Parameterized Complexity of Weighted Local Hamiltonian Problems and the Quantum Exponential Time Hypothesis

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Abstract

We study a parameterized version of the local Hamiltonian problem, called the weighted local Hamiltonian problem, where the relevant quantum states are superpositions of computational basis states of Hamming weight k. The Hamming weight constraint can have a physical interpretation as a constraint on the number of excitations allowed or particle number in a system. We prove that this problem is in QW[1], the first level of the quantum weft hierarchy, and that it is hard for QM[1], the quantum analogue of M[1]. Our results show that this problem cannot be fixed parameter quantum tractable (FPQT) unless certain natural quantum analogue of the exponential time hypothesis (ETH) is false.

1 Introduction

Parameterized complexity theory [DF13] aims to analyze problems in a more refined manner than classical complexity theory by creating tools for comparing complexity over multiple parameters, as opposed to simply the input size. In principle, this opens up more possibilities for the analysis of real-world problems that may have complicated parameter dependencies. The key idea of the theory is to confine the possible super-polynomial dependence of the runtime to the parameters only. Tractability in the parameterized setting is described by the fixed parameter tractable class FPT, which can be thought of as a parameterized version of P where the dependence on the parameter of the runtime can be any computable function while the dependence on the input size is still a polynomial.

One of the most useful results in parameterized complexity theory is that problems which are complete for the class W[1], one of many natural parameterized generalizations of NP, are

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not fixed-parameter tractable unless the exponential time hypothesis (ETH) fails (see [CHKX06] for the initial result, [DF13] for further exposition and [IP99, IPZ01] where ETH is defined). More concisely, if W[1] = FPT then ETH is false. This links parameterized intractability to classical intractability, tying the intractability of SAT with the parameterized intractability of W[1]-complete problems such as k-INDEPENDENT SET and k-CLIQUE.

The LOCAL HAMILTONIAN PROBLEM [KSV02] has been one of the most studied problems in *quantum* complexity theory over the last two decades. There have been many interesting recent works studying variants of this problem with relevance for quantum chemistry. In [OIWF22], the authors establish the QMA-completeness of a variant of the local Hamiltonian problem considering a fixed basis describing the orbitals of the electronic structure problem, inspired by the problem posed in [WLAG13]. Another work in this direction is that of [GL21], where the authors study the so called GUIDED LOCAL HAMILTONIAN PROBLEM in which the instance description includes a local Hamiltonian H and a state vector u promised to be close to the ground state of H. In this work it is shown that when the Hamiltonian is 6-local then the decision problem is BQP-hard, further work [GHLM22, CFW22] has shown that the problem remains BQP-hard when considering 2-local Hamiltonians.

In this paper we link the complexity of the WEIGHT- $k \ \ell$ -LOCAL HAMILTONIAN problem to the classical ETH and quantum variants of it, QETH and QCETH. It is shown that if the WEIGHT $k \ \ell$ -LOCAL HAMILTONIAN problem can be solved in FPT or FPQT (the quantum generalization of FPT introduced in [BJM⁺22]) then versions of these hypotheses will fail. The weight in this problem refers to the Hamming weight of the states in the promise of the local Hamiltonian problem, either there is a weight-k state with a small eigenvalue, or all weight-k states are above a certain energy. The restriction of the weight on the states considered in the problem finds a physical interpretation when considering the 1s in the computational basis as particle excitations and thus the weight corresponds to fixing the particle number to k. We remark that when considering Fermionic Hamiltonians, using the Jordan-Wigner transform in general makes the Hamiltonian non-local, in this sense our results should be considered as a first step towards the goal of studying the complexity of problems with fixed particle number.

Mirroring the situation with the parameterized complexity class W[1], we establish our main result, that the WEIGHT- $k \ \ell$ -LOCAL HAMILTONIAN is contained in QW[1], a quantum generalization of W [1] that was introduced in [BJM⁺22]. These classes are parameterized analogues of NP and QMA where the promise has bounded weight and the verifier is limited to the use of "weft-1" circuits – circuits that have at most a single "large" gate on any path from the input to the output. In the quantum case this is defined to be a single multiply controlled Toffoli gate acting on O(n) qubits. Analogous to the classical case, the link to the exponential-time hypotheses is made via introducing a miniaturized version of the circuit satisfiability problem that is used to define a class QM[1] and proving that the WEIGHT- $k \ \ell$ -LOCAL HAMILTONIAN is QM[1]-hard.

By establishing that the WEIGHT- $k \ell$ -LOCAL HAMILTONIAN is contained in QW[1] and that it can be linked to the quantum exponential time hypotheses, we have resolved a clear question that emerged from [BJM⁺22], whether or not there was evidence of any problems in QW[1] that were likely outside both W[1] and FPQT.

We believe that there are a number of interesting open problems emerging from our work. The first is that it remains an open question as to whether this problem is complete for QW[1], or even if there are any natural problems at all that are complete for this class. One of the roadblocks to proving this is the absence of a *normalization* theory for quantum weft circuits. In an attempt to take the first steps towards resolving this problem, we identify a class inside QW[1](called $SQW_1[1]$) that has a normal form and we prove that the FRUSTRATION-FREE WEIGHTED HAMILTONIAN problem is complete for that class. From a physical perspective, in this paper we have established the likely intractability of the weighted local Hamiltonian problem in an attempt to better connect parameterized complexity theory to problems of interest in quantum chemistry. Moving forward it would be of significant interest if there are additional structures for this problem that would place it in either FPQT or FPT. A natural variation to consider would be to further restrict the locality conditions on the Hamiltonian, for example to consider lattice problems or other regular graph structures. Another direction would be to further restrict the set of potential promises on this problem.

1.1 Summary of Main results

Our main contribution in this paper is to initiate the study of local Hamiltonian problems in the context of parameterized complexity theory. In classical complexity theory, many important parameterized problems such as k-VERTEX COVER, k-INDEPENDENT SET, and WEIGHT-k SAT consider problems parameterized by the Hamming weight of the solution. Local Hamiltonian problems are natural generalizations of constraint satisfiability problems (CSP) and it is natural to study the complexity of the local Hamiltonian problem restricting the solution Hilbert space to the span of basis strings of Hamming weight k (for the definition of weight see Definition 2.3 and for the weighted local Hamiltonian see Definition 2.9). This setup has some physical relevance as the weight k may be interpreted as the particle number or the number of excitations in a physical system.

We establish the likely intractability of the weighted local Hamiltonian problem. Our first result puts this constrained-weight local Hamiltonian problem in QW[1], a quantum version of the classical parameterized intractability class W[1], which uses "weft-1" verifiers, these are circuits of constant depth and at most one large gate (see Definitions 2.5 and 2.7 for more details).

Theorem 1.1 (Informal version of Theorem 3.1). WEIGHT- $k \ell$ -LOCAL HAMILTONIAN *is in* QW[1].

We have not been able to prove the QW[1]-completeness of this problem. It is known that this problem is contained in XP in contrast to the weighted quantum circuit satisfiability problem which cannot be in XP unless P = BQP [BJM⁺22]. In [BJM⁺22] it is shown that QW[1] is not a subset of XP unless P = BQP, however as it is also shown that FPQT is not a subset of W[1] under the same assumption, it may be that although WEIGHT-*k* ℓ -LOCAL HAMILTONIAN is in XP, its closure under FPQT reductions could be QW[1] (i.e., it could be QW[1]-complete).

Nonetheless, the proof technique leveraged in Theorem 1.1 enabled us to show that the weighted local Hamiltonian problems are QM[1]-hard, where QM[1] is the natural quantum analogue of M[1], a class introduced in classical parameterized complexity theory to develop intractability results in reference to parameterized sub-exponential time algorithms, and thence to the exponential time hypothesis (ETH). This establishes the intractability of the weighted local Hamiltonian problem under quantum analogues of the ETH we introduce in this paper.

In this work we consider a *weak* version of the ETH as given in [DF13] (see Definition 5.1). This version states that there are no classical algorithms solving the circuit satisfiability problem in subexponential time. Moreover, we establish that quantum parameterized complexity is connected to two quantum generalizations of the ETH which we define as QCETH and QETH (See Definitions 5.2 and 5.3). The first variant, QCETH, says that there are no quantum algorithms solving classical circuit satisfiability in subexponential time, while QETH roughly states that no such subexponential quantum algorithms exist for a version of the quantum circuit satisfiability problem. We show that, as expected, QCETH implies QETH (see Proposition 5.1).

In Section 5 we make these connections explicit, first by recalling a theorem known in classical parameterized complexity which states that if W[1] = FPT then ETH is false. A simple reduction from INDEPENDENT SET to the WEIGHTED LOCAL HAMILTONIAN problem shows that the existence of FPT algorithms for the local Hamiltonian problem would also lead to refuting ETH, though we remark there could be FPT algorithms under different, but still interesting, parameterizations. Moreover, we prove the following result concerning QETH.

Theorem 1.2 (Informal version of Theorem 5.7). If WEIGHT- $k \ell$ -LOCAL HAMILTONIAN is in FPQT then QETH is false.

Theorem 1.2 gives evidence to the claim that the weighted local Hamiltonian is intractable as depicted in Fig. 1. Moreover, as stated in Theorem 1.2, the tractability of the weighted local Hamiltonian problem for quantum algorithms has implications regarding the existence of subexponential quantum algorithms for a quantum QMA-complete problem. As QETH is implied by QCETH, Theorem 1.2 implies that if WEIGHT- $k \ell$ -LOCAL HAMILTONIAN is in FPQT then QCETH is false.



Figure 1: Summary of main results in our work. We prove that WEIGHT- $k \ell$ -LOCAL HAMILTO-NIAN is in QW[1] (previously it was proven to be in XP [BJM⁺22]). Moreover, we show that if this problem lies in FPT, then ETH is false and if it is in FPQT this implies that both QETH and QCETH are false.

Finally, we also consider the FRUSTRATION-FREE WEIGHTED HAMILTONIAN problems and the equivalent WEIGHTED QUANTUM SAT problems. Reusing the reductions in the proof of Theorem 1.1, we are able to show that frustration-free weighted local Hamiltonian problems are contained in $QW_1[1]$, the counterpart of QW[1] with perfect completeness condition. In fact, we prove a stronger statement that the problem is in $SQW_1[1]$, a class contained in $QW_1[1]$ that models the weft-1 quantum circuit with the large gate being the final AND gate. Furthermore, we are able to show that the FRUSTRATION-FREE WEIGHTED HAMILTONIAN and WEIGHTED QUANTUM SAT problems are actually complete for the class $SQW_1[1]$.

Theorem 1.3 (Informal version of Theorem 4.1). *The* FRUSTRATION-FREE WEIGHT-*k* HAMIL-TONIAN *and* WEIGHT-*k* QUANTUM SAT *problems are* $SQW_1[1]$ -*complete*.

The completeness of the weighted quantum SAT problems for $SQW_1[1]$ raises the questions like whether $SQW_1[1]$ equals $QW_1[1]$ and whether SQW[1] equals $QW_1[1]$. The resolution of

these questions relies on further investigations on whether there is a quantum equivalent to the classical normalization theorems which holds for quantum weft-1 circuits. We leave it as an interesting open problem.

1.2 Proof Techniques

It is very natural to expect that, as the well-known quantum analogues of constraint satisfaction problems, local Hamiltonian problems parameterized using the weight are related to quantum analogues of the weft hierarchy. Yet, even though classical WEIGHT-k SAT problems are trivially contained in W[1], to show that WEIGHT-k LOCAL HAMILTONIAN problems are in QW[1] is not easy.

The main technical challenge in proving this result is that, by definition, any problem in QW[1] is characterized by being verifiable by a weft-1 circuit, which implies we must check if the energy terms are satisfied using a circuit with constrained depth. A direct approach is by measuring the energy of each Hamiltonian term and then either summing up or taking the average of the local energies. However, this does not work as quantum measurements may perturb the witness state, rendering later measurements problematic. Even if we ignore this issue and assume somehow we can measure the local energies without disturbing the state (probably like in the case of a quantum SAT problem and we have perfect completeness), it is not easy to perform these measurements in constant depth. Classically, we can use fanout gates to wire the inputs to each checking term, yet this is not possible in the quantum case! Our solution to this is a long chain of reductions (shown in Fig. 2) that gradually normalizes the form of the Hamiltonian.

In the first step, we reduce the local Hamiltonian problem to a *weight-preserving* circuit satisfiability problem. A weight-preserving quantum circuit is a circuit that consists of only unitary gates that preserve the weight of any state. It is the key technical tool we rely on in this chain of reductions. On one hand, we can use weight-preserving circuits to perform energy measurements for arbitrary Hamiltonian problems, including those whose local terms are not weight-preserving operators (see Section 3.2). On the other hand, we can convert the weight-preserving circuit problem back to a weighted local Hamiltonian problem using Kitaev's construction [KSV02, KKR06]. As the history state of the Kitaev construction has the form

$$|\psi_{\text{history}}\rangle = \frac{1}{\sqrt{T+1}} \sum_{t=0}^{T} (U_t U_{t-1} \cdots U_1 |\psi\rangle_{\text{witness}}) \otimes |t\rangle_{\text{clock}},$$

the weight-preserving property is essential to maintain the weight condition of the resulting Hamiltonian problem. If the circuit is weight-preserving and we choose to use the indicator clock (of the form $|0 \cdots 010 \cdots 0\rangle$ where the 1 is at *t*-th position), the history state is a state of weight k + 1. The Kitaev-type quantum proof checking technique is crucial for checking the propagation without blowing up the weight and without resorting to any normalization theorem about the circuit. If we were checking the propagation classically, the weight will be multiplied by a factor of T, an overhead we won't be able afford. Another advantage that the weight preserving circuit and Kitaev constructions bring is that we do not need to have constant depth circuits yet, providing a lot of flexibility to manipulate the circuit structure and the error reduction we need to perform.

To the best of our knowledge, however, no previous work did a systematic study on weightpreserving circuits. Therefore, we first prove several basic facts about weight-preserving quantum circuits. Following the standard approach [BBC⁺⁹⁵] that established the universal gate sets for standard quantum circuits, we show in Section 3.1 that there is similar theory of universal gate sets for weight preserving circuits. The result is of independent interest and may be useful elsewhere.

As the second step in the chain of reductions, we also show how to perform strong QMA completeness and soundness error reduction for weight-preserving verification circuits. For standard quantum circuits, this is first proved by Marriott and Watrous [MW05]. The Marriott and Watrous procedure utilizes the fact that the post-measurement state of the verifier still contains "most" of the information about the witness state and it is possible to perform a series of measurements with outcomes y_1, y_2, \ldots, y_N so that $z_i = y_{i-1} \oplus y_i$ are identically independently distributed according to the acceptance probability. Thus by doing simple statistics over z_i , we can reduce the error exponentially. This standard Marriott-Watrous construction does not seem to work in the weight preserving setting. The reason is that it is very hard to store or to "forget" the measurement outcomes y_i . For a weight-preserving circuit starting from a weight-k state, the size of the space it can possibly explore is at most $O(n^k)$ dimensional and it is incapable of retaining all measurement outcomes y_i . There is also no easy way to "forget" about the y_i information as quantum computing is reversible and, in order to perform counting over z_i , we need to remember the previous outcome y_{i-1} . Fortunately for us, it is possible to adapt the fast QMA error reduction based on quantum singular value transformation [Gil19] in our setting. This is a rare example where the standard Marriott-Watrous error reduction won't work while the fast QMA error reduction will. What we utilize here is not that the fast QMA error reduction is quadratically faster, but that it does not perform many measurements and that it only uses a single extra qubit and easy-to-implement extra gates.

In the third step, we construct a weighted local Hamiltonian from the weight-preserving circuit SAT problem. Compared to the weighted local Hamiltonian problem we started with, this new Hamiltonian problem has the nice property that the terms are (almost) spatially sparse, meaning that each qubit is involved in a constant number of terms and it is possible to partition the Hamiltonian into a finite number of groups of non-overlapping terms. This paved the way to constructing the final constant-depth weft-1 quantum circuit. This step is made possible by the spatially sparse construction introduced in [OT08]. There are two important changes we made to this construction in [OT08]. First, we need to use *indicator* clock to ensure the weight of the ground state of the Hamiltonian is fixed to some number. The checking of the indicator clock is much harder than the usual unary clock and it will *violate* the spatially sparse condition for the clock register. We will see that this won't be a problem as for the checking of the clock, we can measure all the clock qubits and perform a classical W[1] computation on the measurement outcome. Second, we introduced another simple encoding on the computational qubits, mapping $|0\rangle$ to $|00\rangle$ and $|1\rangle$ to $|11\rangle$ for each qubit in the computational register. This ensures that the weight of the computational qubits is always an even number. So when we ask for a weight-(2k + 1) in the Hamiltonian problem, we do need to check that the clock register is not in the all 0 state $|0^{T+1}\rangle$. This simple trick simplifies the checking of the indicator clock so that we only need a big AND gate. Otherwise, to rule out the all 0 case, we needed a big OR gate and check that the output is 1. This modification is crucial later on for obtaining the completeness result of a weighted quantum satisfiability problem.

We note that due to a technical requirement in the next step, we needed a stronger gap condition on the energy bounds a, b of the local Hamiltonian. The standard condition on the energy bounds says that b - a is necessarily at least $1/\operatorname{poly}(n)$. We will require that $b/n^2 - a$ is also at least $1/\operatorname{poly}(n)$. This strong gap condition is made possible by the weight-preserving strong error reduction procedure (given in Section 3.3).

In the final step of the proof that weighted local Hamiltonian is in QW[1], we construct constant depth "weft-1" circuits for the almost spatially sparse Hamiltonian problems we get from the third step. This is now an easy step given all the preparations already performed. The idea is that the circuit can measure constant number of groups of non-overlapping terms in parallel. To check the clock register, the circuit measures it and makes the decision using classical fanout gates, NAND gates of fanin-2 and a classical big AND gate. Because we are measuring multiple Hamiltonian terms simultaneously, it is necessary to consider the approximation between Hamiltonian of the form $\sum_{i=1}^{n} |0\rangle \langle 0|_i$ and the project $I - |1^n\rangle \langle 1^n|$. This is why we need the stronger gap condition in the previous step.



Figure 2: Reductions used to prove that the weighted local Hamiltonian problem is in QW[1] and QM[1]-hard. Hamiltonian problems are the left while quantum circuit problems are on the right.

Having shown that the WEIGHT-k LOCAL HAMILTONIAN is in QW[1], a natural question is whether this problem is complete. The main motivation to prove QW[1]-completeness is to establish the likely intractability of the problem. Despite not showing this, we manage to give evidence of intractability in Theorem 1.2. A proof sketch of this theorem is as follows. First, a "miniaturized" version of the quantum circuit satisfiability problem is defined where the instance is described by a weft-t quantum circuit with at most $k \log n$ inputs and gates, which we call MINI-QCSAT. From this problem we define the QM-hierarchy as those problems FPQTreducible to MINI-QCSAT and show that $QM[1] \subseteq FPQT$ if and only if QETH is false. The next step is to prove that WEIGHT-k ℓ -LOCAL HAMILTONIAN is QM[1]-hard. For this, we reduce MINI-QCSAT to WEIGHT-k WEIGHT-PRESERVING QUANTUM CIRCUIT SATISFIABILITY, the proof of this reduction is inspired by the so called " $k \log n$ " trick used in classical parameterized complexity (see Corollary 3.13 in [FG06]). More precisely, we reduce the miniaturized quantum circuit over $k \log n$ qubits to a quantum circuit over O(kn) qubits by encoding the $k \log n$ qubits in k groups of $\log n$ qubits and then further encode these with k groups of n qubits with weight-1. This last reduction allows us to leverage the reduction (Step 2 of the proof) from Theorem 1.1 to reduce the weight-preserving circuit satisfiability to weight-k circuit satisfiability with constantdepth. This implies that if the weighted Local Hamiltonian problem is in FPQT then QETH is false, which further implies that QCETH is false.

We also explored the complexity of the weighted quantum SAT problems and frustration-free weighted quantum Hamiltonian problems. This corresponds to the special case of the discussion on the containment of weighted local Hamiltonian in QW[1] where we enforce perfect completeness. It is easy to verify that along the chain of the reductions in Fig. 2, the perfect completeness condition is always kept and, as we only used a big AND gate in the QW[1] verification circuit,

this proves that the weighted quantum SAT problems are in $SQW_1[1]$, a special case of $QW_1[1]$ with the additional structural requirement that the big gate is the last gate and it is the classical AND gate acting on measurement outcomes of a constant-depth quantum circuit.

What is interesting is that for the quantum SAT problems, we are able to prove that they are complete for $SQW_1[1]$ even though the same idea fails for the local Hamiltonian problems. For this, we use a light cone argument to show that it is possible to directly read off a set of local projectors forming a quantum SAT problem from the constant depth $SQW_1[1]$ circuit. The light cone argument contracts the difference between classical fanout and quantum fanout gates. For an arbitrary classical fanout gate, the number of gates and wires needed to be included in the light cone is always a constant while for a quantum fanout, this number will include all qubits involved in the fanout gate and thus unbounded.

1.3 Organization

The organization of this work is as follows. Section 2 introduces the main parameterized complexity classes used in our work together with the notations used. Section 3 provides the proof that the weighted local Hamiltonian problem is in QW[1]. In Section 4 some results on the Frustration Free Local Hamiltonian problems are given and Section 5 presents the results regarding the Exponential Time Hypothesis and the QW-hierarchy. Finally, we give some discussion of open problems and future directions in Section 6

2 Background and Notation

Here we present the main parameterized complexity classes defined in [BJM⁺22] together with some important problems that will be used. For a more extended discussion of these definitions we direct the reader to [BJM⁺22] with regard the quantum parameterized classes and for a discussion of classical parameterized complexity we suggest [DF95, DF99, DF13]. We begin by recalling the definition of a parameterized problem, these are problems where the description of the instance includes a parameter describing certain property of the instance.

Definition 2.1 (Parameterization). A parameterization of a finite alphabet Σ is a mapping κ : $\Sigma^* \to \mathbb{Z}^+$ that is polynomial-time computable. The trivial parameterization κ_{trivial} is the parameterization with $\kappa_{\text{trivial}}(x) = 1$ for all $x \in \Sigma^*$.

We now define a parameterized problem.

Definition 2.2 (Parameterized problem). A parameterized problem over a finite alphabet Σ is a pair (L, κ) where $L \subseteq \Sigma^*$ is a set of strings over Σ and κ is a parameterization of Σ . We say that a parameterized problem (L, κ) over the alphabet Σ is *trivial* if either $L = \emptyset$ or $L = \Sigma^*$.

The complexity class of tractable bounded-error quantum parameterized problems is FPQT (see [BJM⁺22] for a formal definition). While in quantum complexity theory the class QMA is considered as the "quantum version" of NP, quantum parameterized complexity has many analogues of NP (as in the classical parameterized case). The analogues of NP we will focus on here are the classes QW[P] and the those in QW-hierarchy [BJM⁺22]. Here we give a modified definition of the QW-hierarchy adequate for proving our results. The notion of Hamming weight is fundamental in parameterized complexity to define intractability. We will base our definitions in the following notion of weight for quantum states.

Definition 2.3 (Weight of a quantum state). A quantum state $|\psi\rangle = \sum_{x \in \{0,1\}^n} \alpha_x |x\rangle$ on *n* qubits is said to have *weight* k if $\alpha_x = 0$ for all x not of Hamming weight k.

A second notion important for defining the intractable W-hierarchy in the classical case is that of weft.

Definition 2.4 (Circuit weft). Given a Boolean circuit C comprising generalised Toffoli gates and one and two bit fan-in gates. The *weft* of C is the maximum number of Toffoli gates that act on any path from input bit to output bit.

This notion of weft is generalized to the quantum case:

Definition 2.5 (Quantum circuit weft). Given a quantum circuit C comprising generalised Toffoli gates, one and two-qubit gates, and unbounded *classical* fanout. The *weft* of C is the maximum number of Toffoli gates that act on any path from input qubit to output qubit.

We remark that the fanout gate allowed in a weft-1 quantum circuit is classical. In a quantum circuit, a fanout gate is called classical if all of the target qubits are initialized to the $|0\rangle$ state and no other gates acted on them before the fanout gate. After the fanout gate, a unitary gate can only act on the fanout qubits by using them as controls. The equivalence between this definition of classical fanout gates and the standard definition follows from the principle of delayed measurements. Because quantum fanout gates are very powerful and can simulate big Toffoli and threshold gates [HS05], they should be avoided when defining weft-*t* quantum circuits.

To define the QW-hierarchy we proceed similarly as in [MW05] for the class QMA. For functions $c, s : \mathbb{N} \to [0, 1]$ we define the following problem

Definition 2.6 (WEIGHT-k WEFT-t DEPTH-d QUANTUM CIRCUIT SATISFIABILITY (c, s)).

Instance:	A weft-t depth-d quantum circuit C on n witness qubits and $poly(n)$ ancilla qubits.
Parameter:	A natural number k.
Yes:	There exists an <i>n</i> -qubit weight- <i>k</i> quantum state $ \psi\rangle$, such that $\Pr[\mathcal{C}(\psi\rangle)$ accepts] \geq
	С.
No:	For every <i>n</i> -qubit weight-k quantum state $ \psi\rangle$, $\Pr[\mathcal{C}(\psi\rangle)$ accepts] $\leq s$.

Definition 2.7 ($QW_{c,s}[t]$). For $t \in \mathbb{N}$, the class $QW_{c,s}[t]$ consists of all parameterized problems that are FPQT reducible to WEIGHT-*k* WEFT-*t* DEPTH-*d* QUANTUM CIRCUIT SATISFIABILITY(*c*, *s*) for some constant depth $d \geq t$.

Due to the constant depth requirement of weft-t quantum circuits, it is not clear if this class has the error reduction property. These classes are most relevant when c and s have a polynomial gap, i.e., c - s > 1/poly(n). Based on this, we define the QW-hierarchy as

Definition 2.8. Define QW[t] as

$$\mathsf{QW}[t] := \bigcup_{\substack{c,s\\c-s>1/\mathrm{poly}(n)}} \mathsf{QW}_{c,s}[t].$$

We have considered a slight variation for the definition of QW[t] as compared to that in [BJM⁺22] where this class did not include a possible dependence on n in the completeness and soundness parameters. We have chosen the present definition as we want to allow for the possibility of a polynomial gap. Central to our work is the weighted version of the local Hamiltonian problem

which we prove is in QW[1]. As is mentioned in the introduction, it was shown in [BJM⁺22] this problem is in XP (for a definition of this class see [DF99]), which is in stark contrast to the WEIGHT-*k* QUANTUM CIRCUIT SATISFIABILITY problem whose slices are BQP-hard and hence cannot be in XP unless P = BQP. By proving that the weighted local Hamiltonian problem is in QW[1] we demonstrate in this paper a likely separation between this problem and other parameterized variants of QMA-complete problems such as Quantum Circuit Satisfiability under FPQT reductions.

Define the weighted version of the local Hamiltonian problem [BJM⁺22] as

Definition 2.9 (WEIGHT- $k \ell$ -LOCAL HAMILTONIAN(a, b)).

Instance:	An ℓ -local Hamiltonian $H \coloneqq \sum_i H_i$ on n qubits that comprises at most a polyno-
	mial in n many terms $\{H_i\}$, which each act non-trivially on at most ℓ qubits and
	have operator norm $ H_i \leq 1$.
Parameter:	A natural number k.
Yes:	There exists an n -qubit weight- k quantum state $ \psi\rangle$, such that $\langle\psi H \psi\rangle\leq a.$
No:	For every <i>n</i> -qubit weight-k quantum state $ \psi\rangle$, $\langle\psi H \psi\rangle \ge b$.

3 Weighted Local Hamiltonian is in QW[1]

In this section we prove that the weighted version of the Local Hamiltonian problem is in the class QW[1]. We state this as a theorem.

Theorem 3.1. Given a, b such that b-a > 1/poly(n), then WEIGHT- $k \ell$ -LOCAL HAMILTONIAN(a, b) is in $QW_{c,s}[1]$ for some c, s such that c - s > 1/poly(n).

The proof of Theorem 3.1 consists of a series of reductions. In the first step, we reduce the weighted local Hamiltonian problem to a weight-preserving quantum circuit satisfiability problem defined below. This step is discussed in Section 3.2.

Definition 3.1 (Weight-k Weight-Preserving Quantum Circuit Satisfiability(c, s)).

Instance:	A weight-preserving quantum circuit C on n witness qubits, $poly(n)$ ancilla qubits with circuit size $poly(n)$.
Parameter:	A natural number k.
Yes:	There exists an n -qubit weight- k quantum state $ \psi\rangle$, such that
	$\Pr[\mathcal{C}(\psi\rangle) \text{ accepts}] \geq c.$

No: For every *n*-qubit weight-*k* quantum state $|\psi\rangle$,

 $\Pr[\mathcal{C}(|\psi\rangle) \text{ accepts}] \leq s.$

Remark. Note that when initializing the ancilla qubits, we can set at most f(k) of them to $|1\rangle$, where f is some computable function. This guarantees our reduction still contained in FPQT. Also this problem doesn't require the circuit to be constant depth. We will design a constant depth circuit in the last step (in Section 3.5).

In the second step (Section 3.3), we prove that strong completeness and soundness error reduction is also possible for the weight-preserving circuits using the quantum singular value transformation. This step is necessary for the reductions in the later steps. In the third step we reduce the weight-preserving quantum circuit satisfiability problem to instances of the Local Hamiltonian problem that are *almost spatially sparse*. This notion will be defined below in Section 3.4 of the proof of Theorem 3.1. Finally in the fourth step (Section 3.5), we reduce the weighted almost spatially sparse Hamiltonian to an instance of the weighted constant-depth, weft-1, quantum circuit satisfiability problem. Before proceeding to the proof of these reductions, we will prove some preliminary results about weight-preserving quantum circuits first.

3.1 Universality of Weight-Preserving Circuits

In this section, we will show how the classic proof of quantum universality in [BBC⁺95] can be adapted to show universality of weight-preserving circuits.

Definition 3.2. An operator O acting on $(\mathbb{C}^2)^{\otimes n}$ is weight-preserving if for any k and any computational basis state $|x\rangle$ of weight k, $O|x\rangle$ is a vector in $(\mathbb{C}^2)^{\otimes n}$ of weight exactly k.

Definition 3.3. A circuit C is weight-preserving if its corresponding unitary operator is weight-preserving.

We also define the weight-preserving version of one-qubit gates.

Definition 3.4. For any single qubit gate U, define a two-qubit gate

$$\hat{U} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & U & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

It is easy to check that \hat{U} is always a weight-preserving gate. Note that When U = X, \hat{U} is the SWAP gate, this fact will be used regularly below.

The Fredkin gate (control-SWAP gate) is another example of weight-preserving gate. We will also need in Lemma 3.1 the following weight-preserving gate $E = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\delta} \end{pmatrix}$. This phase gate is necessary for universality as otherwise we will not be able to create relative phases between states such as $|00\rangle$ and $|11\rangle$.

Definition 3.5. A set of weight-preserving gates is *weight-universal* if they can (approximately) generate all weight-preserving unitary transformations.

Lemma 3.1. If a set of single-qubit gates U_1, U_2, \ldots, U_s and CNOTs form a standard universal gate set, then $\hat{U}_1, \hat{U}_2, \ldots, \hat{U}_s$, Fredkin and E gates form a weight-universal gate set when allowed two extra ancilla qubits in the state $|01\rangle$.

Proof. We follow the steps of [NC00, Chapt. 4]. In this proof the first step is to show that two-level unitary gates are universal and can generate any $d \times d$ unitary from the group U(d). Recall that two-level unitaries are gates which only act on the subspace spanned by two computational basis state, for example for d = 3 a two-level unitary could be

$$\begin{pmatrix} a & 0 & b \\ 0 & 1 & 0 \\ c & 0 & d \end{pmatrix}.$$

The authors prove that $d \times d$ unitaries can be obtained using d(d-1)/2 two level unitaries. In our case we simply need to recognize that this proof will hold in any chosen weight-k subspace. Hence we can always use the same inductive steps as those in [NC00, Sec. 4.5.1] where non-trivial unitaries are limited to this subspace. This requires at most $\binom{n}{k} \binom{n}{k} - 1/2$.

Then, by following the proof in [NC00, Sec. 4.5.2] it can be shown that if we can implement all \hat{U} operators (where U is a single qubit gate), E, and Fredkin, then we can implement any two-level unitary.

Recall that in [NC00, Sec. 4.5.2] the authors use the Gray code, which given two bitstrings generates a sequence of strings that differ by a single bit. That is the hamming weight changes by one in each step of the sequence. This sequence is used to generate a circuit of multiply-controlled single-qubit gates to define an arbitrary two-level unitary.

In our case, we cannot use this construction as it is not weight-preserving. However, note that we have the Fredkin gate in our gate set, which allows controlled swaps, and also note that we are operating in a weight-preserving space. Hence, we only need a sequence of operations that controllably swap qubits in this space and then will ultimately perform \hat{U} gate. Suppose we want to implement a two level operator in the subspace of $|s\rangle = |10001\rangle$ and $|t\rangle = |11000\rangle$. We can consider the following transformations $10001 \rightarrow 10100 \rightarrow 11000$. Essentially, we want to place (k-1) of the 1's from $|s\rangle$ in the same positions of (k-1) 1's in $|t\rangle$. The remaining non-swapped 1 of $|s\rangle$ is placed in a position next to the remaining 1 in $|t\rangle$, for instance in the previous example we performed the transformation $10001 \rightarrow 10100$ placing the last 1 in the third position, next to the second position where the last 1 of $|t\rangle$ is located. This can be implemented in the same way as in [NC00, Sec. 4.5.2] with the difference that now we apply controlled SWAP operators controlled on the rest of the qubits, see Fig. 3. Finally the operator \hat{U} acts on qubits 2 and 3 (corresponding to the second and third bits from left to right). This operator is controlled on the rest of the qubits grow left to right). This operator is controlled on the rest of the qubits from left to right). This operator is controlled on the rest of the qubits from left to right). This operator is controlled on the rest of the qubits from left to right). This operator is controlled on the rest of the qubits from left to right). This operator is controlled on the rest of the qubits and finally we revert the SWAP operations. For weight-k states we will require at most 2k SWAP gates plus the controlled \hat{U} .



Figure 3: Circuit implementing a two-level unitary between states $|s\rangle = |10001\rangle$ and $|t\rangle = |11000\rangle$. The transformation represented by the controlled SWAP gates is $10001 \rightarrow 10100$. The controlled \hat{V} gate implements the two-level transformation in the subspace spanned by $|s\rangle$ and $|t\rangle$. The black dots denote the control operations activated if the qubit is in the state $|1\rangle$ and white dots denote controls activated when the qubit is in state $|0\rangle$. The crosses indicate SWAP operations.

We now show that we can implement weight-1 two-qubit gates \hat{V} with multiple controls using only weight-1 two-qubit gates, the Fredkin, and E gates. We follow the technique employed in [BBC⁺⁹⁵] to prove this. First, by Lemma 5.1 of [BBC⁺⁹⁵], it's known that a controlled version of $W \in SU(2)$ can be implemented by considering $A, B, C \in SU(2)$ such that ABC = I and AXBXC = W. Directly employing the same decomposition in the case where \hat{W} is a weight-1 two-qubit gate, by noting that $\hat{A}\hat{B}\hat{C} = I$ and $\hat{A}(\text{SWAP})\hat{B}(\text{SWAP})\hat{C} = \hat{W}$. Note that in our case the CNOT gates become Fredkin gates. To implement a single control version of $W \in U(2)$, a controlled phase gate is included, in the weight-preserving case we use the gate E.

To construct a multiple controlled version of a unitary \hat{W} with $W \in U(2)$, consider the construction from Lemma 6.1 in [BBC⁺95]. We can create a weight-preserving version of this construction as in Fig. 4, which includes two ancilla qubits set to $|0\rangle|1\rangle$ and requires finding \hat{V} such that $\hat{V}^2 = \hat{W}$. This qubits can be reused for each gate we want to construct and thus only increases the weight of all the qubits in 1. The intuition behind the circuit is that we use the $|01\rangle$ ancilla to decide if we should apply the controlled V^{\dagger} , since in the original construction there are two CNOTs, we can replace them with SWAPs and the ancilla system. When considering more control qubits, the construction generalizes in the same way, by considering more Fredkin gates acting on ancillas instead of CNOTs. Note that if we want the controls to be activated by $|0\rangle$ instead of $|1\rangle$, we can simply introduce SWAPs in the ancilla system. With these considerations we can implement any two-level unitary constructed from circuits such as the one in Fig. 3 using only weight-1 two qubit gates, the Fredkin gate, and *E*. If we want to use the discrete set $\hat{U}_1, \dots, \hat{U}_s$ instead of all weight-1 preserving two qubit gates, then the Solovay-Kitaev theorem applies in this case and thus proves the result.

Remark. The proof above shows that to implement a two-level unitary over the weight-k subspace requires $O(2^n)$ gates from our weight-universal gate set. This exponential comes mainly from the implementation we used for the controlled \hat{W} gate. For our work in this paper, this exponential dependence is sufficient. We remark that a more efficient construction is possible, with caveat that it includes non-trivial operations outside the weight-k subspace which might be of interest to some readers. In [BBC+95] a more efficient construction is offered which scales like $O(n^2)$. We can adapt our proof to improve the scaling in the same way provided that we don't care how the two-level unitary acts outside the weight-k subspace of dimension 2. This improvement is obtained by noticing that circuits implementing two-level unitaries as in Fig. 3 only require k controls since we need to check the position of the 1's. This will imply that outside the weight-k subspace the action of the unitary will be non-trivial, but if we only care about this subspace, then the dependence will be on k rather than n for implementing them. Even more improvements can be obtained using the techniques from Lemma 7.2 and Lemma 7.3 in [BBC+95].



Figure 4: Circuit implementing a controlled version of \hat{W} with two controls. This requires two ancillas initiated in the state $|01\rangle$ and can be reused in the construction of other gates. In this circuit $\hat{V}^2 = \hat{W}$.

The following lemmas will be necessary for our proof of Theorem 3.1.

Lemma 3.2. Let $n = 2^r$ be an integer power of 2. The W state

$$|W_n\rangle = \frac{1}{\sqrt{n}} \left(|10\cdots0\rangle + |01\cdots0\rangle + \cdots + |00\cdots1\rangle\right)$$

of n qubits can be computed from $|0^{n-1}1\rangle$ by a weight-preserving quantum circuit efficiently.

Proof. We prove by induction on r that there is such circuits C_n such that $C_n|0^n\rangle = |0^n\rangle$ and $C_n|0^{n-1}1\rangle = |W_n\rangle$. First for r = 1, the result follows by applying the gate

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$
 (1)

Assume the claim is proved for $n = 2^{r-1}$ and we shall show the same for $n' = 2^r$. Notice that

$$|W_{n'}\rangle = \frac{1}{\sqrt{2}} (|W_n\rangle|0^n\rangle + |0^n\rangle|W_n\rangle),$$

which can be prepared by first apply the gate in Eq. (1) to the n + 1 and the last qubit followed by two C_{n-1} circuits acting on the first and second half of the qubits.

3.2 Weight-Preserving Quantum Circuit Satisfiability

In this section, we construct a weight-preserving verification circuit from the local Hamiltonian problem. We emphasize that the Hamiltonian does *not* need to be weight-preserving and that the resulting circuit is not of constant depth yet.

Lemma 3.3. Given a weight- $k \ell$ -local Hamiltonian problem $H = \sum_{j=1}^{m} H_j$ of m terms on n qubits and energy bounds a and b with gap $b - a > 1/\operatorname{poly}(n)$. Suppose also that $||H_j|| \le 1$ for all $j = 1, 2, \ldots, m$. Then there is a weight-preserving circuit W_H of $\operatorname{poly}(n)$ size on n + M + k + 2 qubits that accepts with probability

$$1 - \frac{m + \langle \psi | H | \psi \rangle}{2M}$$

where $|\psi\rangle$ is the input witness state and $M = 2^{\lceil \log_2 m \rceil}$, the smallest integer power of 2 larger than m.

Proof of Lemma 3.3. We use $P_m^{(k)}$ to denote the projector onto the subspace of weight-k basis states of length m. By convention, If k > m then $P_m^{(k)}$ is the zero operator. We first show how we can implement a weight-preserving unitary circuit that accepts with probability $\langle \psi | (I - H_j) | \psi \rangle / 2$. Assume for simplicity that the term H_j acts on the first ℓ qubits and let $O = (I - H_j)/2$ be a positive semi-definite operator. We are interested in the quantity $\langle \psi | O | \psi \rangle$ and we claim the following identity

$$\langle \psi | O \otimes I_{n-l} | \psi \rangle = \sum_{w=0}^{l'} \langle \psi | O^{(w)} \otimes P_{n-l}^{(k-w)} | \psi \rangle$$

for state $|\psi\rangle$ of weight k where $O^{(w)} = P_l^{(w)} O P_l^{(w)}$, $l' = \min(k, l)$. This follows by computing the matrix entries of $O \otimes I_{n-l}$ with indices i, i' of weight k. Alternatively, one can see that

$$\begin{split} \langle \psi | O \otimes I | \psi \rangle &= \left\langle \psi \Big| \left(\sum_{w=0}^{l'} P_l^{(w)} \otimes P_{n-l}^{(k-w)} \right) O \otimes I \left(\sum_{w'=0}^{l'} P_l^{(w')} \otimes P_{n-l}^{(k-w')} \right) \Big| \psi \right\rangle \\ &= \left\langle \psi \Big| \sum_{w=0}^{l'} P_l^{(w)} O P_l^{(w)} \otimes P_{n-l}^{(k-w)} \Big| \psi \right\rangle \\ &= \left\langle \psi \Big| \sum_{w=0}^{l'} O^{(w)} \otimes P_{n-l}^{(k-w)} \Big| \psi \right\rangle \end{split}$$

Now we introduce two ancilla qubits starting in state $|01\rangle$. Then the following matrix

$$U^{(w)} = \begin{pmatrix} I & 0 & 0 & 0\\ 0 & \sqrt{O^{(w)}} & \sqrt{I - O^{(w)}} & 0\\ 0 & \sqrt{I - O^{(w)}} & -\sqrt{O^{(w)}} & 0\\ 0 & 0 & 0 & I \end{pmatrix}$$

is unitary and weight-preserving. It is unitary as $U^{(w)}(U^{(w)})^{\dagger} = I$ follows by direct calculations. The weight-preserving property follows from the weight-preserving property of $O^{(w)}$, and therefore also $\sqrt{O^{(w)}}$ and $\sqrt{I - O^{(w)}}$. The ancilla qubits in the state $|01\rangle$ are chosen such that $U^{(w)}|\psi\rangle|01\rangle = \sqrt{O^{(w)}}|\psi\rangle|01\rangle + \sqrt{I - O^{(w)}}|\psi\rangle|10\rangle$. We want to act with $U^{(w)}$ conditioned on the remaining n - l qubits having weight k - w. We can do this by adding k + 1 ancillas in the state $|100 \cdots 0\rangle$ and then act on this ancilla registers with controlled gates that perform a cyclic shift of the registers controlled by the original n - l qubits. We define S as the circular shift operator that act as $S|i_1i_2 \dots i_n\rangle = |i_ni_1 \dots i_{n-1}\rangle$. We can define the circuit V_{weight} formally as

$$V_{\text{weight}} = \sum_{i=0}^{l'} P_{n-l}^{(k-i)} \otimes S^i.$$

The circuit is drawn in Fig. 5. If the remaining n-l have weight k-i, then the k+1 ancillas gets rotated from $|10^k\rangle$ to $|0^i10^{k-i}\rangle$. Consider the probability that we measure the first group of ancillary qubits in basis $|01\rangle$,

$$\begin{split} \left\| (|01\rangle\langle 01|\otimes I)U|01\rangle|\psi\rangle|10^{l'}\rangle \right\|^2 &= \left\| |01\rangle \otimes \left(\sum_{w=0}^{l'} \sqrt{O^{(w)}} \otimes P_{n-l}^{(k-w)}|\psi\rangle \otimes |0^w 10^{k-w}\rangle\right) \right\|^2 \\ &= \left\langle \psi|\sum_{w=0}^{l'} O^{(w)} \otimes P_{n-l}^{(k-w)}|\psi\rangle \\ &= \left\langle \psi|O|\psi \right\rangle. \end{split}$$

We are now ready to construct the weight-preserving circuit for the local Hamiltonian H. It consists of two registers of qubits. The first is the term selection register of $M = 2^{\lceil \log_2 m \rceil}$ qubits. The second register contains n qubits representing the witness state to the Hamiltonian problem.



Figure 5: Circuit implementing the observable $O = (I - H_j)/2$ described in the text. The unitary V_{weight} writes the weight of the n - l qubits on the counting registry $|10 \cdots 0\rangle$. The circuit acts on the ℓ qubits (and the pair of ancillas) depending on this weight.

The circuit starts with the preparation of the M-qubit $|W\rangle$ state in the term selection register. For all j = 1, 2, ..., m and conditioned on the *j*-th qubit in the term selection register being in state $|1\rangle$, we perform the network of SWAP gates that moves the qubits that H_j acts on to the first ℓ qubits, apply the weight-preserving energy measurement circuit for $O = (I - H_j)/2$ as described above, note that the measurement performed depends on the chosen *j* as well. For all $j = m + 1, \ldots, M$, the circuit accepts immediately.

It is easy to check that all gates used in the circuit are weight-preserving and the circuit accepts with probability

$$\frac{M-m}{M} + \sum_{j=1}^{m} \frac{1-\langle \psi | H_j | \psi \rangle}{2M} = 1 - \frac{m+\langle \psi | H | \psi \rangle}{2M}.$$

3.3 Weight-Preserving Marriott-Watrous Amplification

In this section, we prove that it is possible to amplify the completeness and soundness gap for weight-preserving verification circuits with one copy of the witness state. For standard QMA verifiers, this is known as the strong completeness-soundness gap amplification first established by Marriott and Watrous in [MW05], where the construction iteratively measures the post-measurement states of the verifier circuit in some structured way and makes the final decision by performing a counting procedure on the measurement outcomes. This standard construction does not fit will in the weight-preserving scenario as it is difficult to encode polynomially many measurement outcome bits in a Hilbert space of dimension roughly n^k .

For this reason, we would use the fast QMA reduction [NWZ09, Gil19]. We employ a version inspired by the quantum singular value transformation (QSVT) algorithm in the following [Gil19] to amplify the error gap of the verification circuit in a weight preserving manner.

Theorem 3.2. Given a verifier circuit V for a language $L \in \mathsf{QMA}$ with acceptance probability thresholds (a, b), we can construct a new verifier circuit V' with threshold $a' = \epsilon, b' = 1 - \epsilon$ with

one extra ancillary qubit, and $m = O\left(\frac{1}{\max\left[\sqrt{b}-\sqrt{a},\sqrt{1-a}-\sqrt{1-b}\right]}\log\left(\frac{1}{\epsilon}\right)\right)$ calls to V and V[†] as in Fig. 6.

By careful examination of the new circuit constructed in [Gil19], we can show that the circuit could be implemented in a weight preserving manner, giving us the following corollary:

Corollary 3.1. Given an instance circuit C of weight-k weight preserving quantum circuit with completeness and soundness $c, s, (c-s > 1/\operatorname{poly}(n))$, we can construct a new weight preserving circuit C' with threshold $c' = 1 - \epsilon, s' = \epsilon$, by making $\operatorname{poly}(n) \log(1/\epsilon)$ calls to the circuit C.

Now we explicitly write out the circuit in the previous construction. If we assume $V|\psi\rangle|1^{f(k)}0^p\rangle = \alpha|1\rangle|\varphi_1\rangle + \beta|0\rangle|\varphi_0\rangle$, let $\Phi \in \mathbb{R}^{2m}$, define the following circuit U_{Φ} :

$$U_{\Phi} = \prod_{j=1}^{n} \left(e^{i\phi_{2j-1}(2\Pi - I)} V^{\dagger} e^{i\phi_{2j}(2\Pi - I)} V \right),$$

where the $\Pi = I \otimes |1^{f(k)} 0^p \rangle \langle 1^{f(k)} 0^p |$ is the projector that checks the ancillary qubits are correctly initialized, and $\tilde{\Pi} = |1\rangle \langle 1| \otimes I$ is the accepting projector on the output qubit of V.

It is shown in [Gil19] that there exists some $\Phi \in \mathbb{R}^{2m}$, where *m* is set as in Theorem 3.2, such that

$$\begin{aligned} ||(\langle +|\otimes \Pi)(|0\rangle\langle 0|\otimes U_{\Phi}+|1\rangle\langle 1|\otimes U_{-\Phi})(|+\rangle\otimes |\psi\rangle)||^{2} &\geq 1-\epsilon, \quad \text{ if } ||\Pi V|\psi\rangle||^{2} \geq b; \\ ||(\langle +|\otimes \Pi)(|0\rangle\langle 0|\otimes U_{\Phi}+|1\rangle\langle 1|\otimes U_{-\Phi})(|+\rangle\otimes |\psi\rangle)||^{2} &\leq \epsilon, \quad \text{ if } ||\Pi V|\psi\rangle||^{2} \leq a; \end{aligned}$$

To implement the controlled U_{Φ} in the previous formula, we only need to implement the gates $\sum_{b} |b\rangle \langle b| \otimes e^{i(-1)^{b}\phi(2\Pi - I)}$ as in Fig. 6.



Figure 6: Implementing $\sum_{b} |b\rangle \langle b| \otimes e^{i(-1)^{b} \phi(2\Pi - I)}$.

The C_{Π} NOT gate is defined as $\Pi \otimes X + (I - \Pi) \otimes I$. In our weight preserving reduction, we replace the circuit V with our weight preserving instance C, and encode the ancillary qubit in the $\{|01\rangle, |10\rangle\}$ space as before, replacing all operations on the ancilla with their weight preserving counterpart. In the end, we measure the circuit with $\left(\frac{(|01\rangle+|10\rangle)(\langle01|+\langle10|)}{2}\otimes\Pi\right)$, and accept if the output is 1. The projector Π could be implemented by counting the weight of first f(k) qubits and rest of qubits using the shifting trick.

We complete the proof by examining that each gate in the new constructed circuit is weight preserving.

3.4 Spatially Sparse Weighted Local Hamiltonian

We now show that any weight-preserving circuit W with R gates acting on n qubits and with a weight-k witness state can be transformed to a weight-(2k + 1) local Hamiltonian problem that is *almost spatially sparse* (defined below). The almost spatially sparsity will be used in the end to prove that the problem is in QW[1].

Definition 3.6 (Spatially Sparse Local Hamiltonian). A local Hamiltonian problem is *spatially sparse* if each qubit is only acted by O(1) Hamiltonians.

Definition 3.7 (Almost Spatially Sparse Local Hamiltonian). A local Hamiltonian problem is *almost* spatially sparse with respect to a register of qubits if the Hamiltonian becomes spatially sparse if we remove all terms acting only on qubits in this register.

The spatially sparse local Hamiltonian is proven to be QMA complete in [OT08], their key lemma is stated as follows:

Lemma 3.4. Given a verifier circuit V_x for a language $L \in \mathsf{QMA}$, there exists a spatially sparse local Hamiltonian $H = \sum_i H_i$ and $T = \mathrm{poly}(n)$ that satisfies the following conditions:

- If V_x accepts some state $|\xi\rangle$ with probability 1ϵ , there exists state $|\psi\rangle$ that $\langle \psi|H|\psi\rangle \leq \frac{\epsilon}{T+1}$.
- If V_x accepts any state $|\xi\rangle$ with probability no larger than ϵ , then all eigenvalues of H is larger than $\frac{c(1-\sqrt{\epsilon}-\epsilon)}{T^3}$, where c is some constant.

We closely follow the construction in [OT08] to prove our weight preserving variant of Lemma 3.4. We first transform the original verification circuit V_x to an equivalent circuit U_{sp} on a grid, such that each qubit on the grid is only acted upon by a constant number of gates. Then we apply a modified version of Kitaev's circuit-to-Hamiltonian construction to obtain our Hamiltonian.

Assume $V_x = U_R \dots U_2 U_1$ acts on *n* qubits, where U_i are local gates from a universal gate set. We introduce a grid with R + 1 layers, each consists of *n* qubits. Intuitively, we want to use the *i*th qubit in layer *j* in U_{sp} to simulate the state of *i*th qubit at time step *j* in V_x .

To simulate V_x , we initialize the qubits corresponding to the input qubits of V_x in the first layer as the witness state $|\psi\rangle$ for V_x , and rest of qubits in the first layer as V_x initial work space. For the qubits in other layers, we initialize them as $|0\rangle$. On the *i*th layer, we perform the nontrivial gate U_i on the corresponding qubits, then perform SWAP gates on the qubits in the same column between layer *i* and *i* + 1. In the *R* + 1 layer, we perform the measurement on the output qubit. It is easy to verify that the circuit U_{sp} simulates V_x faithfully.

Moreover, if our V_x is a weight preserving circuit, since every gate U_i and SWAP gate are weight preserving, we can see that U_{sp} is also weight preserving. The order of actual computational gates, and SWAP gates are applied in the same order as specified in [OT08].

Now we follow [OT08] and consider a variant of Kitaev's circuit-to-Hamiltonian construction. We denote Q_{in} the set of qubits which correspond to the witness, Q_{out} the output qubit and $C = C_1, \dots, C_T$ the clock registers. The necessary change we make here is that we shall use an indicator clock instead of the unary clock for maintaining the weight property. Let $U_{sp} = W_T \dots W_2 W_1$, the clock register will have T + 1 qubits and valid clock basis states have the form $|0^{t-1}10^{T-t}\rangle_C$ for $t = 1, 2, \dots, T$. Analogous to Kitaev's reduction, our legal history state for the circuit U_{sp} is

$$|\phi\rangle = \frac{1}{\sqrt{T+1}} \sum_{t=0}^{T} |0^{t-1} 10^{T-t}\rangle_C \otimes |\xi_t\rangle,$$



Figure 7: Reproduced Figure 1 of [OT08]. Each row of the qubits has the same number as the starting circuit. The number of rows is one more than the number of gates in the starting circuit. The R-th gate is performed on the R-th row and then all qubuts are swappwd with those in the (R + 1)-th row. This lazy simulation of the circuit will ensure that each qubit is acted on by a gate at most three times.

where $|\xi_t\rangle = W_t |\xi_{t-1}\rangle, |\xi_0\rangle = |\psi\rangle \otimes |1^{f(k)}0^m\rangle$. Since each W_t is weight preserving, the initial state $|\xi_0\rangle$ for U_{sp} has weight k + f(k) = k'.

First, we recall the Hamiltonian construction in [OT08], by replacing the clock checking term for unary clock to indicator clock, we have the following construction:

$$\begin{split} H_{\rm in}' &= \sum_{q \not\in Q_{\rm in}} |\bar{i}_q\rangle \langle \bar{i}_q|_q \otimes |1\rangle \langle 1|_{C_{t_q-1}} \,, \\ H_{\rm out}' &= |0\rangle \langle 0|_{Q_{\rm out}} \otimes |1\rangle \langle 1|_{C_T} \,, \\ H_{\rm prop}' &= \sum_{t=1}^T H_{\rm prop,t}', \end{split}$$

and

$$H'_{\text{prop},t} = (|10\rangle\!\langle 10| + |01\rangle\!\langle 01|)_{C_{t,t+1}} - W_t \otimes |01\rangle\!\langle 10|_{C_{t,t+1}} - W_t^{\dagger} \otimes |10\rangle\!\langle 01|_{C_{t,t+1}}$$

where t_q stands for the earliest time step when qubit q is actually used, \bar{i}_q is the inverse value in which ancillary qubit q should be initialized (so if qubit q should be initialized in state $|0\rangle$, then $\bar{i}_q = 1$). Here we omitted the clock checking term, since it will be reconstructed in our final construction.

In the next step, we will perform the following isometry \mathcal{U} on the state registers of our Hamiltonian: for each qubit q, we will duplicate it in the computational basis: $\mathcal{U}|0\rangle_q = |00\rangle_{I_q}, \mathcal{U}|1\rangle_q = |11\rangle_{I_q}$, where I_q are two qubits indicating the original qubit q, and $I_q \cap I_{q'} = \emptyset$ for $q \neq q'$.

Thus our new Hamiltonian H could be constructed by conjugating \mathcal{U} over the previous construction $H = \mathcal{U}H'\mathcal{U}^{\dagger}$. Our final local Hamiltonian will have the form $H = H_{\text{in}} + H_{\text{out}} + H_{\text{prop}} +$ $H_{\text{clock}} + H_{\text{state}}$ where

$$\begin{split} H_{\rm in} &= \sum_{q \notin Q_{\rm in}} |\bar{i}_q \bar{i}_q \rangle \langle \bar{i}_q \bar{i}_q |_{I_q} \otimes |1\rangle \langle 1|_{C_{t_q-1}} \\ H_{\rm out} &= |00\rangle \langle 00|_{I_{Q_{\rm out}}} \otimes |1\rangle \langle 1|_{C_T} , \\ H_{\rm clock} &= \sum_{t < t'} |11\rangle \langle 11|_{C_{t,t'}} \\ H_{\rm state} &= \sum_q |01\rangle \langle 01|_{I_q} + |10\rangle \langle 10|_{I_q}, \\ H_{\rm prop} &= \sum_{t=1}^T H_{\rm prop,t}, \end{split}$$

and

$$H_{\text{prop},t} = (|10\rangle\langle 10| + |01\rangle\langle 01|)_{C_{t,t+1}} - W_t' \otimes |01\rangle\langle 10|_{C_{t,t+1}} - (W_t')^{\dagger} \otimes |10\rangle\langle 01|_{C_{t,t+1}},$$

where $W'_t = \mathcal{U}|_{Q_{W_t}} W_t \mathcal{U}|^{\dagger}_{Q_{W_t}} \otimes I_{2n-2|Q_{W_t}|}$, Q_{W_t} for the qubits that W_t acts on. In our new construction, our history state could be defined as $|\phi'\rangle = (I \otimes \mathcal{U})|\phi\rangle$. We can observe that \mathcal{U} doubles the weight on the state registers, the weight of our new witness state is 2k' + 1.

The difference between our construction and Oliveira-Terhal [OT08] is in the clock design and checking terms. We use the indicator clock and it is easy to see H_{clock} and H_{state} are 2-local Hamiltonians. H_{state} guarantees the two mapped qubits in I_q always have the same value, thus all legal witness should have even weight on the state registers. Since we require the weight of witness state to be odd, the clock registers must have non-zero weight, and H_{clock} guarantees the only valid clock states are the indicator states $|0^{t-1}10^{T-t}\rangle_C$.

For the completeness part, observe that if original V accepts $|\psi\rangle$ with probability $1 - \epsilon$, the history state $|\phi\rangle$ would be projected to 0 for all hamiltonian terms but H_{out} . Since U_{sp} simulates V faithfully, we obtain that $\langle \phi | H_{out} | \phi \rangle \leq \frac{\epsilon}{T+1}$.

For the soundness part, observe that \hat{H} preserves the subspace of legal history states $S = \{ |\phi\rangle : H_{\text{clock}} |\phi\rangle = H_{\text{state}} |\phi\rangle = 0 \}$, thus we can discuss the eigenvalue of H on S and S^{\perp} separately. Since any eigenvector in S^{\perp} has eigenvalue at least 1, we can focus on $H|_S$. Define $H' = H'_{\text{in}} + H'_{\text{out}} + H'_{\text{prop}}$, we have that $\mathcal{U}H'\mathcal{U}^{\dagger}|_S = H|_S$. In [OT08], they performed analysis of eigenvalue on $H'|_{\mathcal{U}^{\dagger}S\mathcal{U}}$, which is isometric to $\mathcal{U}H'\mathcal{U}^{\dagger}|_S$, thus we obtain the same eigenvalue lower bound $\frac{c(1-\sqrt{\epsilon}-\epsilon)}{T^3}$.

The resulting Hamiltonian in our reduction is not spatially sparse as in [OT08] because the clock checking Hamiltonian S_{clock} is not sparse. Excluding the clock checking terms, however, all other terms are spatially sparse. Therefore, this Hamiltonian is *almost* spatially sparse with respect to the clock register. Note that if we use Lemma 3.1 and a finite gate set, the types of resulting Hamiltonian terms will also be finite. We conclude with the following corollary:

Corollary 3.2. Given a weight-k weight-preserving quantum circuit satisfiability instance C with parameter $(\epsilon, 1 - \epsilon)$, we can construct a weight-2k' + 1 almost spatially sparse local hamiltonian instance with energy thresholds $a = \frac{\epsilon}{T+1}$, $b = \frac{c(1-\sqrt{\epsilon}-\epsilon)}{T^3}$. Furthermore, if we assume C acts on n qubits, we have $T \leq 3n(|\mathcal{C}| + 1)$, the resulting Hamiltonian would act on $2n(|\mathcal{C}| + 1) + T + 1$ qubits, and k' = f(k) for some computable function f.

3.5 QW[1] Verification for Almost Spatially Sparse Hamiltonian Problems

We are now ready to show that the almost spatially sparse Hamiltonian problem we end up with in the last subsection is in QW[1]. We shall design the constant depth circuit verifying the Hamiltonian problem using a combination of two techniques described in the following.

First, we show how to check the spatially sparse terms in constant depth. To do so, we color the terms using constant number of different colors so that all terms having the same color act on different sets of qubits, a condition that leads to constant-depth energy measurements of many Hamiltonian terms in parallel. This is easy to do by observing the structure of the terms in H_{in} , H_{out} , H_{prop} , and Fig. 7.

Second, for the checking of the indicator clock format, we can simply measure the clock register and perform classical W[1] computation to check the result. Thanks to the simplification of the clock checking term using the weight constraint, it suffices to check that there are no two 1's in the measurement outcome of the clock register. This can be done in W[1], and therefore simulated by a constant depth quantum circuit with one big AND gate.

We need the following lemma to relate parallel measurements and Hamiltonian sum later on.

Lemma 3.5. Let M_1, M_2, \ldots, M_m be *m* commuting operators satisfying $0 \le M_j \le I$, then we have

$$I - \sum_{j=1}^{m} M_j \le \prod_{j=1}^{m} (I - M_j) \le I - \frac{1}{m} \sum_{j=1}^{m} M_j.$$

Proof. By the commutativity of the m operators and the spectral decomposition theorem, this problem reduces to the scalar case. For real numbers $x_j \in [0, 1]$ where j = 1, 2, ..., m,

$$1 - \sum_{j=1}^{m} x_j \le \prod_{j=1}^{m} (1 - x_j)$$

follows from a simple induction on m and

$$1 - \frac{1}{m} \sum_{j=1}^{m} x_j \ge \prod_{j=1}^{m} (1 - x_j)$$

follows from the geometric and arithmetic mean inequality

$$\frac{\sum_j (1-x_j)}{m} \ge \left(\prod_j (1-x_j)\right)^{1/m} \ge \prod_j (1-x_j).$$

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Lemma 3.6. Let $H = \sum_{j} H_{j}$ be a local Hamiltonian problem that acts on n qubits. The energy thresholds a and b for the problem satisfies $b/n^{2} - a \ge 1/\operatorname{poly}(n)$. Suppose that Hamiltonian H is almost spatially sparse with respect to a clock register of n_{clock} qubits and that each term H_{j} in the Hamiltonian is a projector. That is, except clock checking terms $|11\rangle\langle 11|_{C_{t,t'}}$ acting on qubits C_{t} and $C_{t'}$ in the clock register, all other Hamiltonian terms in H are spatially sparse. Then, there is a QW[1] verification circuit V and $c, s \in \mathbb{R}$ satisfying $c - s \ge 1/\operatorname{poly}(n)$ such that if the ground state energy of H is at most a, V accepts with probability c while if the ground state energy of H is at least b, V accepts with probability s. Furthermore, V can be chosen so that the big gate is a classical AND gate and it is the last gate in V. *Proof.* As the Hamiltonian is almost spatially sparse, it is possible to color the terms using $n_{color}+1$ colors where n_{color} is a constant. We use $G^{(h)}$ to denote the set of terms of color h. For the first n_{color} sets $G^{(h)}$ where $h = 0, 1, \ldots, n_{color} - 1$, the terms $H_i^{(h)}$ in the color group

$$G^{(h)} = \left\{ H_j^{(h)} \mid j = 1, 2, \dots, m_h \right\}$$

acts on different qubits for all *j*. Here, m_h is the number of terms in group $G^{(h)}$. For the last group $G^{(n_{\text{color}})}$, the terms are $H_j^{(n_{\text{color}})} = |11\rangle\langle 11|_{C_{t,t'}}$ acting on all pairs of qubits $C_t, C_{t'}$ in the clock register. The number of terms in this group is $m_{n_{\text{color}}}$. Define

$$m_{\max} = \max\{m_i \mid i = 0, 1, \dots, n_{\text{color}}\}$$

For each $h = 0, 1, ..., n_{color} - 1$, the size m_h is at most n as the terms in $G^{(h)}$ all act on different qubits. For $h = n_{color}, m_h$ is at most n^2 as $n_{clock} \le n$ and the terms run over a pair of clock qubits. This implies that $m_{max} \le n^2$.

We now present the QW[1] verification circuit V as follows.

- 1. First the circuit samples a random integer $h \in \{0, 1, \dots, n_{color}\}$.
- 2. Conditioned on h the circuit checks all the terms in the group $G^{(h)}$. In particular,
 - (a) If $h < n_{color}$, the circuit performs measurements

$$\{M_{j,1}^{(h)} = I - H_j^{(h)}, M_{j,0}^{(h)} = H_j^{(h)}\},\$$

for all $j = 1, 2, ..., m_h$. The circuit outputs the AND of all measurement outcomes.

(b) If $h = n_{color}$, the circuit performs computational basis measurements on all the clock qubits. The circuit outputs the AND of all pairwise NAND of the measurement outcomes.

Next, we argue that the circuit V can be implemented as a QW[1] verification circuit where the big gate is an AND gate at the end of the circuit. First, we note that the sampling of the integer h can be done using a constant size quantum circuit and computational basis measurement. We can fanout the measurement outcomes to control the later parts in the circuit. Second, as the Hamiltonian terms in each group $G^{(h)}$ act on different qubits for all $h = 0, 1, \ldots, n_{color} - 1$, the measurements $\{M_{j,0}, M_{j,1}\}$ can be implemented in parallel. These measurements output x_h , an m_h -bit vector of classical information. For $h = n_{color}$, the circuit first measures all the clock qubits and computes the pairwise NAND of the outcome. We denote this vector of classical bits as $x_{n_{color}}$, its length is $m_{n_{color}}$. So far, all gates involved are constant size quantum circuits and the classical fanout gates. Finally, the output of the circuit V is the AND of x_h for the sampled integer h. It is easy to reuse the AND gate in all $n_{color} + 1$ cases as we can use fanout of input 1 to pad short x_h 's so that they all have length m_{max} . Then we use controlled SWAP gates to move the bits in x_h to the same register that can hold m_{max} qubits and output their AND.

To complete the proof, we will relate the acceptance probability of V to the promise conditions we have for the Hamiltonian problem. Notice that when $h = n_{color}$, the circuit accepts with probability

$$\langle \psi | \Big(\sum_{x:|x| \le 1} |x\rangle \langle x| \Big) |\psi\rangle = \langle \psi | \prod_{k,l} (I - |11\rangle \langle 11|)_{k,l} |\psi\rangle = \langle \psi | \prod_j (I - H_j^{(n_{\text{color}})}) |\psi\rangle.$$

Therefore, we can write the overall probability that this circuit accepts as

$$\Pr(V \text{ accepts}) = \frac{1}{n_{\text{color}} + 1} \sum_{h=0}^{n_{\text{color}}} \langle \psi | \bigotimes_{j=1}^{m_h} (I - H_j^{(h)}) | \psi \rangle.$$
(2)

In the yes case, the Hamiltonian has ground state energy at most a, which means that there is a witness state $|\psi\rangle$

$$\langle \psi | H | \psi \rangle = \langle \psi | \sum_{j=1}^{m} H_j | \psi \rangle \le a.$$

Hence, continuing on Eq. (2), we have

$$\begin{aligned} \Pr(V \text{ accepts}) &\geq \frac{1}{n_{\text{color}} + 1} \sum_{h=0}^{n_{\text{color}}} \langle \psi | \left(I - \sum_{j=1}^{m_h} H_j^{(h)} \right) | \psi \rangle \\ &= 1 - \frac{\langle \psi | H | \psi \rangle}{n_{\text{color}} + 1} \geq 1 - \frac{a}{n_{\text{color}} + 1}, \end{aligned}$$

where the inequality follows from Lemma 3.5.

In the no case, we have for all state $|\psi\rangle$ of certain weight

$$\langle \psi | H | \psi \rangle = \langle \psi | \sum_{j=1}^{m} H_j | \psi \rangle \ge b$$

So from Eq. (2), this gives

$$\begin{split} \Pr(V \text{ accepts}) &\leq \frac{1}{n_{\text{color}} + 1} \sum_{h=0}^{n_{\text{color}}} \langle \psi | \left(I - \frac{1}{m_h} \sum_{j=1}^{m_h} H_j^{(h)}\right) | \psi \rangle \\ &\leq \frac{1}{n_{\text{color}} + 1} \sum_{h=0}^{n_{\text{color}}} \langle \psi | \left(I - \frac{1}{m_{\text{max}}} \sum_{j=1}^{m_h} H_j^{(h)}\right) | \psi \rangle \\ &= 1 - \frac{\langle \psi | H | \psi \rangle}{m_{\text{max}}(n_{\text{color}} + 1)} \leq 1 - \frac{b}{n^2(n_{\text{color}} + 1)}, \end{split}$$

where the first inequality follows from Lemma 3.5. That is, we can choose

$$c = 1 - \frac{a}{n_{\text{color}} + 1}, \quad s = 1 - \frac{b}{n^2(n_{\text{color}} + 1)}.$$

The condition on the gap $c - s = (b/n^2 - a)/(n_{color} + 1) \ge 1/poly(n)$ follows from the strong gap condition on a, b for the Hamiltonian problem.

From this proof we can also conclude that WEIGHT- $k \ell$ -LOCAL HAMILTONIAN and WEIGHT-k WEIGHT-PRESERVING QUANTUM CIRCUIT SATISFIABILITY can be reduced to each other.

Corollary 3.3. Given a, b with b - a > 1/poly(n), WEIGHT- $k \ell$ -LOCAL HAMILTONIAN(a, b) reduces to WEIGHT-k WEIGHT-PRESERVING QUANTUM CIRCUIT SATISFIABILITY(c, s) under FPT reduction for some c, s such that c - s > 1/poly(n). The same is true when reducing WEIGHT-k WEIGHT-PRESERVING QUANTUM CIRCUIT SATISFIABILITY(c, s) to WEIGHT- $k \ell$ -LOCAL HAMILTONIAN(a, b).

Proof. That WEIGHT- $k \ell$ -LOCAL HAMILTONIAN(a, b) reduces to WEIGHT-k WEIGHT-PRESERVING QUANTUM CIRCUIT SATISFIABILITY(c, s) has been already shown. It has been shown also that WEIGHT-k WEIGHT-PRESERVING QUANTUM CIRCUIT SATISFIABILITY(c, s) reduces to almost spatially sparse weighted Local Hamiltonians.

Finally combining the beyond sections together, we could provide a proof for Theorem 3.1.

Proof. By Lemma 3.3, given a WEIGHT-k LOCAL HAMILTONIAN (a, b) instance $H = \sum_{j=1}^{m} H_j$ on n qubits with $b - a > 1/\operatorname{poly}(n)$, we can obtain a WEIGHT-k WEIGHT-PRESERVING QUAN-TUM CIRCUIT SATISFIABILITY instance W with size $O(km \operatorname{poly}(n)) = O(k \operatorname{poly}(n))$, acting on $O(n + M + k) = \operatorname{poly}(n) + k$ qubits, completeness $1 - \frac{m+a}{M}$ and soundness $1 - \frac{m+b}{M}$.

 $O(n + M + k) = \operatorname{poly}(n) + k \text{ qubits, completeness } 1 - \frac{m+a}{M} \text{ and soundness } 1 - \frac{m+b}{M}.$ Now we can apply Corollary 3.1 to amplify the gap to $(2^{-n}, 1 - 2^{-n})$, and the new circuit has size $|\mathcal{C}| = O\left(\frac{m}{b-a}|W|\log(2^n)\right) = O(k \operatorname{poly}(n))$ acting on $n' = \operatorname{poly}(n) + k$ qubits. Using the parameters in Corollary 3.2, we can construct a weight 2k' + 1 almost spatially sparse local Hamiltonian instance H_{sp} with following parameters: $k' = k + O(1), T \leq 3n'(|\mathcal{C}|+1) = O(k^2 \operatorname{poly}(n)), a = \frac{1}{(T+1)2^n}, b = \frac{c(1-2^{-n/2}-2^{-n})}{T^3}$. The Hamiltonian H_{sp} acts on $n_f = O(k^2 \operatorname{poly}(n))$ qubits.

Finally we apply Lemma 3.6 to obtain our final QW[1] circuit. We can check that the energy thresholds a, b we obtained in the step beyond satisfies $b/n_f^2 - a \ge 1/\operatorname{poly}(n)$. Thus our QW[1] circuit constructed in Lemma 3.6 has probability gap $c - s \ge 1/\operatorname{poly}(n)$ since $k \le n$.

4 Frustration-Free Weighted Hamiltonian Problems

In Hamiltonian complexity theory, there is a variant of the local Hamiltonian problems with physical relevence called frustration-free Hamiltonian problems. A Hamiltonian $H = \sum_j H_j$ is frustration-free if its ground state $|\psi\rangle$ has the lowerest possible energy for each term H_j . That is, $\langle \psi | H_j | \psi \rangle = \lambda_{\min}(H_j)$. In this case, it is convenient to shift the spectrum of the local terms so that $H_j \ge 0$ and require that $\langle \psi | H_j | \psi \rangle = 0$. We define a weighed version of the frustration-free Hamiltonian problem as follows.

Definition 4.1 (FRUSTRATION-FREE WEIGHT- $k \ell$ -LOCAL HAMILTONIAN PROBLEM).

Instance:	A local Hamiltonian $H = \sum_{j} H_{j}$ on n qubits and a real number $b \ge 1/\operatorname{poly}(n)$.
	Each term H_j acts non-trivially on at most ℓ qubits and satisfies that $0 \le H_j \le I$
	for all <i>j</i> .
Parameter:	A natural number k.
Yes:	There exists an <i>n</i> -qubit weight- <i>k</i> quantum state $ \psi\rangle$, such that $\langle \psi H_j \psi\rangle = 0$ for all <i>j</i> .
No:	For all <i>n</i> -qubit weight-k quantum state $ \psi\rangle$, $\langle\psi H \psi\rangle \ge b$.

It is evident that the frustration-free weighted Hamiltonian problems are equivalent to the weighted quantum satisfiability problems defined below.

Definition 4.2 (WEIGHT-k QUANTUM ℓ -SAT PROBLEM).

Instance:	A set of projectors Π_j for $j = 1, 2,, m$ and a real number $b \ge 1/\operatorname{poly}(n)$. Each term Π_j acts on at most ℓ qubits.
Parameter:	A natural number k.
Yes:	There exists an <i>n</i> -qubit weight- <i>k</i> quantum state $ \psi\rangle$, such that $\Pi_j \psi\rangle = 0$ for all <i>j</i> .

No: For all *n*-qubit weight-*k* quantum state $|\psi\rangle$, $\sum_{j=1}^{m} \langle \psi | \Pi_j | \psi \rangle \ge b$.

We will show that the weighed quantum SAT problems are complete for $SQW_1[1]$, a variant of $QW_1[1]$.

Definition 4.3 (SPECIAL WEIGHT-*k* WEFT-1 DEPTH-*d* QUANTUM CIRCUIT SATISFIABILITY).

Instance:	A weft-1 depth-d quantum circuit C on n witness qubits and $poly(n)$ ancilla qubits where the only big gate is an AND gate and it is the last gate of the circuit.
Parameter:	A natural number k.
Yes:	There exists an n -qubit weight- k quantum state $ \psi\rangle$, such that
	$\Pr[\mathcal{C}(\psi\rangle) \text{ accepts}] \geq c.$

No: For every *n*-qubit weight-*k* quantum state $|\psi\rangle$, $\Pr[\mathcal{C}(|\psi\rangle)$ accepts] $\leq s$.

Definition 4.4. The class SQW[1] consists of all parameterized problems that are FPQT reducible to SPECIAL WEIGHT-k WEFT-1 DEPTH-d QUANTUM CIRCUIT SATISFIABILITY for some constant d and completeness and soundness c, s satisfying $c - s \ge 1/\operatorname{poly}(n)$. The class SQW₁[1] consists of all parameterized problems that are FPQT reducible to SPECIAL WEIGHT-k WEFT-1 DEPTH-d QUANTUM CIRCUIT SATISFIABILITY for some constant d, c = 1, and $s \le 1 - 1/\operatorname{poly}(n)$.

It is obvious that $W[1] \subseteq SQW[1] \subseteq QW[1]$ and $W[1] \subseteq SQW_1[1] \subseteq QW_1[1]$. This is because the big AND gate can be simulated by a big Toffoli gate.

Theorem 4.1. WEIGHT-*k* QUANTUM ℓ -SAT problem and FRUSTARTION-FREE WEIGHT-*k* ℓ -LOCAL HAMILTONIAN problem are complete problems for SQW₁[1] for some constant ℓ .

Proof. The fact that these two problems are in $SQW_1[1]$ follows from the proof that the weighted local Hamiltonian problem is in SQW[1] and that perfect completeness is preserved in the chain of reductions from the local Hamiltonian to the SQW[1] circuit problem.

We now prove that the weighed quantum ℓ -SAT problem is SQW₁[1]-hard. Let V be the constant-depth circuit representing an SQW₁[1] circuit. The decision of the circuit is made by first measuring some or all output qubits and then outputing the AND of the measurement outcomes.

The weighted quantum SAT problem we construct consists of n projectors

$$\Pi_j = V^{\dagger}(|0\rangle \langle 0|_j \otimes I) V.$$

Let $|\psi\rangle$ be the witness state to the verification circuit and assume without loss of generality that the first *n* qubits are measured. The acceptance probability of the circuit is then

$$\Pr(V \text{ accepts}) = \langle \psi, 0^{\ell} | V^{\dagger}(|1^n\rangle \langle 1^n| \otimes I) V | \psi, 0^{\ell} \rangle.$$

It is straightforward to rewrite it as

$$\Pr(V \text{ accepts}) = \left\langle \psi, 0^{\ell} \right| \prod_{j=1}^{n} (I - \Pi_j) \left| \psi, 0^{\ell} \right\rangle.$$



Figure 8: An illustration of computing the light cone of a quantum circuit with classical fanout gates. In the exmaple, there is a classical fanout gate in the dashed box. The qubits 2, 3, 4, 5, 6 and gates V_1 to V_6 are in the light cone.

Then, for yes-cases, we have

$$\begin{split} \left\langle \psi, 0^{\ell} \right| \frac{1}{n} \sum_{j=1}^{n} \Pi_{j} \left| \psi, 0^{\ell} \right\rangle &= 1 - \left\langle \psi, 0^{\ell} \right| \left(I - \frac{1}{n} \sum_{j=1}^{n} \Pi_{j} \right) \left| \psi, 0^{\ell} \right\rangle \\ &\leq 1 - \left\langle \psi, 0^{\ell} \right| \prod_{j=1}^{n} (I - \Pi_{j}) \left| \psi, 0^{\ell} \right\rangle \\ &= 1 - \Pr(V \text{ accepts}) = 0. \end{split}$$

In the above equation, the inequality follows from Lemma 3.5. As each Π_j is a projector, this implies that $\langle \psi, 0^{\ell} | \Pi_j | \psi, 0^{\ell} \rangle = 0$.

For the no-cases,

$$\begin{split} \left\langle \psi, 0^{\ell} \right| \sum_{j=1}^{n} \Pi_{j} \left| \psi, 0^{\ell} \right\rangle &= 1 - \left\langle \psi, 0^{\ell} \right| \left(I - \sum_{j=1}^{n} \Pi_{j} \right) \left| \psi, 0^{\ell} \right\rangle \\ &\geq 1 - \left\langle \psi, 0^{\ell} \right| \prod_{j=1}^{n} (I - \Pi_{j}) \left| \psi, 0^{\ell} \right\rangle \\ &= 1 - \Pr(V \text{ accepts}) \geq 1/\operatorname{poly}(n). \end{split}$$

To complete the reduction, we need to show that the projectors Π_j act on constant number of qubits. We prove this using a light-cone argument.

For a quantum circuit Q with classical fanouts, we define the light cone of an output qubit to be the gates and qubits that output qubit j depends on. More precisely, we need to model quantum circuits as a network of gates and wires and the light cone of a qubit is defined inductively by

tracing the circuit backwards from the output wire as follows. Special care needs to be taken when the gate is a classical fanout gate. As Q is a constant depth circuit, it is posssible to partition the gates into d layers of gates, each consisting of non-overlapping unitary gates. To simplify the presentation, we notice that if a gate uses classical fanout input as in the gate V_4 in Fig. 8, the measurement on the witness state that the circuit represents will not change if we change the fanout-target control qubit (qubit 9) of V_4 to qubit 6, the control qubit of the fanout gate. This change may increase the depth but will not change the final measurement outcome of the circuit by the principle of delayed measurements. We implement this type of change for such cases as a preprocessing step and let the resulting circuit be Q'. By the above discussion, Q and Q' define the same measurement Π_j . We now consider the light cone of qubit j for Q' inductively. In the base case, only the output qubit wire j is in the light cone. If we find a gate that is not a classical fanout gate and has an output wire in the light cone, we include all of its input wires to the light cone. As when we trace back each of the d levels of Q', the number of wires in the light cone is at most multiplied by a constant, there will be at most $2^{O(d)}$ input qubit wires in the light cone.

In the above, we see the crucial difference between a classical fanout and a quantum fanout. For a classical fanout gate, we only incluce control qubit in the light cone and do not count the target qubits as the classical information on these target qubits can be inferred from the measurement of the control qubit. While if a quantum fanout gate is in the light cone, then all of its input qubits should be included because of the entangling power of the quantum fanout gate.

Several remarks are in order. First, we remark that the above proof does not seem to work if we want to show weighted local Hamiltonian problems are complete for SQW[1]. The difficulty is due to the fact that we will need a strong gap condition that the energy thresholds a = n(1 - c) and b = (1 - s) satisfy $b - a \ge 1/\text{poly}(n)$. The condition holds automatically if c = 1 but may be false if c < 1 and there is no easy way to amplify the gap as we are working with constant depth circuits.

Second, because of the completeness proof of weighted quantum SAT for $SQW_1[1]$, the problem of whether weighted quantum SAT problems are complete for $QW_1[1]$ is essentially equivalent to whether there is a normalization theorem for weft-1 quantum circuits like in the classical case. Classically, the normalization theorem says that any classical weft-1 circuit can be reduced to a circuit where the only big gate is the last gate and it is the AND gate. The quantum case is technically challenging. The simplest case we don't know how to normalize is when the circuit has a big NAND gate in the end. In the classical case, the technique is that we can ask the prover for some extra information like where a 0 input to the NAND gate is (using a weight-1 indicator) that ensures the acceptance by the final NAND gate. In the quantum case however, the meaurement before the classical big NAND gate has intrinsic randomness and the prover is not able to predict the place of the 0's beforehand.

Third, in the above proof, the locality of the resuling Hamiltonian depends on the depth d of the circuit, which is at most $O(2^d)$. It is possible to bring down the locality by going throught the chain of reduction in Section 3 so that it is independent of the depth d.

5 QW-hierarchy and ETH

As mentioned in the introduction, one of the most important uses of parameterized complexity theory is in the fine-grained complexity analysis. In particular, there are important connections between W[1] and the exponential time hypothesis (ETH), some of which are presented in the book Fundamentals of Parameterized Complexity by Downey and Fellows (2013) [DF13]. We

use the version of ETH that can be found in Section 29.4 of [DF13]. In what follows we say that a circuit C has total description size D if the number of inputs and total number of gates are bounded by D.

Definition 5.1 (Exponential Time Hypothesis (ETH)). We define the Exponential Time Hypothesis as follows. There is no algorithm with running time $2^{o(n)}$ that decides for a weft 1 Boolean circuit C of total description size n, whether there is an input vector x such that C(x) = 1.

Note that this is a weaker definition than the typical one for ETH. The reason for the slight weakening of the hypothesis is done in order to make connections between fine-grained complexity and parameterized complexity (see Chapter 16, in [FG06]). In this section we shall consider a quantum version of ETH together with a quantum-classical version.

Definition 5.2 (Quantum Exponential Time Hypothesis (QETH)). We define the QETH as follows. For some c, s with $c - s > 1/\operatorname{poly}(n)$, there is no *quantum* algorithm running in time $2^{o(n)}$ that decides for a weft-1 quantum circuit Q of total description size n whether (i) there is an input witness state $|\psi\rangle$ such that $\Pr(Q(|\psi\rangle) \operatorname{accepts}) \ge c$ or (ii) for all input witness states $|\psi\rangle$, $\Pr(Q(|\psi\rangle) \operatorname{accepts}) \le s$.

Definition 5.3 (Quantum-Classical Exponential Time Hypothesis (QCETH)). We define the QCETH as follows. There is no *quantum* algorithm running in time $2^{o(n)}$ that decides for a weft-1 Boolean circuit C of total description size n, whether there is an input vector x such that C(x) = 1.

We have defined QETH as a hypothesis about *some* pair c, s with polynomial gap rather than all such pairs c, s. The reason for this choice is that we want to show that if certain problems are tractable given any polynomial gap, then QETH is false. This will be evident later in this section. Nonetheless, we remark that by changing the definition of QETH, Proposition 5.1 would not be affected and Theorem 5.6 would require some minimal modification. A natural question is the relationship between these two hypothesis just defined. We prove first QETH is a weaker statement than QCETH.

Proposition 5.1. QCETH *implies* QETH.

Proof. Assume that QETH is false, then there is a quantum algorithm \mathcal{A} deciding the problem in Definition 5.2. We shall construct a quantum circuit \mathcal{Q} and show that the satisfiability problem on \mathcal{C} reduces to the satisfiability problem on \mathcal{Q} . Let \mathcal{C} be a weft-1 classical circuit of total description size n, we can assume that C has n gates of bounded fan-in f with gate basis {AND, OR, NOT}. First, we modify \mathcal{C} into a reversible circuit by adding an ancilla bit initialized at 0 for each AND and OR gate, including the weft-1 gates. Note that this increases the number of input bits by n since there are at most n gates. For the fan-out gates in the classical circuit, these can be replaced by reversible CNOTs. Note that there are at most $f \cdot n$ possible inputs to the bounded fan-in gates, which implies we require at most O(n) CNOT gates. After this procedure, we end up with at reversible circuit which can be transformed easily into a quantum circuit \mathcal{Q} with O(n) inputs and O(n) gates and generalized Toffoli for weft-1 gates. We also include in \mathcal{Q} a procedure to check that the ancilla qubits are all set to $|0\rangle$, which requires O(n) measurements and gates.

Now we show the decision problem in Definition 5.3 with circuit C reduces to the promise problem with circuit Q in Definition 5.2 with completeness b = 1 and soundness a = 0. If there is $x \in \{0,1\}^n$ such that C(x) = 1, then consider the state $|x0^{cn}\rangle$ where c > 0 and (c+1)n is the number of inputs to Q. We have then $Q|x0^{cn}\rangle = \sum_{u \in \{0,1\}^{(c+1)n-1}} \beta_y |1y\rangle$, where $\beta_y \in \mathbb{C}$ and $\sum_{y} |\beta_y|^2 = 1$. Letting $\Pi_1^{(0)} = |1\rangle\langle 1|$ be the projector onto the state $|1\rangle$ for the first qubit, we have that

$$\Pr\left(\mathcal{Q} \text{ accepts } |x0^{cn}\rangle\right) = \left\|\Pi_{1}^{(0)}\mathcal{Q}|x0^{cn}\rangle\right\|^{2}$$
$$= \left\|\sum_{y \in \{0,1\}^{(c+1)n-1}} \beta_{y}|1y\rangle\right\|^{2}$$
$$= 1$$
(3)

Suppose now that for all $x \in \{0,1\}^n$, $\mathcal{C}(x) = 0$. We have that $\mathcal{Q}|x0^{cn}\rangle = \sum_{y \in \{0,1\}^{(c+1)n-1}} \beta_{y,x}|0y\rangle$ and thus $\Pi_1^{(0)}\mathcal{Q}|x0^{cn}\rangle = 0$. Any state passing the initial verification of the ancillae qubits has the form $|\psi\rangle = \sum_{x \in \{0,1\}^n} \gamma_x |x0^{cn}\rangle$, with $\sum_x |\gamma_x|^2 = 1$. Then we have that

$$\Pr\left(\mathcal{Q} \text{ accepts } |\psi\rangle\right) = \left\|\Pi_{1}^{(0)}\mathcal{Q}|\psi\rangle\right\|^{2}$$
$$= \left\|\Pi_{1}^{(0)}\sum_{x,y}\beta_{y,x}\gamma_{x}|0y\rangle\right\|^{2}$$
$$= 0. \tag{4}$$

This shows the reduction and thus algorithm \mathcal{A} can solve in time $2^{o(n)}$ the decision problem for circuit \mathcal{C} and QCETH is false.

5.1 Miniaturized problems and ETH

In this subsection we shall introduce miniaturized problems which are a key ingredient in connecting results from parameterized complexity and ETH. First, we define the miniature version of the classical circuit satisfiability problem and then we will show how it connects to ETH and QCETH.

Definition 5.4 (MINI-CIRCSAT $_t$).

Instance: Positive integers k and n in unary, and a weft t Boolean circuit C of total description size at most $k \log n$.

Parameter: A natural number k.

Problem: Decide if there is an input binary vector x such that C(x) = 1.

For simplicity, we will refer to $MINI-CIRCSAT_1$ as MINI-CIRCSAT. The following theorem illustrates the connection between the tractability of miniature problems and ETH.

Theorem 5.1 (Theorem 29.4.1 in [DF13]). MINI-CIRCSAT is in FPT if and only if ETH is false.

The MINI-CIRCSAT can be then reduced to WEIGHT-k INDEPENDENT SET which implies the following theorem.

Theorem 5.2 (Section 29.4 of [DF13]). If W[1] = FPT then ETH is false.

Theorem 5.2 establishes a sufficient condition for ETH to be false. In classical parameterized complexity the complexity class M[1] is defined as the closure under FPT reductions of Mini-CircSAT, the claim that this class is tractable for FPT algorithms is equivalent to ETH being false.

Definition 5.5. Define M[t] as the set of problems FPT reducible to MINI-CIRCSAT_t

Theorem 5.3 (Restatement of Theorem 5.1). M[1] = FPT if and only if ETH is false.

As an aside, it is straightforward to see that the weighted local Hamiltonian problem is W[1]-hard, which makes unlikely any FPT algorithms for this problem as implied by the above theorem. To prove this we can simply reduce the weighted independent set problem to the weighted local Hamiltonian problem.

Proposition 5.2. *The* WEIGHT-*k* INDEPENDENT SET *problem reduces to the* WEIGHT-*k* LOCAL HAMILTONIAN PROBLEM(a, b) under FPT reductions, for any a, b with $b > a \ge 0$.

Proof. Let G = (V, E) be a graph with vertex set $V = \{1, 2, \dots, n\}$. For each $i \in V$ define a binary variable x_i and the formula $\varphi(x_1, \dots, x_n) = \bigwedge_{(i,j) \in E} (\neg x_i \lor \neg x_j)$. G has an independent set of size k if and only if φ is satisfiable by a bitstring $x = x_1, \dots, x_n$ of Hamming weight k. We can map φ to a Hamiltonian $H = \sum_i H_i$ acting over n qubits, for this, consider the one qubit projector over qubit i, $\Pi_1^{(i)} = |1\rangle\langle 1|$. We map each term $(\neg x_i \lor \neg x_j)$ to $H_i = \Pi_1^{(i)}\Pi_1^{(j)}$. This Hamiltonian H is an instance of the WEIGHT-k LOCAL HAMILTONIAN and has a ground state of energy 0 with weight-k if and only if graph G has an independent set of size k. Note this reduction works as long as the condition over the a, b in the proposition is as given.

It's known that WEIGHT-k INDEPENDENT SET is W[1]-complete [DF13], thus this implies that the weighted local Hamiltonian problem is W[1]-hard. An immediate consequence is that its unlikely that there are FPT algorithms for the weighted Local Hamiltonian as this would imply that ETH is false by Theorem 5.2. As we show in Theorem 5.7, if this problem can be solved by FPQT algorithms then this implies that QCETH is false.

We can trivially generalize Theorem 5.1 to the quantum case, in particular we will frame the results in terms of the weighted local Hamiltonian problem. We can give a trivial generalization of Theorem 5.1 as follows

Theorem 5.4. $M[1] \subseteq FPQT$ *iff* QCETH *is false*.

Proof. The proof follows from a direct generalization from the proof of Theorem 5.1 in [DF13]. If QCETH is false, then we can solve MINI-CIRCSAT with a quantum algorithm in time $2^{o(k \log n)}$ which is an FPT function, implying that $M[1] \subseteq FPQT$.

Let C be a Boolean circuit of weft 1 and size N and assume there is an FPQT algorithm that solves MINI-CIRCSAT in time $f(k)n^c$ where we assume f to be a growing function in k. We now show that there is an algorithm deciding if C is satisfiable in time $2^{o(n)}$. Take $k = f^{-1}(N)$ and $n = 2^{(N/k)}$, thus, $N = k \log n$. In general, $f^{-1}(N)$ will be a growing function of N and thus N/k = o(N). We can now consider the circuit C as an instance of MINI-CIRCSAT with k and n chosen as before, giving a runtime for the algorithm of $f(f^{-1}(N))(2^{N/k})^c = 2^{\frac{cN}{k} + \log N} = 2^{o(N)}$, thus QCETH is false.

As shown in Proposition 5.2, the weighted independent set problem can be reduced to the weighted local Hamiltonian problem. Moreover, as remarked before, MINI-CIRCSAT reduces to the weighted independent set.

This shows the following

Theorem 5.5. If WEIGHT-k *l*-LOCAL HAMILTONIAN is in FPQT then QCETH is false.

Proof. The WEIGHT-*k* INDEPENDENT SET reduces to the WEIGHT-*k* ℓ -LOCAL HAMILTONIAN, by hypothesis we can solve instances of the Local Hamiltonian problem in FPQT and thus WEIGHT-*k* INDEPENDENT SET as well. By Theorem 5.4 the result follows.

5.2 Miniaturized problems and QETH

Now we turn to a result pertaining to QETH as defined in Definition 5.2. Let us begin by defining a miniature version of the quantum circuit satisfiability problem.

We define the miniature version of the quantum circuit satisfiability MINI-QCSAT_t(a, b) and the class QM[t] as follows

Definition 5.6 (MINI-QCSAT $_t(c, s)$).

Instance:Integers k and n in unary, and weft-t quantum circuit C of description size $k \log n$.Parameter:A natural number k.Yes:There exists an input quantum state $|\psi\rangle$, such that $\Pr[\mathcal{C}(|\psi\rangle)$ accepts] $\geq c$.No:For every input quantum state $|\psi\rangle$, $\Pr[\mathcal{C}(|\psi\rangle)$ accepts] $\leq s$.

Definition 5.7. Define $QM_{c,s}[t]$ as the set of problems FPQT-reducible to MINI-QCSAT_t(c, s) and define QM[t] as

$$\mathsf{QM}[t] := \bigcup_{\substack{c,s \\ c-s > 1/poly(n)}} \mathsf{QM}_{c,s}[t].$$

We denote as Mini-QCSAT(c, s) the problem MINI-QCSAT $_1(c, s)$. Just as in the classical case, we give a theorem connecting the complexity of MINI-QCSAT and QETH from Definition 5.2.

Theorem 5.6. $QM[1] \subseteq FPQT$ *iff* QETH *is false.*

Proof. The argument from Theorem 5.4 can be repeated. First assume QETH is false, then for all c, s with polynomial gap there is an algorithm that solves the quantum circuit satisfiability problem with completeness c and soundness s with c - s > 1/poly(n). Then, given an instance C of Mini-QCSAT(c, s) we can use this algorithm to solve it in time $2^{o(k \log n)}$ which is an FPT function.

Now assume that for all c, s with polynomial gap, Mini-QCSAT(c, s) is solvable in time $f(k)n^{c_0}$ time for some constant $c_0 > 0$. Let C be a weft-1 circuit of size N. Set $k = f^{-1}(N)$ and $n = 2^{(N/k)}$, which implies $N = k \log n$. In general it will be true that N/k = o(N). Using the FPQT algorithm on C, we have a running time $2^{o(N)}$ which solves the decision problem with completeness c and soundness s. Since this is true for all c, s such that c - s > 1/poly(n) then QETH is false.

Now we show that the Mini-QCSAT reduces to the weight-preserving quantum circuit satisfiability problem from Definition 3.1.

Lemma 5.1. WEIGHT-*k* WEIGHT-PRESERVING QUANTUM CIRCUIT SATISFIABILITY(c, s) is $QM_{c,s}[1]$ -hard.

Proof. Let C describe a Mini-QCSAT(c, s) circuit with at most $k \log n$ inputs and $k \log n$ gates. We can decompose these gates into one qubit gates and CNOTs, increasing the number of gates to $poly(k \log n)$. Note that a $k \log n$ qubit state $|\chi\rangle$ can be mapped to a weight-k n-qubit state $|\psi\rangle$ by considering the natural encoding of an n qubit state of weight-k with $k \log n$ qubits. If C has less than $k \log n$ input qubits then we can always add ancillas in the $|0\rangle$ state, and measure at the end of the circuit to check that they are all in the $|0\rangle$ state, we can thus assume that C has $k \log n$ input qubits.

We take the $k \log n$ input qubits and divide them into k groups of $\log n$ qubits and consider the encoding of the log n qubit state into an n qubit state of weight-1. For bitstring $x \in \{0,1\}^{\log n}$ we denote as E(x) the encoding into a bitstring of length n and Hamming weight 1 which preserves lexicographic order. For example, if n = 4 we consider the encoding $|E(00)\rangle = |0001\rangle$, $|E(01)\rangle = |0010\rangle, |E(10)\rangle = |0100\rangle$ and $|E(11)\rangle = |1000\rangle$. This mapping will result in a circuit with kn input qubits. We now explain how to map the gates in circuit C to the encoded version \mathcal{C}' in such a way that the weight is preserved in circuit \mathcal{C}' . A one-qubit gate V in C is mapped to $2^{\log n-1} \hat{V}$ weight-preserving gates acting over two qubits as in Definition 3.4. The qubits over which these gates act can be computed efficiently. Suppose gate V acts over qubit $i \in \{1, 2, \dots, \log n\}$, where the index i runs over the qubits inside some group of $\log n$ qubits. Denote as \tilde{V} the encoded version in the new circuit of gate V. The action of \tilde{V} over basis states is defined as follows. Let $x^{(1)}, x^{(2)} \in \{0, 1\}^{\log n}$ be computational basis states which differ only on the *i*th bit, for example, suppose $x^{(1)} = 0$ and $x_i^{(2)} = 1$. Let p and q be the qubit indices in the new circuit where $E(x^{(1)})$ and $E(x^{(2)})$ have a 1. Then, \tilde{V} will act as gate \hat{V} on qubits p and q. For each such pair $x^{(1)}$ and $x^{(2)}$, \tilde{V} acts on the prescribed pair of qubits as \hat{V} . Thus, in total \tilde{V} requires $2^{\log n-1} \hat{V}$ gates. An example is illustrated in Fig. 9 where n = 8 and k = 1, each group has 3 qubits and is encoded as a group of 8 qubits.



Figure 9: Example of mapping a one-qubit gate to gates acting on 8 qubits for n = 8 and k = 1. The discontinued lines are qubits that are not acted upon by the gates.

It is simple to check that this new circuit preserves the amplitudes of the original miniature circuit. Let $x = (x_1, x_2, \ldots, x_i, \ldots, x_{\log n}) \in \{0, 1\}^{\log n}$ and $V^{(a)} = \sum_{r,s=0}^{1} v_{r,s} |r\rangle \langle s|$ the single-qubit unitary acting on qubit *i*. Then, the action of $V^{(i)}$ over a computational basis state is

$$V^{(i)}|x_1\cdots x_i\cdots x_{\log n}\rangle = v_{0,x_i}|x_1\cdots 0_i\cdots x_{\log n}\rangle + v_{1,x_i}|x_1\cdots 1_i\cdots x_{\log n}\rangle.$$

where 0_i and 1_i denote a 0 or a 1 at the *i*th position respectively. The encoded version of V will act in a similar way by construction

$$\tilde{V}^{(i)}|E(x_1\cdots x_i\cdots x_{\log n})\rangle = v_{0,x_i}|E(x_1\cdots 0\cdots x_{\log n})\rangle + v_{1,x_i}|E(x_1\cdots 1\cdots x_{\log n})\rangle.$$

For CNOT gates we need to consider two different cases, (i) the CNOT is acting between two qubits in the same group and (ii) the CNOT is acting between two qubits in different groups.

For case (i), suppose CNOT acts on control qubit i and target qubit j where i and j are in the same group of $\log n$ qubits. Let $x^{(1)}, x^{(2)} \in \{0, 1\}^{\log n}$, if they differ in the jth qubit and the ith

qubit is 1, then in the new circuit apply a SWAP between the qubits where $E(x_1)$ and $E(x_2)$ have 1s. We add as many SWAPS as pairs $x^{(1)}, x^{(2)}$ fulfilling this condition exist.

For case (ii), we consider control qubit i and target qubit j such that both qubits belong to different groups. To implement this gate in the weight-preserving circuit we will require two ancillae in the state $|01\rangle$. For every $x \in \{0,1\}^{\log n}$ such that qubit i is 1, then we apply a Fredkin gate with control qubit given by the position of 1 in E(x) and with the ancilla qubits as targets. Such an example is given in Fig. 10, inside the green box the Fredkin gates are applied such that if any of the qubits are in state $|1\rangle$ then a SWAP network is applied in the other group, after this the action of the Fredkin gates is undone. The SWAP network consists of SWAP operators acting over qubits as determined by the one-bit case mentioned earlier in our proof, these SWAP gates are controlled by the ancilla qubit.

Note that in the original circuit C the output is given by a single qubit. In the new weightpreserving circuit we can add two more extra ancillas in the state $|01\rangle$ which we assign as the output qubits. Then, after acting with the weight-preserving simulation of C, we can act with several controlled SWAP operators with the output qubits as target and the control qubits corresponding to those that encode states of $\log n$ qubits with the output set to 1.

With the mapping in place, we have constructed a weight-preserving circuit and the last step is to implement measurements to check that each group of n qubits has only one qubit set to $|1\rangle$. In what follows, let $Q_i = \{q_1^{(i)}, \ldots, q_n^{(i)}\}$ be the set of qubits belonging to the *i*th group of qubits, where $i = \{1, \ldots, k\}$. To check that each Q_i is a weight-1 state, we can include k(n+1) ancillas, which we also group into k sets of n qubits and denote as $A_j = \{a_1^{(j)}, \ldots, a_{n+1}^{(j)}\}$ the *j*th group of *n* qubits. First, initialize each A_j in the weight-1 state $|1000\cdots0\rangle$. Next, we will use the qubits in A_i to count the weight of the state in the Q_i register. We construct the following weight-preserving circuit acting between sets A_i and Q_i for each $i \in \{1, \ldots, k\}$. Act with a controlled SWAP on qubits $a_1^{(i)}, a_2^{(i)}$ as targets and qubit $q^{(i)_1}$ as control which we define as $CSWAP_{q_1,a_1,a_2}$ (for simplicity, we surpress the index *i* from now on). Then, act with the gate $CSWAP_{q_2,a_1,a_2} \cdot CSWAP_{q_2,a_2,a_3}$. We act in the same way with succesive qubits in the set Q_i ; for each qubit q_j we act with CSWAP $_{q_i,a_1,a_2}$ · CSWAP $_{q_i,a_2,a_3}$ · · · CSWAP $_{q_j,a_j,a_{j+1}}$. Once applied this circuit, we need to only measure the qubit a_2 which tells us whether the weight of the state in Q_i is 1. Finally, to measure whether all $a_2^{(i)}$ are in the state 1, we add two ancillas in state $|01\rangle$ and act with CSWAP controlled by each $a_2^{(i)}$ and target the two new ancillas, such that we get the state $|10\rangle$ if all $a_2^{(i)}$ are in 1 are $|01\rangle$ otherwise. This construction requires the ancilla states to have in total weight-(k + 1) which together with the input state and two more ancillas for the CNOT gates and other two for the output qubits gives a total of weight-2(k+2) with kn + k(n+1) + 6qubits. We have then a new weight-preserving circuit \mathcal{C}' which is satisfiable by a weight-2(k+1)state if and only if the original circuit C is satisfiable. Moreover, C' simulates C faithfully (at each step the amplitudes are preserved). Let us now show that the reduction works as intended. For completeness, since the simulation is faithfull, then our new weight-preserving circuit preserves the completeness. For soundness, suppose for all states $|\psi\rangle$ we have that $\Pr(\mathcal{C} |\psi\rangle) \leq s$. Let $|\phi\rangle$ be a kn + k(n+1) + 4 qubit state, where the witness has been supplied by the prover and the ancillas have been set as described above. Note that the only way for the prover to cheat is by breaking the encoding we have delineated above, thus we introduce the decomposition $|\phi\rangle =$ $\alpha |\xi_1\rangle + \beta |\xi_2\rangle$, where $|\xi_1\rangle$ is a state respecting the encoding and $|\xi_2\rangle$ is a state that doesn't respect the encoding above. Thus, defining Π_{10} as the projector on the output qubits onto the state $|10\rangle$,

we have that

$$\Pr\left(\mathcal{C}' \text{ accepts } |\phi\rangle\right) = \left\|\Pi_{10}\mathcal{C}'|\phi\rangle\right\|^2$$
$$= |\alpha|^2 \left\|\Pi_{10}\mathcal{C}'|\xi_1\rangle\right\|^2 \tag{5}$$

Since $|\alpha|^2 < 1$ when the prover is cheating, then the accepting probability only diminishes when this is the case. Thus $\Pr(\mathcal{C}' \text{ accepts } |\phi\rangle) \leq s$. This implies the $\mathsf{QM}_{c,s}[1]$ -hardness of WEIGHT-*k* WEIGHT-PRESERVING QUANTUM CIRCUIT SATISFIABILITY(c, s)



Figure 10: Example of mapping a CNOT gate acting between two different groups for n = 8 and k = 2. The gates in the green box implement the control and the SWAP network implements the bit flip part.

We have thus shown the following corollary.

Corollary 5.1. MINI-QCSAT(c, s) is in FPQT if WEIGHT-k WEIGHT-PRESERVING QUANTUM CIRCUIT SATISFIABILITY(c, s) is in FPQT.

Now we can use the reduction from the proof of Theorem 3.1 (Corollary 3.3) to reduce the weigh-preserving circuit to an instance of the almost spatially sparse weight-k local Hamiltonian and thus also can be reduced to the weight-k ℓ -local Hamiltonian.

Theorem 5.7. If for all a, b such that b-a > 1/poly(n), WEIGHT- $k \ell$ -LOCAL HAMILTONIAN(a, b) is in FPQT then QETH is false.

6 Discussion

In this paper we have explored the complexity of the weighted local Hamiltonian problem. We have proven that this problem is in QW[1], but it remains a challenging open question as to whether it is in fact QW[1]-complete. The obstacle when using techniques based on the clock construction

such as in [KSV02, KKR06] is that when reducing from WEIGHT-k WEFT-1 DEPTH-d QUAN-TUM CIRCUIT SATISFIABILITY to the weighted local Hamiltonian problem, the history state is required to be of weight-k. Recall that the circuit in the original instance is not required to be weight-preserving and thus applying the clock construction directly does not work as it takes the history state out of the weight-k subspace. Another possibility is to apply the reduction used in [CGW14], where the authors prove the QMA-completeness of the Bose-Hubbard model. This proof technique for QMA-hardness is based on using the Bose-Hubbard Hamiltonian to simulate the quantum circuit of the verifier. This naturally requires using O(n) particles, yet in our case the number of excitations would be bounded by k. Although the witness of the weighted quantum circuit satisfiability is also bounded, it is not guaranteed to remain bounded as, again, the circuit is not weight preserving. In the proof of Lemma 5.1, we have reduced a miniaturized version of the circuit satisfiability problem to a weight-preserving circuit, one way to prove completeness might be to extend this technique to the case of n qubits and constant depth.

An interesting direction in the future would be to study the local Hamiltonian problem under other parameterizations, one possibility is to consider parameters over the interaction graph of the Hamiltonian. We consider our results here as a first step towards a more fine-grained analysis of the complexity in the local Hamiltonian problem. In classical complexity theory, parameters such as the treewidth or branchwidth play a key role in finding FPT algorithms for graph problems. Studying the interaction of such parameters (or finding new ones) in the quantum setting for local Hamiltonian problems may prove to be a fruitful area of research for finding efficient algorithms.

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