a(p,[p|q]_{(g)})}}} \bar{\varsigma}_{p,[p|q]_{(g)}}{}^{T_{p,[p|q]_{(g)}}(k,k_1)}\\
\times &\prod_{g=1,p\in\mathcal{U}}^{g=s''} \mu_{[p|q]_{(g)}}^{N_{0(p,[p|q]_{(g)})}+\frac{T_{p,[p|q]_{(g)}}(k,k_1)}{\tau_{a(p,[p|q]_{(g)})}}} \bar{\varsigma}_{p,[p|q]_{(g)}}{}^{T_{p,[p|q]_{(g)}}(k,k_1)} \bar{\varsigma}_{\sigma(0)}^{(k_1)} V_{\sigma(0)}(x(0))\\
= &\left\{\prod_{g=1}^{g=s''} \mu_{[p|q]_{(g)}}^{N_{0(p,[p|q]_{(g)})}}\right\}\left\{\prod_{g=1,p\in\mathcal{S}}^{g=s''} (\mu_{[p|q]_{(g)}}^{\frac{1}{\tau_{a(p,[p|q]_{(g)})}}} \bar{\varsigma}_{p,[p|q]_{(g)}})^{T_{p,[p|q]_{(g)}}(k,k_1)}\right\}\\
\times &\left\{\prod_{g=1,p\in\mathcal{U}}^{g=s''} (\mu_{[p|q]_{(g)}}^{\frac{1}{\tau_{a(p,[p|q]_{(g)})}}} \bar{\varsigma}_{p,[p|q]_{(g)}})^{T_{p,[p|q]_{(g)}}(k,k_1)}\right\}\left\{\bar{\varsigma}_{\sigma(0)}^{(k_1)} V_{\sigma(0)}\right\}(x(0)).
\end{aligned} \tag{66}$$

We make the following definitions:

$$T^- \triangleq \begin{cases} \sum\limits_{p\in\mathcal{S}} T_{p,[p|q]}(k,k_1) + k_1, \sigma(0)\in\mathcal{S}, \\ \sum\limits_{p\in\mathcal{S}} T_{p,[p|q]}(k,k_1), \sigma(0)\in\mathcal{U}, \end{cases} \triangleq \sum_{p\in\mathcal{S}} T_p(k,0), \tag{67}$$

$$T^+ \triangleq \begin{cases} \sum\limits_{p\in\mathcal{U}} T_{p,[p|q]}(k,k_1) + k_1, \sigma(0)\in\mathcal{U}, \\ \sum\limits_{p\in\mathcal{U}} T_{p,[p|q]}(k,k_1), \sigma(0)\in\mathcal{S}, \end{cases} \triangleq \sum_{p\in\mathcal{U}} T_p(k,0), \tag{68}$$

$$\gamma_s^- \triangleq \max_{p\in\mathcal{S}}\left\{\mu_{[p|q]}^{\frac{1}{\tau_{a(p,[p|q])}}} \bar{\varsigma}_{p,[p|q]}\right\}, \tag{69}$$

$$\gamma_s^+ \triangleq \max_{p\in\mathcal{U}}\left\{\mu_{[p|q]}^{\frac{1}{\tau_{a(p,[p|q])}}} \bar{\varsigma}_{p,[p|q]}\right\}, \tag{70}$$

$$K_5 \triangleq \prod_{g=1}^{g=s''} \mu_{[p|q]_{(g)}}^{N_{0(p,[p|q]_{(g)})}} \gamma^{-k_1} \bar{\varsigma}_{\sigma(0)}^{k_1}. \tag{71}$$

According to (66)-(70) and the SBASDT condition (64), one can have

$$\begin{aligned}
&V_{\sigma(k)}(k)\\
\leq &K_5 \gamma_s^{-T^-} \gamma_s^{+T^+} V_{\sigma(0)}(0)\\
\leq &K_5 \gamma^{(k-k_0)} V_{\sigma(0)}(0),
\end{aligned} \tag{72}$$

Therefore, the switched system is globally uniformly asymptotically stable.

(b) The SBAPDT switching:

According to (6), one can obtain

$$\begin{aligned}
&V_{\sigma(k)}(k)\\
\leq &\prod_{g=1,q\in\mathcal{S}}^{g=s''} \mu_{[p|q]_{(g)}}^{N_{0(q,[p|q]_{(g)})}+\frac{T_{q,[p|q]_{(g)}}(k_i,0)}{\tau_{a(q,[p|q]_{(g)})}}} \bar{\varsigma}_{q,[p|q]_{(g)}}{}^{T_{q,[p|q]_{(g)}}(k_i,0)}\\
\times &\prod_{g=1,q\in\mathcal{U}}^{g=s''} \mu_{[p|q]_{(g)}}^{N_{0(q,[p|q]_{(g)})}+\frac{T_{q,[p|q]_{(g)}}(k_i,0)}{\tau_{a(q,[p|q]_{(g)})}}} \bar{\varsigma}_{q,[p|q]_{(g)}}{}^{T_{q,[p|q]_{(g)}}(k_i,0)} \bar{\varsigma}_{\sigma(k_i)}^{(k-k_i)} V_{\sigma(0)}(x(0))\\
= &\left\{\prod_{g=1}^{g=s''} \mu_{[p|q]_{(g)}}^{N_{0(q,[p|q]_{(g)})}}\right\}\left\{\prod_{g=1,q\in\mathcal{S}}^{g=s''} (\mu_{[p|q]_{(g)}}^{\frac{1}{\tau_{a(q,[p|q]_{(g)})}}} \bar{\varsigma}_{q,[p|q]_{(g)}})^{T_{q,[p|q]_{(g)}}(k_i,0)}\right\}\\
\times &\left\{\prod_{g=1,p\in\mathcal{U}}^{g=s''} (\mu_{[p|q]_{(g)}}^{\frac{1}{\tau_{a(q,[p|q]_{(g)})}}} \bar{\varsigma}_{q,[p|q]_{(g)}})^{T_{q,[p|q]_{(g)}}(k_i,0)}\right\} \bar{\varsigma}_{\sigma(k_i)}^{(k-k_i)} V_{\sigma(0)}(x(0)).
\end{aligned} \tag{73}$$

We make the following definitions:

$$T^{-} \triangleq \begin{cases} \sum\limits_{q\in\mathcal{S}} T_{q,[p|q]}(k_i,0)+k-k_i, \sigma(k_i)\in\mathcal{S}, \\ \sum\limits_{q\in\mathcal{S}} T_{q,[p|q]}(k_i,0), \sigma(k_i)\in\mathcal{U}, \end{cases} \triangleq \sum_{q\in\mathcal{S}} T_q(k,0), \tag{74}$$

$$T^{+} \triangleq \begin{cases} \sum\limits_{q\in\mathcal{U}} T_{q,[p|q]}(k_i,0)+k-k_i, \sigma(k_i)\in\mathcal{U}, \\ \sum\limits_{q\in\mathcal{U}} T_{q,[p|q]}(k_i,0), \sigma(k_i)\in\mathcal{S}, \end{cases} \triangleq \sum_{q\in\mathcal{U}} T_q(k,0), \tag{75}$$

$$\overset{-}{\gamma_p} \triangleq \max_{q\in\mathcal{S}} \left\{ \mu_{[p|q]}^{\frac{1}{\tau_{a(q,[p|q])}}} \bar{\varsigma}_{q,[p|q]} \right\}, \tag{76}$$

$$\overset{+}{\gamma_p} \triangleq \max_{q\in\mathcal{U}} \left\{ \mu_{[p|q]}^{\frac{1}{\tau_{a(q,[p|q])}}} \bar{\varsigma}_{q,[p|q]} \right\}, \tag{77}$$

$$K_6 \triangleq \prod_{g=1}^{g=s''} \mu_{[p|q]_{(g)}}^{N_{0(q,[p|q]_{(g)})}} \gamma^{-(k-k_i)} \bar{\varsigma}_{\sigma(k_i)}^{(k-k_i)}. \tag{78}$$

According to (73)-(78) and the SBAPDT condition (65), one can get

$$\begin{aligned} & V_{\sigma(k)}(k) \\ & \leq K_6 \overset{-}{\gamma_p}^{T^{-}} \overset{+}{\gamma_p}^{T^{+}} V_{\sigma(0)}(0) \\ & \leq K_6 \gamma^{(k-k_0)} V_{\sigma(0)}(0), \end{aligned} \tag{79}$$

Therefore, the switched system is globally uniformly asymptotically stable.
Combining (a) and (b), this lemma is proved.□
If the considered system is linear, we have Theorem 5.

Theorem 5 *Consider the system (3). Let* $\varsigma_p > 0$, $\varsigma_q > 0$, $\mu_{[p|q]} > 1$ *be given constants. Suppose that there exist matrices* $P_p > 0$, $P_q > 0$, *such that* $\forall(\sigma(k_i)=p, \sigma(k_i-1)=q)\in\mathcal{M}\times\mathcal{M}$ *and* $p\neq q$,

$$\begin{bmatrix} (\varsigma_p+1)P_p & A_p^T P_p \\ * & P_p \end{bmatrix} > 0, \tag{80}$$

$$\begin{bmatrix} (\varsigma_q+1)P_q & A_q^T P_q \\ * & P_q \end{bmatrix} > 0, \tag{81}$$

$$V_p(k_i) \leq \mu_{[p|q]} V_q(k_i). \tag{82}$$

Then for any switching signals with SBASDT satisfying (64) or with SBAPDT satisfying (65), the system (3) is globally uniformly asymptotically stable.

3.2.2 continuous systems

For the continuous-time systems, We can obtain Lemma 6 and Theorem 6 below. The proofs are similar to the discrete systems.

Lemma 6 *Consider the system (2). Let* α_p, α_q *and* $\mu_{[p|q]} > 1$ *be given constants. Suppose that there exist* $\mathbb{C}^1$ *functions* $V_{\sigma(t)} : \mathbb{R}^n \to \mathbb{R}$, *and class* $\mathcal{K}_\infty$ *functions* $\kappa_{p1}, \kappa_{p2}, \kappa_{q1}, \kappa_{q2}$, *such that* $\forall(\sigma(t_i)=p, \sigma(t_i^-)=q)\in\mathcal{M}\times\mathcal{M}$ *and* $p\neq q$,

$$\begin{cases} \kappa_{p1}(\| x(t) \|) \leq V_p(x(t)) \leq \kappa_{p2}(\| x(t) \|), \\ \kappa_{q1}(\| x(t) \|) \leq V_q(x(t)) \leq \kappa_{q2}(\| x(t) \|), \end{cases} \tag{83}$$

$$\begin{cases} \dot{V}_p(x(t)) \leq \alpha_p V_p(x(t)), \\ \dot{V}_q(x(t)) \leq \alpha_q V_q(x(t)), \end{cases} \tag{84}$$

$$V_p(x(t_i)) \leq \mu_{[p|q]} V_q(x(t_i)). \tag{85}$$

Then for any switching signals with SBASDT

$$\begin{cases} \tau_{a(p,[p|q])} > \tau^*_{a(p,[p|q])} \triangleq -ln\mu_{[p|q]}/\alpha_p, (\alpha_p < 0, p\in\mathcal{S}), \\ \tau_{a(p,[p|q])} > \tau^*_{a(p,[p|q])}, (\forall\tau^*_{a(p,[p|q])} > 0, \alpha_p > 0, p\in\mathcal{U}), \\ \frac{T_s^-}{T_s^+} > \frac{\gamma_s^+ + \gamma^*}{\gamma_s^- - \gamma^*}, (0 < \gamma^* < \gamma_s^-), \end{cases} \tag{86}$$

or with SBAPDT

$$\begin{cases} \tau_{a(q,[p|q])} > \tau^*_{a(q,[p|q])} \triangleq -ln\mu_{[p|q]}/\alpha_q, (\alpha_q < 0, q \in \mathcal{S}), \\ \tau_{a(q,[p|q])} > \tau^*_{a(q,[p|q])}, (\forall \tau^*_{a(q,[p|q])} > 0, \alpha_q > 0, q \in \mathcal{U}), \\ \frac{T_p^-}{T_p^+} > \frac{\gamma_p^+ + \gamma^*}{\gamma_p^- - \gamma^*}, (0 < \gamma^* < \gamma_p^-), \end{cases} \tag{87}$$

the system (2) is globally uniformly asymptotically stable with marginal γ^*, where T^- and T^+ stand for the total running time of stable and unstable subsystems. The parameters of γ_s^+,γ_p^+, γ_s^-,γ_p^- are given as:

$$\gamma_s^- \triangleq \max_{p\in\mathcal{S}} \left\{ \alpha_p + \frac{ln\mu_{[p|q]}}{\tau_{a(p,[p|q])}} \right\}, \tag{88}$$

$$\gamma_s^+ \triangleq \max_{p\in\mathcal{U}} \left\{ \alpha_p + \frac{ln\mu_{[p|q]}}{\tau_{a(p,[p|q])}} \right\}, \tag{89}$$

$$\gamma_p^- \triangleq \max_{q\in\mathcal{S}} \left\{ \alpha_q + \frac{ln\mu_{[p|q]}}{\tau_{a(q,[p|q])}} \right\}, \tag{90}$$

$$\gamma_p^+ \triangleq \max_{q\in\mathcal{U}} \left\{ \alpha_q + \frac{ln\mu_{[p|q]}}{\tau_{a(q,[p|q])}} \right\}. \tag{91}$$

Theorem 6 *Consider the system (4). Let α_p, α_q and $\mu_{[p|q]} > 1$ be given constants. Suppose that there exist matrices $P_p > 0$, $P_q > 0$, such that $\forall(\sigma(t_i) = p, \sigma(t_i^-) = q) \in \mathcal{M} \times \mathcal{M}$ and $p \neq q$,*

$$A_p^T P_p + P_p A_p \leq \alpha_p P_p, \tag{92}$$

$$A_q^T P_q + P_q A_q \leq \alpha_q P_q, \tag{93}$$

$$P_p \leq \mu_{[p|q]} P_q. \tag{94}$$

Then the system (4) is globally uniformly asymptotically stable for any switching signals with SBASDT satisfying (86) or with SBAPDT satisfying (87).

If the sequence information is ignored, the presented SBASDT/SBAPDT sets degenerate into a mode-dependent average dwell time set. If the sequence and mode information is ignored, the presented SBASDT/SBAPDT sets degenerate into an average dwell time set. The presented SBASDT/SBAPDT sets can cover the dwell time set, the average dwell time set, etc.

In the past years, many control systems have been studied based on various dwell time methods, such as common linear systems and T-S fuzzy systems [26], but almost all of them ignore the significance of sequence. Our method can be extended to these systems, such as the T-S switched fuzzy systems with unstable subsystems.

4 Numerical Example

In order to verify our approach, we present an illustrative numerical example here. Comparison with the existing method in [17] is also provided. In particular, comparison is made with respect to Theorem 1 in [17] , which studied a mode-dependent average dwell time approach. It proves that the system (3) is globally uniformly asymptotically stable for any switching signal satisfying some proper conditions of mode-dependent average dwell time. Considering the similarity between the discrete-time and continuous-time cases, as well as the space limitation, we only verify Theorem 3 for discrete-time systems.

Let us consider the linear discrete-time switched systems with $A_1 = \begin{bmatrix} -0.012 & -0.022 \\ -0.11 & 0.012 \end{bmatrix}$, $A_2 = \begin{bmatrix} -0.5 & -0.9 \\ -0.7 & 0.2 \end{bmatrix}$, $A_3 = \begin{bmatrix} 0.011 & -0.013 \\ -0.23 & 0.031 \end{bmatrix}$. The first and third subsystem are Schur stable. The second subsystem is Schur unstable.

Parameters and the results are provided in Table 1 for Theorem 1 in [17] and Theorem 3 in this paper for these three subsystems.

From Table 1, we can get the following observations.

(a) For stable subsystems:

(a.1) According to Theorem 1 in [17]

$$\begin{cases} \tau_{a1} > 0.9350 \\ \tau_{a3} > 1.1564 \end{cases};$$

Table 1: Parameters and results for the numerical example for verifying Theorem 3 in this paper and Theorem 1 in [17].

	Theorem 1 in [17]	Theorem 3 in this paper
Given constants	λ_1=-0.98;	λ_1=-0.98;
	$\lambda_2 = 0.036$;	$\lambda_2 = 0.036$;
	$\lambda_3 = -0.95$;	$\lambda_3 = -0.95$;
	$\mu_1 = 2.5$;	$\mu_{[1\|2]} = 2.5$; $\mu_{[1\|3]} \to 1^+$;
	$\mu_2 = 0.41$;	$\mu_{[2\|1]} = 0.41$;$\mu_{[2\|3]} = 0.40$;
	$\mu_3 = 3.0$.	$\mu_{[3\|1]} = 1.2$;$\mu_{[3\|2]} = 3.0$.
Time Thresholds(stable subsystems)	τ^*_{a1}=0.9350;	$\tau^*_{a(1,[1\|2])} = 0.9350$;
	τ^*_{a3}= 1.1564.	$\tau^*_{a(1,[1\|3])} \to 0^+$;
		$\tau^*_{a(3,[3\|2])} = 1.1564$;
		$\tau^*_{a(3,[3\|1])} = 0.1919$.
Time Thresholds(unstable subsystems)	τ^*_{a2}=24.7666.	$\tau^*_{a(2,[2\|1])} = 24.7666$;
		$\tau^*_{a(2,[2\|3])} = 25.4525$.

(a.2) According to Theorem 3 in this paper

$$\begin{cases} \tau_{a(1,[1|2])} > 0.9350; \tau_{a(1,[1|3])} > 0 \\ \tau_{a(3,[3|2])} > 1.1564; \tau_{a(3,[3|1])} > 0.1919 \end{cases}.$$

According to (a.1) and (a.2), the proposed methods in this paper have a less conservative condition for stable subsystems. It has been shown in the literature [17] that, for the sequence that the first subsystem is activated immediately after the third subsystem, stability cannot be guaranteed if the average dwell time is less than 0.9350. In this paper, we show that stability can be guaranteed if the average dwell time is less than 0.9350, when the order of sequence is known. In the same way, we can guarantee the stability if the average dwell time of the sequence, when the third subsystem is activated immediately after the first subsystem, is between 0.1919 and 1.1596.

(b) For unstable subsystems:

(b.1) According to Theorem 1 in [17]

$$\tau^c_{a2} < 24.7666,$$

(b.2) According to Theorem 3 in this paper

$$\tau^c_{a(2,[2|1])} < 24.7666; \tau^c_{a(2,[2|3])} < 25.4525.$$

According to (b.1) and (b.2), the proposed methods in this paper have a less conservative condition for unstable subsystems. For the sequence that the second subsystem is activated immediately after the third subsystem, stability cannot be guaranteed if the average dwell time is between 24.7666 and 25.4525 according to the past conclusion in [17]. But in this paper, we show that stability can be guaranteed if the average dwell time is between 24.7666 and 25.4525 under this circumstance, when the order of sequence is known.

Combining (a) and (b), we can know that our sequence-based approach provides a less conservative condition for ensuring the stability of switched systems with unstable subsystems.

According to the above discussion, we can enlarge the dwell time ranges of subsystems for both stable subsystems and unstable subsystems which lowers conservativeness.

This new approach and conventional methods have some common limitations, such as when the dimensions of state vectors or the number of subsystems are large, more computing resources are needed or the time required to calculate results will increase exponentially.

In papers [22, 24], stability analysis of unstable switched subsystems is important for multi-agent systems under switched topologies. If we consider the order of sequence and use our proposed methods, the estimation accuracy of multi-agent systems can be improved [10, 21, 27].

5 Conclusion

The stability problem of switched systems with unstable subsystems was studied using a sequence-based average dwell time approach, similar to how switched systems with stable subsystems have been reviewed in the past with great success. Switching behaviours of both slow-switching and fast-switching were included in the study. When all subsystems are unstable, the constraints are further relaxed. If parts of the subsystems are unstable, a better threshold value can be obtained when we use the new approach. These methods lead to results that are less conservative than those of existing methods.

6 Data Availability

There is no associated data.

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