2-D TWO-FOLD SYMMETRIC CIRCULAR SHAPED FILTER DESIGN WITH HOMOMORPHIC PROCESSING APPLICATION

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ABSTRACT

A design method of a linear-phased, two-dimensional (2-D), two-fold symmetric circular shaped filter is presented in this paper. Although the proposed method designs a non-separable filter, its implementation has linear complexity. The shape of the passband and the stopband is expressed in terms of level sets of second order trigonometric polynomials. This enables the transformation of the filter specifications to a Semi-Definite Program (SDP) of moderate dimension. The proposed filter outperforms currently available filter design methods. We present a performance comparison, as well as a homomorphic processing image enhancement example to illustrate the effectiveness of this method.

***Index Terms*—** Two-fold symmetric, Two-dimensional Circular shaped filter, Semi-definite program, Trigonometric polynomials,

1. INTRODUCTION

There are many different approaches in designing 2-D filters, each with its pros and cons. The simplest method is in the form of a separable filter [1] which is developed as a product of two 1-D filters. They are favored in applications which require very high order filters since their complexity is very low. However there is limited freedom with the design variables and it is confined to rectangular shape. Non-separable filters are preferred when designing differently shaped filters but their design is much more challenging. McClellan transform [2], [3] can be used to develop non-separable filters using 1-D filters. Their complexity is very low but it is not possible to control the filter characteristics such as passband shape and cut-off frequency. This is mainly because there is no direct way to find the 1-D filter that, when transformed, will produce the desired 2-D filter.

Frequency sampling method [4] gives better control over the passband shape and produces lower peak ripples. However the complexity is very high and many difficulties such as singularities and numerical instability arise in the implementation of these filters. The greatest barrier in designing filters with desired characteristics is in the handling the semi-infinite constraints that arise due to the peak passband and stopband ripple constraints. Classical minmax filters [5] deal with this by approximating them by linear constraints calculated at samples of the frequency plane.

Recently SDP has been employed to deal with these semi-infinite constraints. There have been continues interest in representing the passband and stopband by trigonometric polynomials that are positive in the region of interest. Initially first order trigonometric polynomials have been used to design two-fold symmetric filters [6]. However they did not consider the circular shape and they have not presented a direct relationship between the filter specifications and the coefficients of the polynomials. The resulting SDP formulation is of very high dimension.

SDP formulations of moderate dimension have been achieved in four-fold symmetric filter designs [7], [8]. Although they have linear complexity in implementation, the trigonometric polynomials used do not approximate the passband and stopband shapes as well as one would hope for. This problem has been addressed in [9] where circular shaped passband and stopband has been successfully expressed by level sets of simple second order trigonometric polynomials. They have clearly presented the relationship between the desired filter characteristics and coefficients of the trigonometric polynomials.

The paper [9] is based on four-fold symmetric filters and we will extend this to a more general case of two-fold symmetric filters in this paper. Two-fold symmetric filters present more flexibility in the design variables and, therefore, the design procedure becomes more challenging. However since the optimization is performed over greater number of variables the filter performance can be improved in terms of frequency selectivity. Circular filters are of great importance in two-dimensional signal processing but they have not been given proper attention so far. Thus design and implementation of two-fold symmetric circular shaped filter are discussed in this paper followed by an example of homomorphic processing application. Standard mathematical notations are used in this paper, except that $\langle \boldsymbol{A} \rangle$ refers to the trace of a matrix and the elements of a matrix are given as $\boldsymbol{A} = [a_{ij}]_{i,j=0}^{n}$.

2. TWO-FOLD SYMMETRIC 2-D FILTER DESIGN

The frequency response of a zero-phased digital filter is a real valued function and its impulse response is symmetric about the origin $h(n, n) = h(-n, -n)$ [10]. Thus the Z-transform of a $(2n + 1) \times (2n + 1)$, zero-phase, two-fold symmetric, two-dimensional filter can be written as follows

$$\begin{aligned}\mathcal{H}(z_1, z_2) =& h_{nn} + \sum_{l=1}^{n} h_{(n+l)n}(z_1^l + z_1^{-l}) \\ &+ \sum_{l=-n}^{n}\sum_{i=1}^{n} h_{(n+l)(n+i)}(z_1^l z_2^i + z_1^{-l} z_2^{-i}).\end{aligned} \quad (1)$$

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Its Fourier transform can be obtained by evaluating on the unit circle $\{(z_1, z_2) = (e^{j\omega_1}, e^{j\omega_2}), \Omega := (\omega_1, \omega_2) \in [-\pi, \pi] \times [0, \pi]\}$,

$$H(\Omega) = \sum_{i=0}^{n}\sum_{l=0}^{n} x_{il}\cos(i\omega_1 + l\omega_2) + \sum_{i=0}^{n}\sum_{l=0}^{n} y_{il}\cos(i\omega_1 - l\omega_2), \quad (2)$$
$$= \langle \boldsymbol{X}, \boldsymbol{M_1}(\Omega)\rangle + \langle \boldsymbol{Y}, \boldsymbol{M_2}(\Omega)\rangle, \quad (3)$$

where $\boldsymbol{M_1}(\Omega) = [\cos(i\omega_1 + l\omega_2)]_{i,l=0}^{n}$, $\boldsymbol{M_2}(\Omega) = [\cos(i\omega_1 - l\omega_2)]_{i,l=0}^{n}$, $\boldsymbol{X} = [x_{il}]_{i,l=0}^{n}$ and $\boldsymbol{Y} = [y_{il}]_{i,l=0}^{n}$.

The design objective is to find the coefficients of the matrix $\boldsymbol{X}$ and $\boldsymbol{Y}$ such that the desired frequency response is obtained. There are $2(n+1)^2$ design variables, which is twice as that of the four-fold symmetric filter. Although this makes the design of a two-fold symmetric filter more challenging, the filter performance can be improved in terms of frequency selectivity since the optimization is performed over a larger number of variables. In this paper the filter order is taken as n. Depending on the application and the data to be processed the filter specifications vary considerably. Generally filter specifications can be formulated as a minimum stopband attenuation problem as follows:

$$\min_{\boldsymbol{X},\boldsymbol{Y},\delta_s} \quad \delta_s \quad (4a)$$
$$\text{s.t.} \quad |H(\Omega) - 1| \le \delta_p, \quad \forall \Omega \in \Omega_p \quad (4b)$$
$$|H(\Omega)| \le \delta_s, \quad \forall \Omega \in \Omega_s, \quad (4c)$$

where δ_p and δ_s are the passband ripple and the stopband attenuation respectively. The constraints given in (4b) and (4c) are semi-infinite constraints and pose the greatest challenge in designing the desired filter. In the next section we consider an effective method of transforming this optimization problem in to an SDP of linear complexity.

3. CIRCULAR SHAPED FILTER DESIGN

The passband and the stopband of a circular shaped filter can be specified as follows:

$$\Omega_p = \{(\omega_1, \omega_2) \in [-\pi, \pi] \times [0, \pi] : \omega_1^2 + \omega_2^2 \le \omega_p^2\}, \quad (5)$$
$$\Omega_s = \{(\omega_1, \omega_2) \in [-\pi, \pi] \times [0, \pi] : \omega_1^2 + \omega_2^2 \ge \omega_s^2\}. \quad (6)$$

It is very difficult to represent the circular shape in the form of trigonometric polynomials. Circular shape has not been attempted in [6] and the polynomials used in [8] do not produce the desired shape. However it has been proven in [9] that the passband and the stopband of the form (5) and (6) can be exactly described by level sets of second order trigonometric polynomials of the following form,

$$T(\Omega) = a(\cos 2\omega_1 + \cos 2\omega_2) + b(\cos\omega_1 + \cos\omega_2) + c. \quad (7)$$

Coefficients a,b and c in Equation (7) have to be found such that its level sets closely represents the circular shape given in (5) and (6). Since the trigonometric polynomial of the form of (7) has been proven to be a successful method of representing the circular shape, the same mask will be used in this paper. The following least squares optimization problem can be used to find the coefficients of (7).

$$\min_{x=(a,b,c)^T, ||x||=1} \int_0^{\omega_\alpha/\sqrt{2}} x^T \begin{bmatrix} \cos 2\omega_1 + \cos 2\sqrt{\omega_\alpha^2 - \omega_1^2} \\ \cos\omega_1 + \cos\sqrt{\omega_\alpha^2 - \omega_1^2} \\ 1 \end{bmatrix} \begin{bmatrix} \cos 2\omega_1 + \cos 2\sqrt{\omega_\alpha^2 - \omega_1^2} \\ \cos\omega_1 + \cos\sqrt{\omega_\alpha^2 - \omega_1^2} \\ 1 \end{bmatrix}^T x d\omega_1, \quad (8)$$

with $\alpha \in \{p, s\}$. The normalized eigenvector of the minimum eigenvalue of the following matrix will give the solution $x = (a, b, c)$ which minimizes the optimization problem defined in (8).

$$\int_0^{\frac{\omega_\alpha}{\sqrt{2}}} \begin{bmatrix} \cos 2\omega_1 + \cos 2\sqrt{\omega_\alpha^2 - \omega_1^2} \\ \cos\omega_1 + \cos\sqrt{\omega_\alpha^2 - \omega_1^2} \\ 1 \end{bmatrix} \begin{bmatrix} \cos 2\omega_1 + \cos 2\sqrt{\omega_\alpha^2 - \omega_1^2} \\ \cos\omega_1 + \cos\sqrt{\omega_\alpha^2 - \omega_1^2} \\ 1 \end{bmatrix}^T d\omega_1. \quad (9)$$

Generally the minimum eigenvalue of the matrix (9) is zero or nearly zero. This confirms the fact that the second order trigonometric polynomials of the form (7) exactly describes the circular shaped passband and the stopband. Thus two trigonometric polynomials can be derived to represent the passband and the stopband.

$$T_\alpha(\Omega) = a_\alpha(\cos 2\omega_1 + \cos 2\omega_2) + b_\alpha(\cos\omega_1 + \cos\omega_2) + c_\alpha, \quad (10)$$

with $\alpha \in \{p, s\}$. The next step is to define a family of trigonometric polynomials using the Chebyshev recursion.

$$\text{If} \quad T_0(\Omega) = 1,$$
$$T_1(\Omega) = A + B\cos\omega_1 + C\cos\omega_2 + D\cos(\omega_1 + \omega_2) + E\cos(\omega_1 - \omega_2), \quad (11)$$

with predefined coefficients A,B,C,D and E, then $T_j(\Omega), j = 2, 3, ...$ can be derived using Chebyshev recursion as follows,

$$T_j(\Omega) = 2T_{(j-1)}(\Omega)T_1(\Omega) - T_{(j-2)}(\Omega). \quad (12)$$

Here it should be noted that more complicated class of trigonometric polynomials of the form $\{\cos(j\omega_1 + k\omega_2), j, k = -n, .., -1, 0, 1, ...n\}$ are used in this paper than that used in [9] to facilitate the two fold symmetry. However our results show that this leads to improvement of the performance of the filter. Using the set of trigonometric polynomials derived using (11) and (12) a moment matrix is defined as follows,

$$\boldsymbol{\Psi}(\Omega) = \begin{bmatrix} 1 \\ T_1(\Omega) \\ \vdots \\ T_m(\Omega) \end{bmatrix} \begin{bmatrix} 1 \\ T_1(\Omega) \\ \vdots \\ T_m(\Omega) \end{bmatrix}^T, \quad m = [(n-2)/2]. \quad (13)$$

Since $T_\alpha(\Omega) \ge 0, \forall \Omega \in \Omega_\alpha$ and since $\boldsymbol{\Psi}(\Omega)$ is a positive definite matrix, $T_\alpha(\Omega)\boldsymbol{\Psi}(\Omega) \ge 0, \forall \Omega \in \Omega_\alpha$. This result can be used to define cone constraints which represent the passband and the stopband of the filter,

$$C_\alpha = \{(\boldsymbol{X}, \boldsymbol{Y}), \boldsymbol{X} \in R^{(n+1)\times(n+1)}, \boldsymbol{Y} \in R^{(n+1)\times(n+1)} :$$
$$\langle \boldsymbol{X}, \boldsymbol{M_1}(\Omega)\rangle + \langle \boldsymbol{Y}, \boldsymbol{M_2}(\Omega)\rangle \equiv \langle \hat{\boldsymbol{X}}_{\boldsymbol{\alpha}}, T_\alpha(\Omega)\boldsymbol{\Psi}(\Omega)\rangle,$$
$$\hat{\boldsymbol{X}}_{\boldsymbol{\alpha}} \ge 0, \alpha \in \{p, s\}\}. \quad (14)$$

Thus the semi-infinite program given by (4a),(4b) and (4c) can be transformed in to a semi-definite program as follows:

$$\min_{\boldsymbol{X},\boldsymbol{Y},\delta_s} \quad \delta_s \quad (15a)$$
$$\text{s.t.} \quad \big(\boldsymbol{X} - (1-\delta_p)\boldsymbol{E}, \boldsymbol{Y}\big) \in C_p \quad (15b)$$
$$\big(-\boldsymbol{X} + (1+\delta_p)\boldsymbol{E}, -\boldsymbol{Y}\big) \in C_p \quad (15c)$$
$$\big(\boldsymbol{X} - (\delta_s)\boldsymbol{E}, \boldsymbol{Y}\big) \in C_s \quad (15d)$$
$$\big(-\boldsymbol{X} + (\delta_s)\boldsymbol{E}, -\boldsymbol{Y}\big) \in C_s, \quad (15e)$$

where $\boldsymbol{E} \in R^{(n+1)\times(n+1)}$ has zero entries except $\boldsymbol{E}(0, 0) = 1$. The passband defined by (5) can be sampled, and for each sample

Table 1. Lowpass circular shaped filters with $(\omega_p, \omega_s) = (0.4\pi, 0.6\pi)$

Specifications	Our method	[9]	[8]	[3]
Filter order	12	12	25	25
Filter complexity	30	30	96	625
δ_p	0.0106	0.0124	0.0173	0.0257
δ_s	0.0019	0.0036	0.017	0.0248

two constraints can be derived using (15b) and (15c). Similarly two constraints can be derived for each sample of the stopband (6) by evaluating expressions (15d) and (15e). Then the coefficients of the matrix $\boldsymbol{X}$ and $\boldsymbol{Y}$ have to be found such that it minimizes δ_s subject to the constraints derived by sampling the passband and the stopband.

By considering equations (3) and by expanding the expression given in (14), the following interesting result can be derived to represent the frequency response of the passband and the stopband.

$$H(\Omega) = T_p \sum_{j=0}^{2m} a_j T_j(\Omega) + (1 - \delta_p) \quad \forall \Omega \in \Omega_p \tag{16}$$

$$H(\Omega) = T_s \sum_{j=0}^{2m} a'_j T_j(\Omega) - \delta_s \quad \forall \Omega \in \Omega_s. \tag{17}$$

It is clear from equations (16) and (17) that the passband and stopband can be described by $2m + 1$ waveforms each, which are of the form $\{T_\alpha T_j, \alpha \in \{p, s\}, j = 0, .., 2m\}$. The number of waveforms required to describe the passband and stopband of the two-fold symmetric filters presented here is the same as that of the four-fold symmetric filters presented in [9] and therefore the digital complexity of both cases are the same$(2n+6)$. However due to the selection of a more complicated polynomial for $T_1(\Omega)$ in Chebyshev recursion, the filter performance is improved in two-fold symmetric filters.

4. SIMULATION

In this section we will consider the simulation of a circular shaped lowpass filter with $\omega_p = 0.4\pi$, $\omega_s = 0.6\pi$ and of order 24. The first step is to derive the second order trigonometric polynomials that represent the passband and the stopband. The eigenvector of the minimum eigenvalue of the matrix given in (9) was calculated for $\omega_\alpha = \omega_p$ and $\omega_\alpha = \omega_s$.

Minimum eigenvalues were zero and the corresponding eigenvectors were $(a_p, b_p, c_p) = (-0.0488, 0.6108, -0.7903)$ and $(a_s, b_s, c_s) = (-0.0922, 0.8269, -0.5547)$ which was then substituted in equation (10). Select δ_p as 0.01 and the coefficients of equation (11) as $(A, B, C, D, E) = (0.5, 0.25.0.25.0.125, 0.25)$. Semi-definite program was derived as described in Section 3 and the simulation was performed using optimization software YALMIP [11] and SDPT3 [12] in MATLAB. Fig. 1 shows the frequency response of the designed filter in log scale and the performance comparison with different design methods is given in Table 1.

Highpass circular shaped filters can be designed analogously. The simulation result of a circular shaped highpass filter with $\omega_p = 0.6\pi$, $\omega_s = 0.4\pi$ and of order 36 is shown in Fig. 2. Performance comparison of different design methods is given in Table 2. In the case of both lowpass and highpass filter designs, it is clearly evident that the filter design method presented in this paper has the lowest complexity and achieves the lowest δ_p and δ_s values.

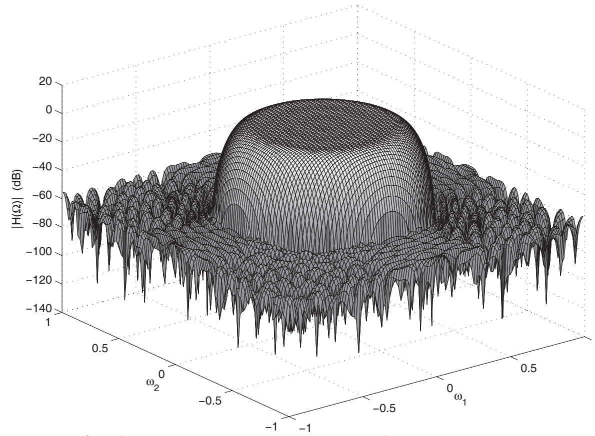

Fig. 1. Lowpass circular shaped filter in log scale.

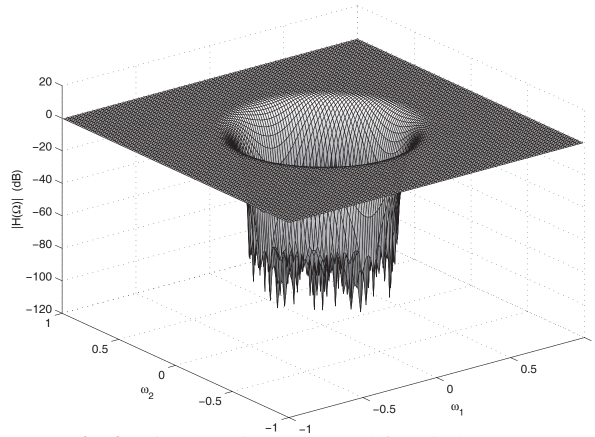

Fig. 2. Highpass circular shaped filter in log scale.

5. HOMOMORPHIC PROCESSING SYSTEM

When images with large dynamic range such as natural scenes are recorded, image contrast can be significantly reduced. Homomorphic processing can be used to reduce the dynamic range and to increase the contrast of such images. An image is formed by recording the light reflected from an object which is illuminated by some light source. Based on this the model $f(n_1, n_2) = i(n_1, n_2) r(n_1, n_1)$ can be used to represent the image, where $i(n_1, n_2)$ and $r(n_1, n_2)$ represent the illumination and reflection respectively [10].

Since they are combined multiplicatively the components are made additive by taking the logarithm of the image intensity $f(n_1, n_2)$. The illumination component $i(n_1, n_2)$ is related to the dynamic range of the image and is generally slow varying. So it can be separated by lowpass filtering $\log f(n_1, n_2)$. The reflectance component $i(n_1, n_2)$ on the other hand is related to the contrast

Table 2. Highpass circular shaped filters with $(\omega_p, \omega_s) = (0.6\pi, 0.4\pi)$

Specifications	Our method	[9]	[8]	[3]
Filter order	18	39	25	25
Filter complexity	42	84	96	625
δ_p	0.0003	0.0062	0.0171	0.0248
δ_s	0.0014	0.0026	0.0178	0.0257

within the image and generally vary rapidly. Highpass filtering of $\log f(n_1, n_2)$ can be used to extract the reflectance component.

Fig. 3 depicts the complete homomorphic processing system. The dynamic range can be reduced by decreasing α and the contrast can be increased by increasing β. This process was implemented using circular shaped lowpass filter with $(\omega_p, \omega_s) = (0.4\pi, 0.8\pi)$ and highpass filter with $(\omega_p, \omega_s) = (0.8\pi, 0.4\pi)$. It was applied on youtube video with $(\alpha, \beta) = (0.99, 1.5)$ and the results are given in Figs. 4 and 5. To see these images more clearly visit http://ee.unsw.edu.au/∼z3265024/enhancement.html. It can be seen that the background details are more highlighted in the enhanced image and people standing at the back of the court can be clearly seen.

To evaluate the effectiveness of the homomorphic process the original image given in Fig. 4 was blurred using a gaussian lowpass filter before applying the homomorphic process. Then we recovered the sharpness and the contrast of the original image by changing the α and β values of the homomorphic process. The PSNR value between the blurred image and the original image, and that between the enhanced image and original image was calculated. Results are given in Fig. 6. This effectively shows that the PSNR value can be considerably improved by the homomorphic process implemented using the filters discussed in this paper.

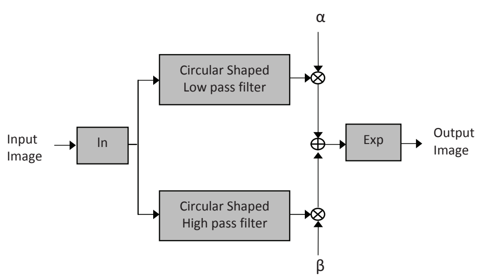

Fig. 3. Homomorphic system for image enhancement.

Fig. 4. Original image.

Fig. 5. Enhanced image.

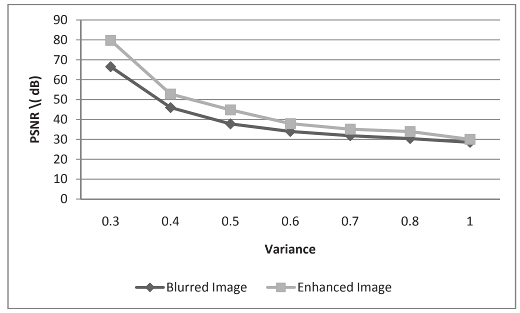

Fig. 6. PSNR improvement using homomorphic processing.

6. CONCLUSION

A very general approach of designing two-fold symmetric circular shaped filters with complexity equal to that of a four-fold symmetric filter was presented in this paper. The main advantage is that it successfully produces the desired circular shape with minimum passband and stopband ripple compared to the currently available design methods. The coefficients of the polynomial $T_1(\Omega)$ in Chebyshev recursion have an influence on the filter performance and a method of selecting optimal values for these coefficients is still open for research.

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