

Asymmetric Quantum Concatenated and Tensor Product Codes with Large Z -Distances

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Abstract

In this paper, we present a new construction of asymmetric quantum codes (AQC) by combining classical concatenated codes (CC) with tensor product codes (TPC), called asymmetric quantum concatenated and tensor product codes (AQCTPC) which have the following three advantages. First, only the outer codes in AQCTPCs need to satisfy the orthogonal constraint in quantum codes, and any classical linear code can be used for the inner, which makes AQCTPCs very easy to construct. Second, most AQCTPCs are highly degenerate, which means they can correct many more errors than their classical TPC counterparts. Consequently, we construct several families of AQC with better parameters than known results in the literature. Third, AQCTPCs can be efficiently decoded although they are degenerate, provided that the inner and outer codes are efficiently decodable. In particular, we significantly reduce the inner decoding complexity of TPCs from $\Omega(n_2 a^{n_1}) (a > 1)$ to $O(n_2)$ by considering error degeneracy, where n_1 and n_2 are the block length of the inner code and the outer

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code, respectively. Furthermore, we generalize our concatenation scheme by using the generalized CCs and TPCs correspondingly.

Index Terms

Asymmetric quantum code, concatenated code, error degeneracy, tensor product code.

I. INTRODUCTION

Quantum noise due to decoherence widely exists in quantum communication channels, quantum gates and quantum measurement. It is one of the biggest challenges in realizing large-scale quantum communication systems and fully fault-tolerant quantum computation. For a quantum state, the two main mechanisms of decoherence are population relaxation and dephasing. The level of noise is usually characterized by the relaxation time \mathcal{T}_1 and the dephasing time \mathcal{T}_2 . Further, dephasing usually generates a single phase flip error, while population relaxation generates a mixed bit-phase flip error. It is shown in almost all quantum systems that, the dephasing rate $1/\mathcal{T}_2$ is much faster than the relaxation rate $1/\mathcal{T}_1$, i.e., $\mathcal{T}_1 \gg \mathcal{T}_2$ [1], [2]. For example, in the trapped ions [1], [3], the ratio $\mathcal{T}_1/\mathcal{T}_2$ can be larger than 10^2 and, in quantum dots systems [4], it can be larger than 10^4 . Such large asymmetry between population relaxation and dephasing indicates that phase flip errors (Z -errors) happen much more frequently than bit flip errors (X -errors).

Steane first saw that prior knowledge of this asymmetry in errors could be leveraged for performance gains and, hence, proposed asymmetric quantum codes (AQCs) in [5]. In the years since, many AQCs have been designed to have a biased error correction towards Z -errors [1], [6]–[8]. For example, AQCs constructed from classical Bose-Chaudhuri-Hocquenghem (BCH) codes [9] and low-density parity-check (LDPC) codes [10]–[14] were proposed in [1], [7]. The BCH codes are used to correct X -errors and the more powerful LDPC codes are used to correct Z -errors. Another approach, devised by Galindo *et al.* [15], is to introduce some preshared

entanglement [16]–[21] to help construct AQCs. More recently, asymmetric errors have been explored as a way to help improve the fault-tolerant thresholds [6], [8], particularly, in topological quantum codes [2], [22], [23]. In [8], a family of asymmetric Bacon-Shor (ABS) codes with parameters $[[mn, 1, m/n]]$, where m and n are positive integers, is used for fault-tolerant quantum computation against highly biased noise. For example, ABS codes with parameters $[[175, 1, 25/7]]$ and $[[315, 1, 35/9]]$ can achieve a very low logical error rate around 10^{-12} with much fewer physical two-qubit gates than symmetric quantum codes. In [22], surface codes defined on a $d \times d$ square lattice of qubits with $d = 12, 14, 16, 18$, and 20 have thresholds exceeding 5% when the asymmetry between Z -errors and X -errors is around 100. Even more, it is shown recently in [24] that thresholds for surface codes can exceed the zero-rate Shannon bound of Pauli channels when the asymmetry is properly large! These results reveal that the large asymmetry in quantum channels has a significant effect to quantum error correction and needs to be further exploited.

However, although there are many different constructions of AQCs in the literature, only a few are made on binary AQCs with a relatively large Z -distance d_Z . This is because the dual-containing constraint in CSS codes often makes constructing an AQC with a large minimum distance d_Z difficult. Aly [25] and Sarvepalli *et al.* [7] derived families of binary asymmetric quantum Bose-Chaudhuri-Hocquenghemms (QBCH) codes with minimum distances d_X and d_Z , both upper bounded by the square root of the block length. Li *et al.* [26] were able to construct a few binary QBCH codes of length $n = 2^m - 1$ with a large minimum distance d_Z . Ezerman *et al.* [27] constructed some binary CSS-like AQCs of length ≤ 40 with best-known parameters by exhaustively searching the database of MAGMA [28]. Additionally, several families of nonbinary AQCs with a large d_Z have been developed, but all have a large field size [29]–[31].

The key to construct an AQC is to find two classical linear codes that satisfy a certain dual-containing relationship. In classical codes, the two most useful combining methods for constructing linear codes from short constituent codes are: concatenated codes (CCs) [32] and tensor product codes (TPCs) [33], [34]. In general, CCs have a large minimum distance because

the distances in the constituent codes are multiplied, while TPCs have a poor minimum distance but a better dimension as a trade-off. In [35], Maucher *et al.* show that generalized concatenated codes (GCCs) are equivalent to generalized tensor product codes (GTPCs).

It is not difficult to apply the concatenation method to the quantum realm, i.e., to construct concatenated quantum codes (CQCs) [36], [37] and quantum tensor product codes (QTPCs) [38], [39], including asymmetric QTPCs [40] and entanglement-assisted QTPCs [41]. CQCs and QTPCs also exhibit some similar characteristics to their classical counterparts. For example, CQCs have a large minimum distance but a relatively small dimension, which is seeing them play an important role in fault-tolerant quantum computation. And, like TPCs, QTPCs have a large dimension but a small minimum distance. However, it is worth noting that CQCs are not constructed from classical CCs directly, but rather by serially concatenating two constituent quantum codes. This means both the inner and outer constituent codes need to satisfy the dual-containing relationship, which limits their construction. The same does not apply to QTPCs, giving them a distinct advantage. But QTPCs usually have a poor minimum distance. Moreover, some CQCs are known to be degenerate codes [37], which is a unique phenomenon in quantum coding theory. Degenerate codes have an advantage in that they can correct more errors than non-degenerate codes, but, in general, they are difficult to decode (see [42]) with the classical decoding algorithms often failing outright.

Hence, in this paper, we propose a novel concatenation scheme called asymmetric quantum concatenated and tensor product codes (AQCTPCs) that combines both CCs and TPCs, where CCs are used to correct Z -errors, and TPCs are used to correct X -errors. Compared to the current methods, this new concatenation scheme has several advantages.

- 1) In AQCTPCs, only the outer constituent codes over the extension field need to satisfy the dual-containing constraint. The inner constituent codes can be any classical linear codes. Then we have much freedom in the choice of the constituent codes.
- 2) It is shown that AQCTPCs can be decoded efficiently provided that the classical constituent

codes can be decoded efficiently. In addition, AQCTPCs are highly degenerate for correcting X -errors and they can correct many more X -errors beyond the error correction ability of the corresponding TPCs. Further, we show that the total inner decoding complexity of TPCs is reduced significantly from $\Omega(n_2 a^{n_1}) (a > 1)$ to $O(n_2)$ due to error degeneracy. To this end, we have developed a syndrome-based decoding algorithm specifically for AQCTPCs.

- 3) The AQCTPCs demonstrated in this paper are better than QBCH codes or asymmetric quantum algebraic geometry (QAG) codes as the block length goes to infinity. We construct a family of AQCTPCs with a very large Z -distance d_Z , of approximately half the block length. Meanwhile, the dimension and the X -distance d_X continue increasing as the block length goes to infinity. If $d_X = 2$, then the Z -distance d_Z is larger than half the block length.

We compare the parameters of AQCTPCs to previous results, and provide a generalized AQCTPC concatenation scheme that uses GCCs and GTPCs. We list AQCTPCs with better parameters than the binary extension of asymmetric quantum Reed-Solomon (QRS) codes. We derive families of AQCTPCs with the largest Z -distance d_Z compared to existed AQC with comparable block length and X -distance d_X .

The rest of this paper is organized as follows. In Section II, we provide the basic notations and definitions needed for the construction of AQCTPCs. In Section III, we present the AQCTPC concatenation scheme and the decoding algorithms. Section IV provides detailed performance comparisons of AQCTPCs against previous constructions, and the discussions and conclusions follow in Section V.

II. PRELIMINARIES

In this section we first review some basic definitions and known results about stabilizer codes and AQC, followed by the introduction of classical CCs and TPCs and their generalizations.

A. Stabilizer Codes and Asymmetric Quantum Codes

Denote by q a power of a prime p and denote by \mathbb{F}_p the prime field. Let \mathbb{F}_q be the finite field with q elements and let the field \mathbb{F}_{q^m} be a field extension of \mathbb{F}_q , where $m \geq 1$ is an integer. Let \mathbb{C} be the field of complex numbers. For a positive integer n , let $V_n = (\mathbb{C}^q)^{\otimes n} = \mathbb{C}^{q^n}$ be the n th tensor product of \mathbb{C}^q . Denote by u and v two vectors of \mathbb{F}_q^n . Define the error operators on V_n by $X(u)|\varphi\rangle = |u + \varphi\rangle$ and $Z(v)|\varphi\rangle = \zeta^{\text{Tr}(v\varphi)}|\varphi\rangle$, where “Tr” stands for the trace operation from \mathbb{F}_q to \mathbb{F}_p , and $\zeta = \exp(2\pi i/p)$ is a primitive p th root of unity. Denote by

$$G_n = \{\zeta^a X(u)Z(v) : u, v \in \mathbb{F}_q^n, a \in \mathbb{F}_q\} \quad (1)$$

the group generated by $E_n = \{X(u)Z(v) : u, v \in \mathbb{F}_q^n\}$. For any $\varepsilon = \zeta^a X(u)Z(v) \in G_n$, where $u = (u_1, \dots, u_n) \in \mathbb{F}_q^n$ and $v = (v_1, \dots, v_n) \in \mathbb{F}_q^n$, the weight of ε is defined by

$$\text{wt}_Q(\varepsilon) = |\{1 \leq i \leq n : (u_i, v_i) \neq (0, 0)\}|. \quad (2)$$

The weight of X -errors and the weight of Z -errors in ε are defined by $\text{wt}_H(u)$ and $\text{wt}_H(v)$, respectively, where “wt_H” stands for the Hamming weight. The definition of quantum stabilizer codes is given below.

Definition 2.1: A q -ary quantum stabilizer code \mathcal{Q} is a q^k -dimensional ($k > 0$) subspace of V_n such that

$$\mathcal{Q} = \bigcap_{\varepsilon \in S} \{|\varphi\rangle \in V_n : \varepsilon|\varphi\rangle = |\varphi\rangle\}, \quad (3)$$

where S is a subgroup of G_n and is called the stabilizer group. \mathcal{Q} has minimum distance d if it can detect all errors $\varepsilon \in G_n$ of weight $\text{wt}_Q(\varepsilon)$ up to $d-1$. Then \mathcal{Q} is denoted by $\mathcal{Q} = [[n, k, d]]_q$. Further, \mathcal{Q} is called non-degenerate if each stabilizer in S has quantum weight at least the minimum distance d , otherwise it is degenerate.

The Calderbank-Shor-Steane (CSS) code in [5], [43] is a special family of quantum stabilizer codes and can be constructed from two classical linear codes which satisfy some dual-containing relationship. Let d_X and d_Z be two positive integers. We define an AQC as a CSS code in V_n

with parameters $Q = [[n, k, d_Z/d_X]]_q$ if it can detect all $\varepsilon \in G_n$ of weight $\text{wt}_X(\varepsilon)$ up to $d_X - 1$ and weight $\text{wt}_Z(\varepsilon)$ up to $d_Z - 1$, simultaneously. The construction in [1], [7], [44] can be used to construct AQCs in which a pair of classical linear codes are used, one for correcting X -errors and the other for correcting Z -errors.

Lemma 2.1 ([44, Theorem 2.4]): Let C_1 and C_2 be two classical linear codes with parameters $[n, k_1, d_1]_q$ and $[n, k_2, d_2]_q$, respectively, and $C_2^\perp \subseteq C_1$. Then there exists an AQC with parameters $Q = [[n, k_1 + k_2 - n, d_Z/d_X]]_q$, where

$$d_Z = \max\{\text{wt}_H(C_1 \setminus C_2^\perp), \text{wt}_H(C_2 \setminus C_1^\perp)\}, \quad (4)$$

$$d_X = \min\{\text{wt}_H(C_1 \setminus C_2^\perp), \text{wt}_H(C_2 \setminus C_1^\perp)\}. \quad (5)$$

If $d_1 = \text{wt}_H(C_1 \setminus C_2^\perp)$ and $d_2 = \text{wt}_H(C_2 \setminus C_1^\perp)$, then Q is non-degenerate, otherwise it is degenerate.

B. Classical Tensor Product Codes

Let $C_1 = [n_1, k_1, d_1]_q$ be a classical linear code whose parity check matrix is given by H_{c_1} , and let $r_1 = n_1 - k_1$ be the number of parity checks. Let $C_2 = [n_2, k_2, d_2]_{q^{r_1}}$ be a linear code over the extension field $\mathbb{F}_{q^{r_1}}$ whose parity check matrix is given by H_{c_2} . Let $r_2 = n_2 - k_2$. Denote by

$$C_T \equiv C_2 \otimes_T C_1 \quad (6)$$

the tensor product code of C_1 and C_2 . The block length and dimension of C_T are given by $[n_1 n_2, n_1 n_2 - r_1 r_2]$. In addition, C_1 and C_2 are known as the inner and outer constituent codes of C_T , respectively. If we regard H_{c_1} as a $1 \times n_1$ matrix with elements over $\mathbb{F}_{q^{r_1}}$, then the parity check matrix H_T of C_T is the Kronecker product of H_{c_1} and H_{c_2} , i.e.,

$$H_T = H_{c_2} \otimes H_{c_1}. \quad (7)$$

Then we can derive a parity check matrix of C_T with elements over \mathbb{F}_q by expanding all the elements of H_T from $\mathbb{F}_{q^{r_1}}$ to \mathbb{F}_q . The error detection/correction ability of C_T is restricted by the constituent codes and is given by:

Lemma 2.2 ([33, Theorem 1]): Partition the codeword of $C_T = C_2 \otimes_T C_1$ into n_2 sub-blocks, where each sub-block contains n_1 elements, and assume that the constituent code C_i can detect or correct an error pattern class ξ_i ($i = 1$ or 2), then the TPC C_T can detect or correct all error-patterns where the sub-blocks containing errors form a pattern belonging to class ξ_2 and the errors within each erroneous sub-block fall within the class ξ_1 .

Here we give an illustrative example for the construction of TPCs.

Example 2.1: Let $C_1 = [3, 1, 3]_2$ be a binary repetition code with a parity check matrix given by

$$H_{c_1} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \omega & \omega^2 \end{pmatrix}, \quad (8)$$

where ω is a primitive element of $GF(2^2)$ such that $\omega^2 + \omega + 1 = 0$. Let C_2 be a 2^2 -ary linear code over $GF(2^2)$, such as we let $C_2 = [5, 3, 3]_{2^2}$ be a maximum-distance-separable (MDS) code with a parity check matrix

$$H_{c_2} = \begin{pmatrix} 1 & 0 & 1 & \omega & \omega \\ 0 & 1 & \omega & \omega & 1 \end{pmatrix}. \quad (9)$$

Then we can derive a TPC C_T of length 15 whose parity check matrix $H_T = H_{c_2} \otimes H_{c_1}$ is given in (10). It is easy to verify, e.g., by using the MAGMA computational software [28], that the dimension and minimum distance of C_T with a parity check matrix H_T in (10) are exact 11 and 3, respectively.

Ref. [35] shows that the parity check matrix of TPCs can also be represented in a companion matrix form. Let $g(x) = g_0 + g_1x + \dots + g_{r_1-1}x^{r_1-1} + x^{r_1}$ be a primitive polynomial over $\mathbb{F}_{q^{r_1}}$ and denote by α a primitive element of $\mathbb{F}_{q^{r_1}}$. The companion matrix of $g(x)$ is defined to be the

$$\begin{aligned}
H_T &= \begin{pmatrix} 1 & \omega & \omega^2 & 0 & 0 & 0 & 1 & \omega & \omega^2 & \omega & \omega^2 & 1 & \omega & \omega^2 & 1 \\ 0 & 0 & 0 & 1 & \omega & \omega^2 & \omega & \omega^2 & 1 & \omega & \omega^2 & 1 & 1 & \omega & \omega^2 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \end{pmatrix}. \tag{10}
\end{aligned}$$

$r_1 \times r_1$ matrix

$$M = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -g_0 & -g_1 & -g_2 & \cdots & -g_{r_1-1} \end{pmatrix}. \tag{11}$$

Then for any element $\beta = \alpha^i$ of $\mathbb{F}_{q^{r_1}}$, the companion matrix of β , denoted by $[\beta] = M^i$, is an $r_1 \times r_1$ matrix with elements over \mathbb{F}_q . Let the parity check matrix of the constituent code C_2 be $H_{c_2} = (a_{ij})_{r_2 \times n_2}$ with elements over $\mathbb{F}_{q^{r_1}}$, i.e., $a_{ij} \in \mathbb{F}_{q^{r_1}}$ for $1 \leq i \leq r_2$ and $1 \leq j \leq n_2$. Following the notations used in [35], we denote by $[H_{c_2}] = ([a_{ij}])_{r_1 r_2 \times r_1 n_2}$, where $[a_{ij}]$ is a companion matrix form. The parity check matrix of C_T can be written as

$$\begin{aligned}
H_T &\equiv [H_{c_2}^t] \otimes H_{c_1} \\
&= \begin{pmatrix} [a_{11}^t]H_{c_1} & [a_{12}^t]H_{c_1} & \cdots & [a_{1n_2}^t]H_{c_1} \\ [a_{21}^t]H_{c_1} & [a_{22}^t]H_{c_1} & \cdots & [a_{2n_2}^t]H_{c_1} \\ \vdots & \vdots & \vdots & \vdots \\ [a_{r_2 1}^t]H_{c_1} & [a_{r_2 2}^t]H_{c_1} & \cdots & [a_{r_2 n_2}^t]H_{c_1} \end{pmatrix} \tag{12}
\end{aligned}$$

in which the matrix $[H_{c_2}^t]$ is obtained by transposing the constituent companion matrices of $[H_{c_2}]$,

and $[a_{ij}^t]$ is the transpose of $[a_{ij}]$. According to [35], [45], if we do not transpose the constituent companion matrices in (12), we can obtain another representation of the parity check matrix H_T as follows

$$H_T \equiv [H_{c_2}] \otimes H_{c_1} = \begin{pmatrix} [a_{11}]H_{c_1} & [a_{12}]H_{c_1} & \cdots & [a_{1n_2}]H_{c_1} \\ [a_{21}]H_{c_1} & [a_{22}]H_{c_1} & \cdots & [a_{2n_2}]H_{c_1} \\ \vdots & \vdots & \vdots & \vdots \\ [a_{r_21}]H_{c_1} & [a_{r_22}]H_{c_1} & \cdots & [a_{r_2n_2}]H_{c_1} \end{pmatrix}. \quad (13)$$

The two representations in (12) and (13) do not make any difference for the parameters and the error correction performance of TPCs. We will use them alternately in the following constructions. It should be noticed that the Kronecker product defined in equations (12) and (13) is a little different from the standard one. In the following, the Kronecker product of matrices follows the definition in (12) and (13).

The generalized tensor product codes are proposed in [35], [46] by combining a series of outer codes and inner codes. Let $A_{\bar{h}} = [n_A, k_{\bar{h}}, d_{\bar{h}}]_q$ and $B_{\bar{h}} = [N_B, K_{\bar{h}}, D_{\bar{h}}]_{q^{r_{\bar{h}}}}$ be L pairs of inner and outer codes, respectively, where $1 \leq \bar{h} \leq L$ and $r_{\bar{h}} = n_A - k_{\bar{h}}$. Let the parity check matrices of $A_{\bar{h}}$ and $B_{\bar{h}}$, respectively, be $H_{\bar{h}}^A$ and $H_{\bar{h}}^B$, $1 \leq \bar{h} \leq L$. Assume that all the rows in $H_{\bar{h}}^A$, $1 \leq \bar{h} \leq L$, are independent with each other. Then the parity check matrix of the GTPCs

$$\mathcal{C}_{\mathcal{T}} = \bigcap_{\bar{h}=1}^L B_{\bar{h}} \otimes_T A_{\bar{h}} \quad (14)$$

is defined by

$$H_{\mathcal{C}_{\mathcal{T}}} \equiv \begin{pmatrix} [H_1^{B^t}] \otimes H_1^A \\ [H_2^{B^t}] \otimes H_2^A \\ \vdots \\ [H_L^{B^t}] \otimes H_L^A \end{pmatrix}, \quad (15)$$

where $[H_h^{B^t}]$ is obtained by transposing the component companion matrices of $[H_h^B]$ for each $1 \leq h \leq L$. The block length and the dimension of GTPCs are given by $\mathcal{C}_T = [N_B n_A, N_B n_A - \sum_{h=1}^L R_h r_h]_q$, where $R_h = N_B - K_h$ for $1 \leq h \leq L$.

C. Classical Concatenated Codes

Concatenated codes can be seen as the dual counterpart of TPCs, which are obtained by concatenating an inner code $C_1 = [n, k, d]_q$ with an outer code $C_2 = [N, K, D]_{q^k}$. Denote the concatenation of C_1 and C_2 by

$$C_C \equiv C_2 \otimes_C C_1, \quad (16)$$

and $C_C = [Nn, Kk, d_{C_C} \geq Dd]_q$ (see [9], [32]). The generator matrix of C_C can also be given in a companion matrix form (see [35])

$$G_C = [G_2] \otimes G_1. \quad (17)$$

where G_1 and G_2 are the generator matrices of C_1 and C_2 , respectively.

In [9], [35], the generalized concatenated codes are obtained by concatenating a serial of outer codes and inner codes. For simplicity, we only consider linear codes here. Let $A_1 = [n_A, k_1, d_1]_q$ be a q -ary linear code with the generator matrix G_1^A , which is partitioned to S submatrices $\mathbf{G}_1^A, \dots, \mathbf{G}_S^A$ such that $k_\ell^A = \text{rank}(\mathbf{G}_\ell^A)$ for $1 \leq \ell \leq S$, and then $k_1 = \sum_{\ell=1}^S k_\ell^A$. Denote by

$$G_1^A = \begin{pmatrix} \mathbf{G}_1^A \\ \mathbf{G}_2^A \\ \vdots \\ \mathbf{G}_S^A \end{pmatrix}, G_\ell^A = \begin{pmatrix} \mathbf{G}_\ell^A \\ \mathbf{G}_{\ell+1}^A \\ \vdots \\ \mathbf{G}_S^A \end{pmatrix}, 2 \leq \ell \leq S, \quad (18)$$

and let G_ℓ^A be the generator matrices of the linear codes $A_\ell = [n_A, k_\ell, d_\ell]_q$, for $2 \leq \ell \leq S$, respectively. Denote by $B_\ell = [N_B, K_\ell, D_\ell]_{q^{k_\ell^A}}$ the outer codes with the generator matrices, respectively, G_ℓ^B , for $1 \leq \ell \leq S$. Then the generator matrix of the GCCs

$$\mathcal{C}_C = \bigcup_{\ell=1}^S B_\ell \otimes_C A_\ell \quad (19)$$

is defined by

$$G_{\mathcal{C}_c} \equiv \begin{pmatrix} [G_1^B] \otimes \mathbf{G}_1^A \\ [G_2^B] \otimes \mathbf{G}_2^A \\ \vdots \\ [G_S^B] \otimes \mathbf{G}_S^A \end{pmatrix}, \quad (20)$$

and the parameters of GCCs are given by

$$\mathcal{C}_c = [N_B n_A, \sum_{\ell=1}^S K_\ell k_\ell^A, d_{\mathcal{C}_c}]_q, \quad (21)$$

where $d_{\mathcal{C}_c} \geq \min\{D_1 d_1, \dots, D_S d_S\}$.

Compared to other types of classical linear codes in [9], [47], the parameters of CCs (GCCs) and TPCs (GTPCs) may not have any advantages. However the encoding and decoding algorithms of CCs (GCCs) and TPCs (GTPCs) usually have low complexity, and can be decoded efficiently in polynomial time. Therefore CCs are widely used in many digital communication systems, e.g., the NASA standard for deep space communications and wireless communications [10], [48], and GCCs show large potential applications, e.g., in data transmission systems [49] and Flash memory [50], [51]. TPCs and GTPCs exhibit large advantages in magnetic storage systems [52]–[55], Flash memory [56], [57] and in constructing locally repairable codes for distributed storage systems [58]–[60]. In [61], it is shown that Polar codes can be treated as GCCs for a fast encoding.

III. MAIN RESULTS

In this section, we present the AQCTPC concatenation framework, where CCs are used to correct Z -errors and TPCs are used to correct X -errors. In our construction, the dimension of the inner codes of CCs needs to be equal to the number of parity checks of the inner codes of TPCs. Let $C_1 = [n_1, k_1, d_1]_q$ denote an arbitrary q -ary linear code and $C_2 = [n_2, k_2, d_2]_{q^{k_1}}$ and $C_3 = [n_2, k_3, d_3]_{q^{k_1}}$ denote two linear codes over the extension field $\mathbb{F}_{q^{k_1}}$. Let $\mathcal{C}_c = C_3 \otimes_C C_1$

be the CC of C_1 and C_3 , and let $\mathcal{C}_T = C_2 \otimes_T C_1^\perp$ be the TPC of C_1^\perp and C_2 . Then we have the following dual-containing relationship between CCs and TPCs.

Lemma 3.1: If $C_3^\perp \subseteq C_2$, then there exists $\mathcal{C}_T^\perp \subseteq \mathcal{C}_C$.

Proof: Let H_{c_1} and G_{c_1} be the parity check matrix and generator matrix of C_1 over \mathbb{F}_q , respectively. Let H_{c_i} and G_{c_i} , $i = 2, 3$, be the parity check matrix and generator matrix of C_i over $\mathbb{F}_{q^{k_1}}$, respectively. It is easy to see that the parity check matrix of the TPC \mathcal{C}_T with transposed companion matrices is given by

$$H_{\mathcal{C}_T} = [H_{c_2}^t] \otimes G_{c_1}. \quad (22)$$

From [35], [45], we know that the parity check matrix of \mathcal{C}_C is given by

$$H_{\mathcal{C}_C} = \begin{pmatrix} [H_{c_3}] \otimes (I_{k_1}, \mathbf{0}) \\ [I_{n_2}] \otimes H_{c_1} \end{pmatrix}, \quad (23)$$

where “ $\mathbf{0}$ ” is a zero sub-block of size $k_1 \times (n_1 - k_1)$. It is not difficult to verify that if $C_3^\perp \subseteq C_2$, then we have $[H_{c_3}][H_{c_2}^t]^T = 0$ and $H_{\mathcal{C}_C} H_{\mathcal{C}_T}^T = 0$. Therefore we have $\mathcal{C}_T^\perp \subseteq \mathcal{C}_C$. ■

By combining $\mathcal{C}_C = C_3 \otimes_C C_1$ and $\mathcal{C}_T = C_2 \otimes_T C_1^\perp$, we have the construction of AQCTPCs as follows.

Theorem 3.1: There exists a family of AQCTPCs with parameters $\mathcal{Q} = [[n_1 n_2, k_1(k_2 + k_3 - n_2), d_Z \geq d_1 d_3 / d_X \geq d_2]]_q$.

The AQCTPC concatenation scheme has several advantages over the current methods. First, only the outer constituent codes C_2 and C_3 over the extension field need to satisfy the dual-containing constraint. Then we have much freedom in the choice of the outer codes. It is generally believed that certain families of linear codes over the extension field can easily satisfy the dual-containing constraint. For example, the dual-containing relationship of ℓ -ary MDS codes has been determined for all possible dimensions and block length less than $\ell + 2$, see e.g., in [62]–[64], where ℓ is a power of a prime. We can let C_2 and C_3 be two MDS codes that satisfy the dual-containing constraint and with reasonable block length. Second, in the following proof

we show that AQCTPCs are highly degenerate in that they can correct more X -errors than a corresponding classical TPC.

Proof of Theorem 3.1: Let $\mathcal{C}_C = C_3 \otimes_C C_1$ denote the CC of C_1 and C_3 , and let $\mathcal{C}_T = C_2 \otimes_T C_1^\perp$ denote the TPC of C_1^\perp and C_2 . Then we have $\mathcal{C}_C = [n_1 n_2, k_1 k_2]_q$ and $\mathcal{C}_T = [n_1 n_2, n_1 n_2 - k_1(n_2 - k_2)]_q$. According to the CSS construction in Lemma 2.1 and Lemma 3.1, if $C_3^\perp \subseteq C_2$, then we can derive an AQCTPC with parameters $\mathcal{Q} = [[n_1 n_2, k_1(k_2 + k_3 - n_2), d_Z/d_X]]_q$.

We still need to compute the minimum distance of \mathcal{Q} . It is easy to see that the minimum distance of the CC is larger than or equal to $d_1 d_3$, and then we have $d_Z \geq d_1 d_3$. Next we determine the X -distance d_X .

Suppose that there is an X -error e_X of length $n_1 n_2$ in the encoded codeword. We divide the error e_X into n_2 sub-blocks e_{X_i} ($1 \leq i \leq n_2$), with each sub-block being of length n_1 (see Fig. 1). We then do the syndrome measurement for X -errors by using the parity check matrix $H_{\mathcal{C}_T}$ given in (22). The syndrome information Φ can be derived by measuring the ancilla, which is given by

$$\begin{aligned} \Phi &\equiv [H_{\mathcal{C}_2}^t] \otimes G_{c_1} \cdot e_X^T \\ &= \begin{pmatrix} [a_{11}^t] \Phi_{1_1} + \cdots + [a_{1n_2}^t] \Phi_{1_{n_2}} \\ [a_{21}^t] \Phi_{1_1} + \cdots + [a_{2n_2}^t] \Phi_{1_{n_2}} \\ \vdots \\ [a_{r_2 1}^t] \Phi_{1_1} + \cdots + [a_{r_2 n_2}^t] \Phi_{1_{n_2}} \end{pmatrix}, \end{aligned} \quad (24)$$

where $H_{\mathcal{C}_2} = (a_{ij})$, $1 \leq i \leq r_2 = n_2 - k_2$, $1 \leq j \leq n_2$. We let $\Phi_{1_l} \equiv G_{c_1} e_{X_l}^T$, $1 \leq l \leq n_2$, which can be regarded as the logical error sequences in the outer code C_2 . Then we have

$$\Phi = \begin{pmatrix} [a_{11}^t] & [a_{12}^t] & \cdots & [a_{1n_2}^t] \\ [a_{21}^t] & [a_{22}^t] & \cdots & [a_{2n_2}^t] \\ \vdots & \vdots & \vdots & \vdots \\ [a_{r_2 1}^t] & [a_{r_2 2}^t] & \cdots & [a_{r_2 n_2}^t] \end{pmatrix} \begin{pmatrix} \Phi_{1_1} \\ \Phi_{1_2} \\ \vdots \\ \Phi_{1_{n_2}} \end{pmatrix}. \quad (25)$$

If the outer decoding can be conducted successfully, then the sequences $\Phi_{l_i}(1 \leq l \leq n_2)$ are used as the inner syndrome information for C_1^\perp .

The outer code C_2 must be decoded by mapping the syndrome information Φ to the symbols over the extension field $\mathbb{F}_{q^{k_1}}$. Here we need a syndrome based decoding [9] of the outer code C_2 , which, if successful, will result in the exact inner syndrome sequence $\Phi_{l_i}(1 \leq l \leq n_2)$. The inner decoding follows using the dual of the inner code C_1 . In general, for any $\Phi_{l_i} \equiv G_{c_1} e_{X_{l_i}}^T (1 \leq l \leq n_2)$, we can always obtain a decoded error sequence $\tilde{e}_{X_{l_i}}$ such that $\Phi_{l_i} = G_{c_1} \tilde{e}_{X_{l_i}}^T$ by using some syndrome based decoder for C_1^\perp , such as a syndrome table-look-up decoder. However, we do not need to do that and just let $\tilde{e}_{X_{l_i}} = (\Phi_{l_i}, \mathbf{0})$ by assuming that G_{c_1} is in a standard form, where “ $\mathbf{0}$ ” is a zero vector of length r_1 . Let $\tilde{e}_X \equiv (\tilde{e}_{X_1}, \dots, \tilde{e}_{X_{n_2}})$ be the decoded error sequence. There must be $G_{C_C}(e_X^T + \tilde{e}_X^T) = 0$, where

$$G_{C_C} = [G_{c_3}^t] \otimes G_{c_1} \quad (26)$$

is the generator matrix of $C_C = C_3 \otimes_C C_1$. There are two cases: (1) $e_X = \tilde{e}_X$ which means that the decoded error \tilde{e}_X is exactly the true error. (2) $e_X \neq \tilde{e}_X$ but they belong to the same coset of C_C^\perp , which means that they are degenerate.

This phenomenon of degeneracy is quite different from the decoding of classical TPCs [33], [35], [52], where the decoding fails if the number of errors in one sub-block exceeds the error correction ability of the inner codes. As such, AQCTPCs can correct many more X -errors than their classical TPC counterparts. If $\text{wt}(e_X) \leq d_2 - 1$, whenever the error is separated into different sub-blocks in Fig. 1, the number of erroneous sub-blocks will be at most $d_2 - 1$. This means that either the error will always be detectable or that the error is undetectable but harmless since it is degenerate. Thus the X -distance d_X is at least d_2 . Therefore we have an AQCTPC with the parameters $\mathcal{Q} = [[n_1 n_2, k_1(k_2 + k_3 - n_2), d_Z \geq d_1 d_3 / d_X \geq d_2]]_q$. ■

In the proof of Theorem 3.1, we have given the decoding of AQCTPCs for correcting X -errors. We summarize and provide the whole decoding process in Algorithm 1.

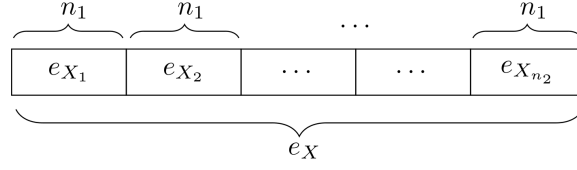


Fig. 1. Dividing the Pauli X -error e_X into n_2 sub-blocks where each sub-block e_{X_i} ($1 \leq i \leq n_2$) is of length n_1 .

Algorithm 1 The Decoding Algorithm of AQCTPCs for Correcting X -errors.

Input: Φ, H_{c_2} ;

Output: The decoded X -error sequence \tilde{e}_X .

- 1: Initialization: $\hat{\Phi} = \emptyset, \tilde{e}_X = \emptyset$;
 - 2: // Divide Φ into r_2 sub-blocks, each sub-block is of length k_1 .
 - 3: $\Phi = (\Phi_1, \dots, \Phi_{r_2}), |\Phi_i| = k_1$;
 - 4: // Map Φ to $\hat{\Phi}$ with elements over the extension field $\mathbb{F}_{q^{k_1}}$.
 - 5: **for** $i \in [1, r_2]$ **do**
 - 6: Map Φ_i into a symbol $\hat{\Phi}_i$ over the field $\mathbb{F}_{q^{k_1}}$;
 - 7: $\hat{\Phi} = (\hat{\Phi}, \hat{\Phi}_i)$;
 - 8: **end for**
 - 9: // Do the outer decoding according to the syndrome information
 $H_{c_2} \hat{\Phi}_1^T = \hat{\Phi}$.
 - 10: Denote by $\hat{\Phi}_1 = (\hat{\Phi}_{1_1}, \dots, \hat{\Phi}_{1_{n_2}})$;
 - 11: **for** $i \in [1, n_2]$ **do**
 - 12: Map $\hat{\Phi}_{1_i}$ into a sequence over field \mathbb{F}_q, Φ_{1_i} ;
 - 13: $\tilde{e}_{X_i} = (\Phi_{1_i}, \mathbf{0})$;
 - 14: $\tilde{e}_X = (\tilde{e}_X, \tilde{e}_{X_i})$;
 - 15: **end for**
 - 16: **return** \tilde{e}_X ;
-

On the other hand, like the serial decoding of classical CCs, the decoding of Z -errors in AQCTPCs can also be done serially, i.e., an inner decoding followed by an outer decoding. However, the decoding algorithm for classical CCs can not be used to decode Z -errors directly. Instead, a modified version of syndrome-based decoding is needed, as explained next.

Before performing the decoding, the ancilla needs to be measured first to determine the syndrome information. Denote the encoded quantum basis states of the AQCTPC \mathcal{Q} by

$$|u + \mathcal{C}_C^\perp\rangle \equiv \frac{1}{\sqrt{|\mathcal{C}_C^\perp|}} \sum_{v \in \mathcal{C}_C^\perp} |u + v\rangle, \quad (27)$$

where $u \in \mathcal{C}_T$. Suppose that a Z -error e_Z happens to the encoded state (27), then

$$|u + \mathcal{C}_C^\perp\rangle \rightarrow \frac{1}{\sqrt{|\mathcal{C}_C|}} \sum_{w \in \mathcal{C}_C} (-1)^{wu^T} |w + e_Z\rangle. \quad (28)$$

First we take the syndrome measurement using the inner parity check matrix H_{c_1} to get the inner syndrome information

$$\Psi_{I_i} \equiv H_{c_1}(w_i + e_{Z_i})^T = H_{c_1}e_{Z_i}^T, 1 \leq i \leq n_2, \quad (29)$$

where $w = (w_1, \dots, w_{n_2})$ and $e_Z = (e_{Z_1}, \dots, e_{Z_{n_2}})$. The inner decodings are done first according to the inner syndrome information $\Psi_{I_i} (1 \leq i \leq n_2)$. They result in n_2 decoded error sequences $\bar{e}_{Z_i} (1 \leq i \leq n_2)$, each of length n_1 . Denote by $\bar{e}_Z = (\bar{e}_{Z_1}, \dots, \bar{e}_{Z_{n_2}})$. We add the decoded result \bar{e}_Z to (28), and then perform the measurement using the parity check matrix $[H_{c_3}] \otimes (I_{k_1}, \mathbf{0})$ to get the outer syndrome information

$$\begin{aligned} \Psi_O &\equiv [H_{c_3}] \otimes (I_{k_1}, \mathbf{0})(w + e_Z + \bar{e}_Z)^T \\ &= [H_{c_3}] \otimes (I_{k_1}, \mathbf{0})(e_Z + \bar{e}_Z)^T. \end{aligned} \quad (30)$$

Discarding the zero part in Ψ_O due to the $\mathbf{0}$ sub-block in $(I_{k_1}, \mathbf{0})$, the punctured Ψ_O is then mapped into a sequence $\bar{\Psi}_O$ with elements over field $\mathbb{F}_{q^{k_1}}$.

The outer decoding is done with a syndrome-based decoding of C_3 according to the outer syndrome information $\bar{\Psi}_O$. If the outer decoding is successful, we can obtain a decoded sequence

$\mathbf{e}'_Z = (e'_{Z_1}, \dots, e'_{Z_{n_2}})$ with elements over $\mathbb{F}_{q^{k_1}}$. Then we map the sequence \mathbf{e}'_Z back to the basis field \mathbb{F}_q , and derive a decoded error sequence $e'_Z = (e'_{Z_1}, \dots, e'_{Z_{n_2}})$ with elements over \mathbb{F}_q , where $e'_{Z_i} (1 \leq i \leq n_2)$ is the sub-sequence of length k_1 . But notice that e'_Z is incomplete due to the $\mathbf{0}$ sub-block in $(I_{k_1}, \mathbf{0})$. In order to derive the fully-decoded error sequence, we only need to do some operations according to the inner syndrome information in (29). Denote by $\tilde{e}_Z = (\tilde{e}_{Z_1}, \dots, \tilde{e}_{Z_{n_2}})$ and $\tilde{e}_{Z_i} = (e'_{Z_i}, f_{Z_i}) (1 \leq i \leq n_2)$, where f_{Z_i} denotes the unknown errors in \tilde{e}_{Z_i} and is of length r_1 . Suppose that $H_{c_1} = (P_1, P_2)$, where P_1 is of size $r_1 \times k_1$ and P_2 is an invertible $r_1 \times r_1$ matrix, then we have $\Psi_{I_i} = H_{c_1} \tilde{e}_{Z_i}^T = P_1 e_{Z_i}^T + P_2 f_{Z_i}^T$ and then $f_{Z_i} = [P_2^{-1}(\Psi_{I_i} - P_1 e_{Z_i}^T)]^T$, where $1 \leq i \leq n_2$.

Similar to classical CCs, no matter how many Z -errors happen in each sub-block of length n_1 , the outer decoding will not be affected provided that the total number of erroneous sub-blocks does not exceed the error correction ability of the outer code C_3 . A summary of the full decoding process is provided in Algorithm 2. A complexity analysis of the whole decoding process follows.

In terms of decoding X -errors with the TPC, we first need to map the outer decoding sequence from $\mathbb{F}_{q^{k_1}}$ to \mathbb{F}_q whose running time complexity is $O(n_2)$ (see Algorithm 1, lines 11-15). And it is easy to see that the complexity of the inner syndrome decoding of $C_1^\perp = [n_1, r_1]$ is $O(1)$ since we just need to do $\tilde{e}_{X_l} = (\Phi_{I_l}, \mathbf{0})$, for $1 \leq l \leq n_2$. Therefore, the inner decoding complexity (IDC) of the TPC is $O(n_2)$. Recall that the IDC of classical TPCs is $\Omega(n_2 a^{n_1}) (a > 1)$ by using the maximum likelihood (ML) decoding in general, which is enormous if n_1 is large. Even though the inner codes can be efficiently decoded, see, e.g., [34], [52], the IDC of TPCs is still $\Omega(n_2 n_1^b) (b > 0)$. Here, in quantum cases, we consider error degeneracy in the inner decoding and significantly reduce the IDC of TPCs to $O(n_2)$ in general.

It is easy to see that the outer decoding complexity (ODC) of TPCs is completely determined by the outer constituent codes. Thus if the outer codes can be decoded efficiently, the whole decoding of TPCs is efficient. For example, we let the outer codes be the Reed-Solomon (RS)

codes or the generalized Reed-Solomon (GRS) codes that satisfy the dual-containing relationship [63], [64]. They can be decoded efficiently in time polynomial to their block length, e.g., by using the Berlekamp-Massey (BM) algorithm, see [9], [10], [52]. Then the whole decoding of TPCs for correcting X -errors can be done efficiently in polynomial time.

Algorithm 2 The Decoding Algorithm of AQCTPCs for Correcting Z -errors.

Input: $\Psi_{I_i} (1 \leq i \leq n_2)$, Ψ_O , $H_{c_1} = (P_1, P_2)$, H_{c_3} ;

Output: The decoded Z -error sequence \tilde{e}_Z .

- 1: Initialization: $\bar{e}_Z = \emptyset, \tilde{e}_Z = \emptyset$;
- 2: **for** $i \in [1, n_2]$ **do**
- 3: // Do the inner decoding according to (29).
- 4: $H_{c_1} \bar{e}_{Z_i}^T = \Psi_{I_i}, \bar{e}_Z = (\bar{e}_Z, \bar{e}_{Z_i})$;
- 5: **end for**
- 6: Map Ψ_O to a sequence $\bar{\Psi}_O$ with elements over field $\mathbb{F}_{q^{k_1}}$;
- 7: //Do the outer decoding according to $\bar{\Psi}_O$ and C_3 .
- 8: $\mathbf{e}'_Z = (\mathbf{e}'_{Z_1}, \dots, \mathbf{e}'_{Z_{n_2}})$;
- 9: **for** $i \in [1, n_2]$ **do**
- 10: Map \mathbf{e}'_{Z_i} into a sequence over field \mathbb{F}_q , e'_{Z_i} ;
- 11: $f_{Z_i} = [P_2^{-1}(\Psi_{I_i} - P_1 e_{Z_i}^T)]^T$;
- 12: $\tilde{e}_{Z_i} = (e'_{Z_i}, f_{Z_i})$;
- 13: $\tilde{e}_Z = (\tilde{e}_Z, \tilde{e}_{Z_i})$;
- 14: **end for**
- 15: **return** \tilde{e}_Z ;

When correcting Z -errors by using the CC, it is easy to see that the decoding complexity is the sum of the complexities of the inner and outer decodings. Thus, the CC is efficiently

decodable provided that the constituent codes C_1 and C_3 can be decoded efficiently, e.g., in time polynomial to the block length [9], [10]. Overall, we can conclude that the entire AQCTPC decoding process for correcting both X -errors and Z -errors is efficient provided that the inner and outer constituent codes are efficiently decodable.

Similar to the generalization of classical CCs and TPCs, we can generalize the concatenation scheme of AQCTPCs by combining GCCs with GTPCs. Let $A_\ell = [n_A, k_\ell, d_\ell]_q (1 \leq \ell \leq L)$ be L q -ary linear codes. Let $B_\ell = [N_B, K_\ell, D_\ell]_{q^{k_\ell}}$ and $C_\ell = [N_B, M_\ell, E_\ell]_{q^{k_\ell}} (1 \leq \ell \leq L)$ be L q^{k_ℓ} -ary linear codes, respectively. Denote $\mathbf{A}_\ell = [n_A, k_\ell^A]_q (1 \leq \ell \leq L)$ by L linear codes obtained by partitioning the generator matrix of A_1 into L submatrices. Then we have the following result about the dual-containing relationship between GCCs and GTPCs.

Lemma 3.2: Let $\mathcal{C}_\mathcal{T}$ be the GTPC of \mathbf{A}_ℓ^\perp and $B_\ell (1 \leq \ell \leq L)$, and let $\mathcal{C}_\mathcal{C}$ be the GCC of A_ℓ and $C_\ell (1 \leq \ell \leq L)$. If $B_\ell^\perp \subseteq C_\ell$ for all $1 \leq \ell \leq L$, then there is $\mathcal{C}_\mathcal{T}^\perp \subseteq \mathcal{C}_\mathcal{C}$.

Proof: We use the notations for GTPCs and GCCs given in Preliminaries. Denote the *collection of duals matrix* (cdm) (see Ref. [35]) of G_1^A in (18) by

$$\hat{H}^A = \text{cdm}(G_1^A) = \begin{pmatrix} \hat{H}_1^A \\ \hat{H}_2^A \\ \vdots \\ \hat{H}_{L+1}^A \end{pmatrix} \quad (31)$$

with $k_\ell^A = \text{rank}(\hat{H}_\ell^A) = \text{rank}(\mathbf{G}_\ell^A)$, for $1 \leq \ell \leq L$, and $k_{L+1}^A = \text{rank}(\hat{H}_{L+1}^A) = n_A - k_1$. Then the parity check matrix of the GCC $\mathcal{C}_\mathcal{C}$ is given by

$$H_{\mathcal{C}_\mathcal{C}} \equiv \begin{pmatrix} [H_1^C] \otimes \hat{H}_1^A \\ \vdots \\ [H_L^C] \otimes \hat{H}_L^A \\ [I_{N_B}] \otimes \hat{H}_{L+1}^A \end{pmatrix}. \quad (32)$$

And the parity check matrix of the GTPC $\mathcal{C}_\mathcal{T}$ is given by

$$H_{\mathcal{C}_\mathcal{T}} \equiv \begin{pmatrix} [H_1^{B^t}] \otimes \mathbf{G}_1^A \\ [H_2^{B^t}] \otimes \mathbf{G}_2^A \\ \vdots \\ [H_L^{B^t}] \otimes \mathbf{G}_L^A \end{pmatrix} \quad (33)$$

According to Ref. [35] and Ref. [45], we know the following two properties about the cdm of G_1^A :

- $\hat{H}_\ell^A \mathbf{G}_h^{A^t} = 0$, for all $1 \leq \ell \leq L+1$, $1 \leq h \leq L$ and $\ell \neq h$.
- $\hat{H}_\ell^A \mathbf{G}_\ell^{A^t}$ is of full rank, for all $1 \leq \ell \leq L$.

Since $\hat{H}_\ell^A \mathbf{G}_\ell^{A^t}$ is of full rank, we can always find an invertible matrix U_ℓ such that $U_\ell \hat{H}_\ell^A \mathbf{G}_\ell^{A^t}$ is an identity matrix, for $1 \leq \ell \leq L$. If $B_\ell^\perp \subseteq C_\ell$, which means that $[H_\ell^C][H_\ell^{B^t}]^T = 0$ for all $1 \leq \ell \leq L$, then there is $H_{\mathcal{C}_c} H_{\mathcal{C}_\mathcal{T}}^T = 0$ and we have $\mathcal{C}_\mathcal{T}^\perp \subseteq \mathcal{C}_c$. ■

Theorem 3.2: There exist generalized AQCTPCs with parameters

$$\mathcal{Q} = [[N_B n_A, \sum_{\ell=1}^L (K_\ell + M_\ell - N_B) k_\ell^A, d_Z/d_X]]_q, \quad (34)$$

where $d_Z \geq \min\{D_1 d_1, \dots, D_L d_L\}$, $d_X \geq \min\{E_1, \dots, E_L\}$.

Proof: By combining Lemma 2.1 and Lemma 3.1, we can obtain the generalized AQCTPCs with parameters

$$\mathcal{Q} = [[N_B n_A, \sum_{\ell=1}^L (K_\ell + M_\ell - N_B) k_\ell^A, d_Z/d_X]]_q. \quad (35)$$

We use the GCCs to correct Z -errors and thus the Z -distance d_Z of the generalized AQCTPC \mathcal{Q} is given by $d_Z \geq \min\{D_1 d_1, \dots, D_L d_L\}$. Next we need to compute the X -distance d_X of \mathcal{Q} .

Suppose that there is an X -error e_X of length N_{Bn_A} in the encoded codeword. Denote

$$\begin{aligned} \Phi_X &\equiv H_{C_T} e_X^T \\ &= \begin{pmatrix} [H_1^{B^t}] \otimes \mathbf{G}_1^A \cdot e_X^T \\ [H_2^{B^t}] \otimes \mathbf{G}_2^A \cdot e_X^T \\ \vdots \\ [H_L^{B^t}] \otimes \mathbf{G}_L^A \cdot e_X^T \end{pmatrix} \end{aligned} \quad (36)$$

by the syndrome information obtained by measuring the ancilla and let $\Phi_{X_\ell} \equiv [H_\ell^{B^t}] \otimes \mathbf{G}_\ell^A \cdot e_X^T$, $1 \leq \ell \leq L$. Suppose that for some $1 \leq \iota \leq L$, we have $E_\iota = \min\{E_1, \dots, E_L\}$. Similar to the proof of Theorem 3.1, if $\text{wt}(e_X) \leq E_\iota - 1$, then we must have $\Phi_{X_\iota} \neq 0$ and then the error can be detected or $\Phi_{X_\iota} = 0$ but the error is degenerate. Therefore we have $d_X \geq \min\{E_1, \dots, E_L\}$. ■

It should be noticed in the proof of Theorem 3.2 that, we only give a minimum limit of the distance d_X . In the practical error correction, e.g., in [50] for classical GCCs, we have L syndrome information Φ_{X_ℓ} ($1 \leq \ell \leq L$) to be used for the decoding and then the generalized AQCTPCs can correct many more X -errors beyond the minimum distance limit in Theorem 3.2 in practice.

IV. FAMILIES OF AQCTPCs

In this section, we provide examples of AQCTPCs that outperform best-known AQC in the literature. Since the inner constituent codes C_1 in AQCTPCs can be chosen arbitrarily, we can get varieties of AQCTPCs by using different types of the constituent codes. Although the construction of AQCTPCs is not restricted by the field size q , in this section, we mainly focus on binary codes which may be more practical in the future application. For simplicity, if $q = 2$, we omit the subscript in the parameters of quantum and classical codes.

Firstly we use classical single-parity-check codes [9] as the inner constituent codes and we have the following result.

Corollary 4.1: There exists a family of binary AQCTPCs with parameters

$$\mathcal{Q} = [[N_{\mathcal{Q}}, K_{\mathcal{Q}}, d_Z \geq 2d_3/d_X \geq d_2]], \quad (37)$$

where $N_{\mathcal{Q}} = (m_1 + 1)n_2$, $K_{\mathcal{Q}} = m_1(n_2 - d_2 - d_3 + 2)$, $m_1 \geq 2$, $2 \leq n_2 \leq 2^{m_1} + 1$, and $2 \leq d_2 + d_3 \leq n_2 + 2$ are all integers.

Proof: Let $C_1 = [m_1 + 1, m_1, 2]$ be a binary single-parity-check code with even codewords, and let $C_2 = [n_2, k_2, d_2]_{2^{m_1}}$ and $C_3 = [n_2, k_3, d_3]_{2^{m_1}}$ be two classical GRS codes. It is shown in [64] that if $2 \leq n_2 \leq 2^{m_1} + 1$ and $2 \leq d_2 + d_3 \leq n_2 + 2$, there exists $C_3^{\perp} \subseteq C_2$. ■

In Corollary 4.1, if we let $d_2 = 2d_3$, then we can also obtain a family of symmetric quantum codes with parameters

$$\mathcal{Q} = [[N_{\mathcal{Q}}, K_{\mathcal{Q}}, d_{\mathcal{Q}} \geq d_2]], \quad (38)$$

where $N_{\mathcal{Q}} = (m_1 + 1)n_2$, $K_{\mathcal{Q}} = m_1(n_2 - 3d_2/2 + 2)$, $2 \leq d_2 \leq 2(n_2 + 2)/3$. We first compare (38) with QBCH codes in [65]. It is known that the minimum distance of QBCH codes of length $\Theta(N_{\mathcal{Q}})$ is upper bounded by $c\sqrt{N_{\mathcal{Q}}}$ ($c > 0$ is a constant). On the other hand, the minimum distance of our codes is upper bounded by $2(n_2 + 2)/3$ which is larger than $c\sqrt{N_{\mathcal{Q}}}$ provided that $n_2 \geq 9c^2(m_1 + 1)/4 - 4$.

For example, let $n_2 = 2^{m_1} + 1$, then the minimum distance of our codes can be as large as $2N_{\mathcal{Q}}/(3 \log(N_{\mathcal{Q}}))$, which is almost linear to the length $N_{\mathcal{Q}}$, while the dimension is larger than $\log(N_{\mathcal{Q}})$. If we let $d_2 = O((N_{\mathcal{Q}})^{c_1})$, where $1/2 < c_1 < 1$ is any constant, then the rate of our codes

$$R_{\mathcal{Q}} = \frac{K_{\mathcal{Q}}}{N_{\mathcal{Q}}} = \frac{m_1}{m_1 + 1} \left(1 - \frac{3d_2}{2n_2} + \frac{2}{n_2}\right) \quad (39)$$

is equal to 1 as $n_2 = 2^{m_1} + 1$ goes to infinity and $d_{\mathcal{Q}} \geq d_2 = O((N_{\mathcal{Q}})^{c_1})$. In [65], the rate of binary QBCH codes of CSS type is given by

$$R_{QBCH} = 1 - \frac{m(\delta - 1)}{N}, \quad (40)$$

TABLE I

COMPARISON OF BINARY AQCTPCs WITH THE BINARY EXTENSION OF ASYMMETRIC QRS CODES IN [29]. THE AQCTPCs ARE DERIVED FROM COROLLARY 4.1. THE “–” IN THE TABLE MEANS THAT THERE DO NOT EXIST AQCS WITH COMPARABLE PARAMETERS IN REF. [29]. IN QUANTUM CODES, AN AQC WITH PARAMETERS $[[n, 0, d_Z/d_X]]$ OF DIMENSION 1 IS A PURE STATE WHICH CAN CORRECT ALL X -ERRORS OF WEIGHT UP TO $\lfloor (d_X - 1)/2 \rfloor$ AND ALL Z -ERRORS OF WEIGHT UP TO $\lfloor (d_Z - 1)/2 \rfloor$ [27], [66]. TO FACILITATE NOTATION, THE NUMBERS OF Z - AND X -DISTANCE OF THE AQCS ARE THE LOWER BOUND.

m_1	AQCTPCs	Ref. [29]	m_1	AQCTPCs	Ref. [29]	m_1	AQCTPCs	Ref. [29]
6	[[378, 6, 104/3]]	–	7	[[888, 7, 218/3]]	–	8	[[2034, 8, 448/3]]	–
6	[[378, 12, 102/3]]	–	7	[[888, 14, 216/3]]	–	8	[[2034, 16, 446/3]]	–
6	[[378, 132, 62/3]]	[[378, 0, 62/3]]	7	[[888, 329, 126/3]]	[[889, 0, 126/3]]	8	[[2034, 784, 254/3]]	[[2040, 0, 254/3]]
6	[[378, 138, 60/3]]	[[378, 12, 60/3]]	7	[[888, 336, 124/3]]	[[889, 14, 124/3]]	8	[[2034, 792, 252/3]]	[[2040, 16, 252/3]]
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
6	[[378, 258, 20/3]]	[[378, 252, 20/3]]	7	[[888, 651, 34/3]]	[[889, 644, 34/3]]	8	[[2034, 1560, 60/3]]	[[2040, 1552, 60/3]]
6	[[378, 6, 100/5]]	–	7	[[888, 7, 214/5]]	–	8	[[2034, 8, 444/5]]	–
6	[[378, 12, 98/5]]	–	7	[[888, 14, 212/5]]	–	8	[[2034, 16, 442/5]]	–
6	[[378, 126, 60/5]]	[[378, 0, 60/5]]	7	[[888, 322, 124/5]]	[[889, 0, 124/5]]	8	[[2034, 776, 252/5]]	[[2040, 0, 252/5]]
6	[[378, 132, 58/5]]	[[378, 12, 58/5]]	7	[[888, 329, 122/5]]	[[889, 14, 122/5]]	8	[[2034, 784, 250/5]]	[[2040, 16, 250/5]]
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
6	[[378, 246, 20/5]]	[[378, 240, 20/5]]	7	[[888, 637, 34/5]]	[[889, 630, 34/5]]	8	[[2034, 1544, 60/5]]	[[2040, 1536, 60/5]]

where N is the block length, $m = \text{ord}_N(2)$ is the multiplicative order of 2 modulo N , and $2 \leq \delta \leq N(2^{\lceil m/2 \rceil} - 1)/(2^m - 1) = O(\sqrt{N})$. It is easy to see that R_{QBCH} is also asymptotic to 1 as N goes to infinity. However our codes have much better minimum distance upper bound than QBCH codes.

Then we compare AQCTPCs in Corollary 4.1 with the extension of asymmetric quantum MDS codes in [29]. For simplicity, we consider the extension of binary asymmetric QRS codes in [29] with parameters

$$[[\tilde{N}_Q, \tilde{K}_Q, \tilde{d}_Z \geq \tilde{d}_1/\tilde{d}_X \geq \tilde{d}_2]], \quad (41)$$

where $\tilde{N}_Q = m_1(2^{m_1} - 1)$, $\tilde{K}_Q = m_1(2^{m_1} - \tilde{d}_1 - \tilde{d}_2 + 1)$, $2 \leq \tilde{d}_1 + \tilde{d}_2 \leq 2^{m_1} + 1$. In order to make a fair comparison between them, we let $n_2 = \lfloor m_1(2^{m_1} - 1)/(m_1 + 1) \rfloor$ in Corollary 4.1 so that they have an equal or a similar block length. Let $\tilde{d}_1 = 2d_3$ and $\tilde{d}_2 = d_2$, then it is easy to see that if $d_3 \geq (2^{m_1} - 1)/(m_1 + 1)$, the dimension of AQCTPCs in (37) is larger than that of AQC in (41). Further, AQCTPCs of length $N_Q = \Theta(m_1(2^{m_1} - 1))$ in Corollary 4.1 can be decoded efficiently in polynomial time and we have the following result.

Corollary 4.2: There exist AQCTPCs of length $N_Q = \Theta(m_1(2^{m_1} - 1))$ which can be decoded in $O(N_Q^2/\log N_Q)$ arithmetic operations.

Proof: First we consider the complexity of the decoding of X -errors. The IDC of TPCs is $O(2^{m_1})$ according to the proof of Theorem 3.1. It is known that GRS codes of length $n_2 = \Theta(2^{m_1})$ can be decoded in $O(n_2^2)$ field operations by using the BM algorithm [9], [67]. Therefore the total decoding of TPCs requires $O(n_2^2)$ arithmetic operations.

Then we consider the decoding of Z -errors by using the CC. If we use Algorithm 2 to do the decoding, we can only decode up to $\lfloor (2d_3 - 1)/4 \rfloor$ numbers of Z -errors. In order to decode any Z -error of weight smaller than half the minimum distance $2d_3$, we use the inner code $C_1 = [m_1 + 1, m_1, 2]$ to do the error detection for each sub-block of the CC. Suppose we can detect t_1 erroneous sub-blocks and suppose that there exist t_2 erroneous sub-blocks which are undetectable. As a result, there are t_1 erroneous positions which are known in the error sequence \mathbf{e}_Z that corresponds to the outer code and, t_2 erroneous positions that are unknown. It is easy to see that if the weight of the Z -error is smaller than d_3 , we must have $0 \leq t_1 + 2t_2 \leq 2d_3 - 1$. Then we can decode the error sequence \mathbf{e}_Z with t_1 errors in known locations and t_2 errors which are unknown by using the BM algorithm in $O(n_2^2)$ arithmetic operations [9], [68]. Therefore the total complexity of decoding Z -errors is $O(n_2^2)$. Overall, the whole decoding of AQCTPCs of length $N_Q = \Theta(m_1(2^{m_1} - 1))$ can be fulfilled in $O(N_Q^2/\log N_Q)$ arithmetic operations. ■

In Table I, we make a comparison between parameters of AQCTPCs and the results in [29]. It is shown that AQCTPCs have a relatively large Z -distance and can have much better parameters

than the codes in [29]. Further, the decoding complexity of AQC's in [29] of length $\tilde{N}_Q = m_1(2^{m_1} - 1)$ is dominated by the RS code decoding whose complexity scales as $O(\tilde{n}_2^2)$ by using the BM decoder, where $\tilde{n}_2 = 2^{m_1} - 1$. Then the decoding complexity of AQC's in [29] is $O(\tilde{N}_Q^2 / \log \tilde{N}_Q)$. It is shown that both AQCTPC's in Corollary 4.2 and AQC's in [29] can be decoded efficiently in polynomial arithmetic operations. However, our codes have much better parameters than the codes in [29].

Next we use classical Simplex codes [9] that have a large minimum distance as the inner constituent codes. We show that Simplex codes can result in AQCTPC's with a large Z -distance d_Z .

Corollary 4.3: There exists a family of binary AQCTPC's with parameters

$$\mathcal{Q} = [[N_{\mathcal{Q}}, K_{\mathcal{Q}}, d_Z \geq 2^{m_1-1}d_3/d_X \geq d_2]], \quad (42)$$

where $N_{\mathcal{Q}} = (2^{m_1} - 1)n_2$, $K_{\mathcal{Q}} = m_1(n_2 - d_2 - d_3 + 2)$, $m_1 \geq 2$, $2 \leq n_2 \leq 2^{m_1} + 1$, and $2 \leq d_2 + d_3 \leq n_2 + 2$.

Proof: The proof proceeds in the same way as in Corollary 4.1 except we use classical Simplex codes $C_1 = [2^{m_1} - 1, m_1, 2^{m_1-1}]$ as the inner constituent code. ■

In particular, if we take $n_2 = 2^{m_1} + 1$ and let $d_2 = O(2^{cm_1})$ and $d_3 = 2^{m_1} + 2 - d_2$, where $0 < c < 1$ is a constant, then we have

$$\mathcal{Q} = [[N_{\mathcal{Q}}, m_1, d_Z/d_X]], \quad (43)$$

where $N_{\mathcal{Q}} = 2^{2m_1} - 1$, $d_Z \geq 2^{m_1-1}(2^{m_1} + 2 - d_2)$, and $d_X \geq d_2$. It is easy to see that $d_Z/N_{\mathcal{Q}} \rightarrow 1/2$ as $m_1 \rightarrow \infty$ and \mathcal{Q} can meet the quantum Gilbert-Varshamov (GV) bound for AQC's in [69]. Therefore we get a family of AQCTPC's with a very large Z -distance d_Z which is of approximately half the block length, at the same time, the dimension and the X -distance d_X can continue increasing as the block length goes to infinity. In Table II, we list several AQCTPC's with a large Z -distance d_Z which is of approximately half the block length.

In particular, if $d_X = 2$, then the Z -distance d_Z of AQCTPCs in Corollary 4.3 could be larger than half the block length.

TABLE II

CONSTRUCTION OF BINARY AQCTPCs WHOSE Z -DISTANCE d_Z IS APPROXIMATELY HALF OF THE BLOCK LENGTH BY USING BINARY SIMPLEX CODES AS INNER CODES C_1 . THE OUTER CODES C_2 AND C_3 ARE DUAL-CONTAINING MDS CODES DERIVED FROM [64]. TO FACILITATE NOTATION, THE NUMBERS OF Z - AND X -DISTANCE OF THE AQCTPCs ARE THE LOWER BOUND.

m_1	d_2	AQCTPCs	m_1	d_2	AQCTPCs
2	2	[[15, 2, 8/2]]	6	4	[[4095, 6, 1984/4]]
2	3	[[15, 2, 6/3]]	6	5	[[4095, 6, 1952/5]]
3	2	[[63, 3, 32/2]]	6	6	[[4095, 6, 1920/6]]
4	2	[[255, 4, 128/2]]	6	7	[[4095, 6, 1888/7]]
5	2	[[1023, 5, 512/2]]	7	2	[[16383, 7, 8192/2]]
5	3	[[1023, 5, 496/3]]	7	3	[[16383, 7, 8128/3]]
5	4	[[1023, 5, 480/4]]	7	4	[[16383, 7, 8064/4]]
5	5	[[1023, 5, 464/5]]	7	5	[[16383, 7, 8000/5]]
6	2	[[4095, 6, 2048/2]]	7	6	[[16383, 7, 7936/6]]
6	3	[[4095, 6, 2016/3]]	7	7	[[16383, 7, 7827/7]]

In addition, if we use linear codes in [47] with best known parameters as the inner codes, we can get many new AQCTPCs with a relatively large Z -distance d_Z and very flexible code parameters. We list some of them in Table III. The Z -distances of the last four codes in Table III are much larger than half the block length, respectively. All the AQCTPCs in Table II and Table III have the largest Z -distance d_Z compared to existed AQC with comparable block length and X -distance d_X .

If we use asymptotically good linear codes that can attain the classical GV bound as the inner codes C_1 , we can get the following asymptotic result about AQCTPCs.

TABLE III

CONSTRUCTION OF BINARY AQCTPCs WITH A LARGE Z -DISTANCE BY USING SOME BEST KNOWN LINEAR CODES IN REF. [47] AS INNER CODES. THE OUTER CODES $C_2 = [n_2, k_2, d_2]_{2^{k_1}}$ AND $C_3 = [n_2, k_3, d_3]_{2^{k_1}}$ ARE DUAL-CONTAINING MDS CODES WITH OPTIMAL PARAMETERS IN [64], RESPECTIVELY. TO FACILITATE NOTATION, THE NUMBERS OF Z - AND X -DISTANCE OF THE AQCTPCs IN THIS TABLE ARE THE LOWER BOUND.

C_1 in Ref. [47]	$\{n_2, d_2, d_3\}$ in Theorem 3.1	AQCTPCs
[7, 3, 4]	{9, 3, 7}	[[63, 3, 28/3]]
[7, 3, 4]	{9, 5, 5}	[[63, 3, 20/5]]
[8, 4, 4]	{17, 3, 15}	[[136, 4, 60/3]]
[8, 4, 4]	{17, 5, 13}	[[136, 4, 52/5]]
[12, 4, 6]	{17, 3, 15}	[[204, 4, 90/3]]
[12, 4, 6]	{17, 5, 13}	[[204, 4, 78/5]]
[15, 4, 8]	{17, 3, 15}	[[255, 4, 120/3]]
[15, 4, 8]	{17, 5, 13}	[[255, 4, 104/5]]
[16, 5, 8]	{33, 3, 31}	[[528, 5, 248/3]]
[16, 5, 8]	{33, 5, 29}	[[528, 5, 232/5]]
[21, 5, 10]	{33, 3, 31}	[[693, 5, 310/3]]
[21, 5, 10]	{33, 5, 29}	[[693, 5, 290/5]]
[22, 6, 9]	{65, 3, 63}	[[1430, 6, 567/3]]
[22, 6, 9]	{65, 5, 61}	[[1430, 6, 549/5]]
[24, 7, 10]	{129, 3, 127}	[[3096, 7, 1270/3]]
[24, 7, 10]	{129, 5, 125}	[[3096, 7, 1250/5]]
[63, 3, 36]	{9, 2, 8}	[[567, 3, 288/2]]
[127, 3, 72]	{9, 2, 8}	[[1143, 3, 576/2]]
[255, 3, 145]	{9, 2, 8}	[[2295, 3, 1160/2]]
[255, 4, 136]	{17, 2, 16}	[[4335, 4, 2176/2]]

Corollary 4.4: There exists a family of q -ary AQCTPCs with parameters

$$Q = [[N_Q = n_1 n_2, K_Q, d_Z/d_X]]_q \quad (44)$$

such that

$$\frac{K_{\mathcal{Q}}}{N_{\mathcal{Q}}} \geq \left(1 - H_q\left(\frac{d_1}{n_1}\right)\right) \left(1 - \frac{d_2}{n_2} - \frac{d_3}{n_2}\right), \quad (45)$$

$$d_Z \geq d_1 d_3, \quad (46)$$

$$d_X \geq d_2, \quad (47)$$

where

$$H_q(x) = x \log_q(q-1) - x \log_q x - (1-x) \log_q(1-x) \quad (48)$$

is the q -ary entropy function, $2 \leq d_1 \leq n_1$, $2 \leq d_2 + d_3 \leq n_2$, and $n_1, n_2 \rightarrow \infty$.

Proof: We choose $C_1 = [n_1, k_1, d_1]_q$ to be asymptotically good linear codes meeting the GV bound, i.e.,

$$\frac{k_1}{n_1} \geq 1 - H_q\left(\frac{d_1}{n_1}\right). \quad (49)$$

Let $C_2 = [n_2, k_2, d_2]_{q^{k_1}}$ and $C_3 = [n_2, k_3, d_3]_{q^{k_1}}$ be two MDS codes such that $C_2^\perp \subseteq C_3$. Denote by $N_{\mathcal{Q}} = n_1 n_2$, $K_{\mathcal{Q}} = k_1(k_2 + k_3 - n_2)$, $d_Z = d_1 d_3$ and $d_X = d_2$. According to Theorem 3.1, we can get the asymptotic result in (45) as $n_1, n_2 \rightarrow \infty$. ■

On the other hand, besides using MDS codes as the outer constituent codes, we can also use AG codes that satisfy the dual-containing constraint [44], [70]. We will adopt the notation of AG codes used in [44], [71].

Theorem 4.1 ([44]): Let \mathcal{X} be an algebraic curve over \mathbb{F}_q of genus g with at least n rational points. For any $2g - 2 < s < l < n$, there exist two q -ary AG codes $C_1 = [n, k_1, d_1]_q$ and $C_2 = [n, k_2, d_2]_q$ with $k_1 = n - k_2 + l - s$ such that $C_2^\perp \subset C_1$, where $d_1 \geq s - 2g + 2$ and $d_2 \geq n - l$.

For $q = 2^m (m \geq 2)$, there is the following asymptotic result about asymmetric QAG codes in [44].

Theorem 4.2 ([44]): Let $q = 2^m$ and let $0 \leq \delta_x, \delta_z \leq 1$ such that $\delta_x + \delta_z \leq 1 - 2/(\sqrt{2^m} - 1)$, then there exists a family of asymptotically good asymmetric QAG codes \mathcal{Q} satisfying

$$R_{\mathcal{Q}}(\delta_x, \delta_z) \geq 1 - \delta_x - \delta_z - \frac{2}{\sqrt{2^m} - 1}. \quad (50)$$

By using similar code extension methods in [70] and the CSS construction of AQC's, one can obtain asymptotically good binary extensions of asymmetric QAG codes as follows.

Corollary 4.5: Let $q = 2^m$ and let $0 \leq \delta_x, \delta_z \leq 1$ such that $\delta_x + \delta_z \leq 1 - 2/(\sqrt{2^m} - 1)$, then there exists a family of asymptotically good binary asymmetric QAG codes \mathcal{Q} satisfying

$$R_{\mathcal{Q}}(\delta_x, \delta_z) \geq 1 - m\delta_x - m\delta_z - \frac{2}{\sqrt{2^m} - 1}. \quad (51)$$

Proof: The asymptotic bound in (51) can be obtained from Ref. [70] and Theorem 4.2. ■

Denote by $C_1 = [n_1, k_1, d_1]$ a binary linear code and let \mathcal{X} be an algebraic curve over $\mathbb{F}_{2^{k_1}}$ of genus g with at least n_2 rational points. Then we have the following result for constructing AQCTPCs by using AG codes as outer codes.

Proposition 4.1: There exists a family of binary AQCTPCs with parameters

$$\mathcal{Q} = [[N_{\mathcal{Q}}, K_{\mathcal{Q}}, d_Z \geq d_1 d_3 / d_X \geq d_2]], \quad (52)$$

where $N_{\mathcal{Q}} = n_1 n_2$, $K_{\mathcal{Q}} = k_1(l - s)$, $2g - 2 < s < l < n_2$, $d_2 \geq s - 2g + 2$ and $d_3 \geq n_2 - l$. As n_2 goes to infinity, the following asymptotic bound of AQCTPCs holds

$$R_{\mathcal{Q}} \geq \frac{k_1}{n_1} \left(1 - \frac{n_1}{d_1} \delta_z - n_1 \delta_x - \frac{2}{\sqrt{2^{k_1}} - 1} \right), \quad (53)$$

where $R_{\mathcal{Q}} = K_{\mathcal{Q}}/N_{\mathcal{Q}}$, δ_X and δ_Z are the relatively minimum distance of \mathcal{Q} .

Proof: According to Theorem 4.1, we know that there exist two 2^{k_1} -ary AG codes $C_2 = [n_2, k_2, d_2]_{2^{k_1}}$ and $C_3 = [n_2, k_3, d_3]_{2^{k_1}}$ such that $C_2^{\perp} \subseteq C_3$, where $k_2 = n_2 - k_3 + l - s$ and $2g - 2 < s < l < n_2$. Then from Theorem 3.1, we can construct a family of binary AQCTPCs with parameters $\mathcal{Q} = [[N_{\mathcal{Q}} = n_1 n_2, K_{\mathcal{Q}} = k_1(l - s), d_Z \geq d_1 d_3 / d_X \geq d_2]]$, where $d_2 \geq s - 2g + 2$ and $d_3 \geq n_2 - l$. Denote by δ_X and δ_Z the relatively minimum distance of \mathcal{Q} , i.e., $\delta_X = d_X/N_{\mathcal{Q}}$ and $\delta_Z = d_Z/N_{\mathcal{Q}}$. The asymptotic result can be obtained by Theorem 4.2. ■

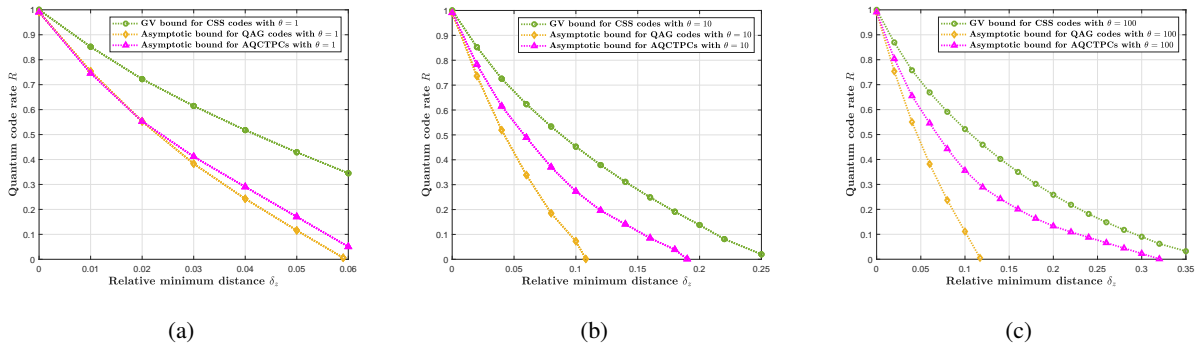


Fig. 2. The comparison among the asymptotic bound for AQCTPCs, the GV bound for CSS codes and the asymptotic bound for asymmetric QAGs. The asymmetry parameter $\theta = d_Z/d_X$ is chosen as 1, 10, 100 in (a), (b), and (c), respectively. We use the relative minimum distance $\delta_z = d_Z/N$ as the horizontal axis and use code rate $R = K/N$ as the vertical axis. In order to optimize the asymptotic curves for AQCTPCs, we use different constituent code parameters in Ref. [47] to generate several piecewise asymptotic curves for AQCTPCs and then we joint them together.

In Fig. 2, we compare the asymptotic bound of AQCTPCs in (53) with that of asymmetric QAG codes in (51). We also give the GV bound of CSS codes for comparisons. In order to get as good as possible asymptotic curves for AQCTPCs, we use different inner constituent codes to generate several piecewise asymptotic curves and then joint them together. In Fig. 2(a), we can see that the asymptotic bound of AQCTPCs is better than that for asymmetric QAG codes when the relative minimum distance $0.02 < \delta_z < 0.06$. As the the asymmetry $\theta = d_Z/d_X$ grows, it is shown in Fig. 2(b) and Fig. 2(c) that AQCTPCs perform much better than asymmetric QAG codes.

V. CONCLUSIONS AND DISCUSSIONS

In this paper, we proposed the construction of asymmetric quantum concatenated and tensor product codes that combine the classical CCs and TPCs. The CCs correct the Z -errors and the TPCs correct the X -errors. Compared to concatenation schemes like CQCs and QTPCs, the AQCTPC construction only requires that the outer constituent codes satisfy the dual-containing constraint; the inner constituent codes can be chosen freely. Further, AQCTPCs are highly

degenerate codes and, as a result, they passively correct many X -errors. To avoid issues with decoding, we present efficient syndrome-based decoding algorithms and show that if the inner and outer constituent codes are efficiently decodable, then the AQCTPC is also efficiently decodable. Particularly, the inner decoding complexity of TPCs is significantly reduced to $O(n_2)$ in general. Further, we generalized the AQCTPC concatenation scheme by using GCCs and GTPCs.

To showcase the power of the method, we constructed many state-of-the-art AQC. Through these constructions, we demonstrate how AQCTPCs can be superior to QBCH codes or asymmetric QAG codes as the block length goes to infinity; how they can have better parameters than the binary extension of asymmetric QRS codes; and how varieties of AQCTPCs with a large Z -distance d_Z can be designed by using some best known linear codes in [47]. In particular, we constructed a family of AQCTPCs with a Z -distance d_Z of approximately half the block length, and meanwhile with dimension and X -distance d_X that continue to increase as the block length goes to infinity. If $d_X = 2$, we obtain the first family of binary AQC with the Z -distance larger than half the block length.

Our codes are practical to quantum communication channels with a large asymmetry and may be used in fault-tolerant quantum computation to deal with highly biased noise. In the next work, we may consider the construction and decoding of AQCTPCs by using some other constituent codes, e.g., the Polar codes.

ACKNOWLEDGMENT

The authors would like to thank the Editor and the referees for their valuable comments that are helpful to improve the presentation of their article. J. Fan thanks Prof. Yonghui Li and Prof. Martin Bossert for some earlier communications about tensor product codes.

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