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SENSITIVITY AND BIFURCATION ANALYSIS OF AN ANALYTICAL MODEL OF A TRAPPED OBJECT IN AN EXTERNALLY EXCITED ACOUSTIC RADIATION FORCE FIELD

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ABSTRACT

Acoustic radiation force (ARF) is a nonlinear acoustic phenomenon for which the acoustic field properties and, to an even greater extent, the explicit dynamics of the object, have received limited attention in the published literature to date. Any oscillations due to the flow field or external perturbations are thereby negligible while the particle is trapped in a stable position. By changing the viewpoint from the acoustic field to the dynamics of a levitated particle, the amplitude and frequency of external oscillation is non-negligible, we ask the question of how external excitation changes the dynamics of the object. We explicitly derive an analytical formulation of a trapped object in the form of a Duffing-like equation with its constants being defined by the object itself, the fluid, the acoustic wave, and the external vibration properties. In this case, the bifurcation behaviour is studied, and we show this together with a sensitivity analysis to represent correct dynamic behaviour in certain regimes of the bifurcation diagram.

INTRODUCTION

The study of acoustic wave propagation in fluids has attracted much research in the recent years [1-4]. One of the most interesting phenomena in the field of nonlinear acoustics is that of the Acoustic Radiation Force (ARF), which is based on the physical interaction of a sonic wave with an object placed along the acoustic wave path. The acoustic radiation force exerted on the object is defined through the integration of the acoustic radiation stress over a time-varying surface. The acoustic radiation stress tensor is extracted by solving the Navier-Stokes equations and the continuity equations. Using the perturbation theory, the non-zero radiation force stems from the higher order terms of the solution. Changing the energy density of an incident acoustic field drives the acoustic radiation force with its magnitude dependent on the amount of energy being absorbed by the object [2].

Until recently, acoustic levitation was applied mainly to statically suspend particles at a fixed position in air [2] which reduce the applicability of acoustic levitation. To overcome this limitation, new levitation techniques have been proposed to manipulate and to transport small samples of solids and liquids in which the dynamic of particles is important. Separation and manipulation of particles suspended in fluid in micro dimensions has taken a major share of recent research and has been developed in many applied fields including biotechnology and medical diagnosis [4]. This manipulation of particles is possible in different ways, including the use of magnetic or electronic fields, dielectric manipulation [5-7]. In this rather field-centric way (the potential field is in the focus), the used samples and particles must have specific electrical or magnetic properties so that they can be well controlled and manipulated. The use of acoustic forces (including acoustic streaming) is another method of manipulating particles, which has attracted special attention due to its different capabilities. The study of acoustic forces and the interaction of the sound waves and particles is called "Acoustophoresis" and it is a form of non-contact particle manipulation in microfluidic devices [3]. The precision of manipulation processes can be enhanced through a better understanding of the acoustic radiation forces and their interactions [3]. Unlike the previously mentioned methods, the use of acoustic waves has no limitations on electrical or magnetic properties and can separate all types of particles from the fluid if they differ from the host fluid in terms of their densities. Also, acoustic waves do not have the negative effects that electric and magnetic fields may have on particles and biological cells, and do not change their properties [8]. Using specifically shaped (acoustic metamaterial) objects would enable to manipulate target specimens, thus pioneering object-centric, acoustic particle control approaches [9, 10]. The number of applications which make use of acoustic radiation forces especially related to the fields of biotechnology, medical diagnosis, or micro-devices are growing and because of the complexity of experimental works dealing with micrometre dimensions an urgent need for analytical studies and theoretical insight in this subject matter is apparent.

Commonly, the fluid domain rather than dynamics of the particle are considered even though acoustophoretic applications are related to particle manipulation, particle sorting etc. However, due to the nonlinear nature of the problem and interaction among the fluid and the particle, the nonlinear dynamics of the particle are essential to better understand these acoustophoretic applications. Complex dynamics is based on studying time series by linking data to the explicit theory of nonlinear dynamics to shed light on the true dynamic behaviour of a particle also extractable from experiments. Fushimi et al [11] recently studied the nonlinear dynamics of an acoustic levitator and extracted a Duffing-like equation from their experiments. Yet, the exact properties of this equation, and their relatedness to acoustics, is yet to be explored. The sensitivity of certain parameters important for nonlinear dynamics and the behaviour of the levitated mass is also not fully understood yet.

Coming from Gor'kov's theory [12] and the acoustic wave equation, herein, a dynamical system and surrogate of single degree-of-freedom nonlinear oscillator model of a solid particle trapped in a one-dimensional acoustic standing wave field, similar to that found in an acoustic levitator, is derived. Using time integration a variance-based analysis method is conducted [13,14] to understand parameter sensitivities to conduct a bifurcation analysis. Using the bifurcation analysis allows comparing the dynamics of the surrogate system with that of the Duffing oscillator in an object-centric approach [11,16] .

NONLINEAR DYNAMICAL MODEL OF THE ACOUSTIC RADIATION FORCES

According to Fushimi et al. [11], the acoustic radiation force, F_{rad} , for a solid particle trapped in a standing wave, caused by two counter-propagating waves, e.g., from two sets of transducers, operating at a given frequency, is

$$F_{\mathrm{rad}} = \alpha \sin\left(\beta\left(z - Z_{\mathrm{o}}(t)\right)\right), \tag{1}$$

where z and $Z_o = A\sin(\omega_o t)$ are the position of the spherical object with respect to the pressure node and the equilibrium position, respectively. Here, α and β are treated as constants and, using King's formulation [15], are given as

$$\alpha = 2\pi\rho_0 \left|\frac{P_a}{\rho_0\omega}\right|^2 (kr)^3 \frac{3+2(1-\tilde{\rho})}{3(2+\tilde{\rho})},\ \tilde{\rho} = \rho_0 / \rho_1, \tag{2a}$$

$$\beta = (2k), \tag{2b}$$

where P_a is the magnitude of the incident pressure field, ω is the frequency of standing acoustic wave, k denotes the acoustic wave number, r is the levitated object's radius, ρ_0 and ρ_1 are the object's and the surrounding fluid's density, respectively. Considering the drag force and by applying Newton's second law, the equation of motion becomes [11]

$$M_o\ddot{z} = F_{rad} + F_{drag} \tag{3}$$

where M_o is the mass of the object. From the Stokes flow theory, the drag force on a sphere can be written as [11]

$$F_{drag} = -C_d \frac{1}{2}\left(\pi r^2\right)\rho_0 |\dot{z}|\dot{z}, \tag{4}$$

where C_d denotes the drag coefficient that depends on the shape of the object and the Reynolds number [11]. Substituting Eqs. (1) and (4) into Eq. (3) provides

$$M_o\ddot{z} = -\alpha\sin(\beta z)\cos(\beta A\sin(\omega_o t)) + \alpha\cos(\beta z)\sin(\beta A\sin(\omega_o t)) \\ -\frac{1}{2}C_d\left(\pi r^2\right)\rho_0|\dot{z}|\dot{z}. \tag{5}$$

Eq. (5) is a nonlinear equation for a single degree-of-freedom system describing the behaviour of an acoustically trapped particle in a standing plane wave subjected to a secondary excitation. By introducing $\theta = \beta z$, and for small input excitation of kA<<1, Eq. (5) can be re-written as

$$\ddot{\theta} + c_1|\dot{\theta}|\dot{\theta} + c_2\sin(\theta) - c_3\cos(\theta)\sin(\omega_o t) = 0, \tag{6}$$

where the constant coefficients are

$$c_1 = \frac{1}{M_o}\frac{1}{2}C_d\frac{\pi}{4}(2r)^2\frac{\rho_0}{\beta},\ c_2 = \frac{\beta}{M_o}\alpha,\ c_3 = \frac{\beta^2}{M_o}\alpha A.$$

Using the Taylor series expansion for $\sin\theta \approx \theta - \theta^3/6$, Eq. (6) can be rewritten as

$$\ddot{\theta} + c_1|\dot{\theta}|\dot{\theta} + c_2\theta - \frac{b}{6}\theta^3 = c_3\cos(\theta)\sin(\omega_o t), \tag{7}$$

Eq. (7) shows a Duffing-like oscillator in too small amplitude excitation, which was previously predicted experimentally for an acoustically levitated spherical object [16] which obtained using Gor'kov's theory. Another way to extract the nonlinear equation of motion is the sparse identification of nonlinear dynamics (SINDy) algorithm which is one approach to discover dynamical systems models from experimental data, directly [17].

STATISTICAL MODEL AND SOBOL SENSITIVITY ANALYSIS

Model inputs always have uncertainties that come from different sources. The input uncertainty is propagated through the model into the output, and therefore, the output will also include uncertainty. The purpose of the sensitivity analysis is to find the important parameters and their significance in the output of the model. To consider the uncertainties in the inputs, the input parameters of the model are introduced using statistical distributions. One of the most famous charts that shows many indicators of descriptive statistics related to data is "Box and Whisker plot" which usually simply called a box plot [13,14]. This chart can give information about the existence of outlier data, symmetry in the data, and the skewness of the data.

In this paper and by using the 4th order Runge-Kutta method, Eq. (6), with a different time step Δt varying from $\Delta t_R = 10^{-4}$ $[\mathrm{s}]$ to a reference time step at $\Delta t_R = 10^{-8}$ $[\mathrm{s}]$, is solved numerically. To investigate the dynamic response of the system, the following values of parameters are extracted from Ref. [11] and in each step the relative error can be calculated as:

$$E_R = |\theta - \theta_R|. \tag{8}$$

Then, the relative errors for these time steps are normalized by dividing to $(\Delta t / \Delta t_R)$, and are plotted in the Fig. 1 which shows 95% Confidence Interval (CI) for the median (notched boxplot diagram) [14]. By using this chart, $\Delta t = 10^{-6}$ $[\mathrm{s}]$ can be chosen as an appropriate time step wit $CI = [9.60\ , 9.81] \times 10^{-4}$.

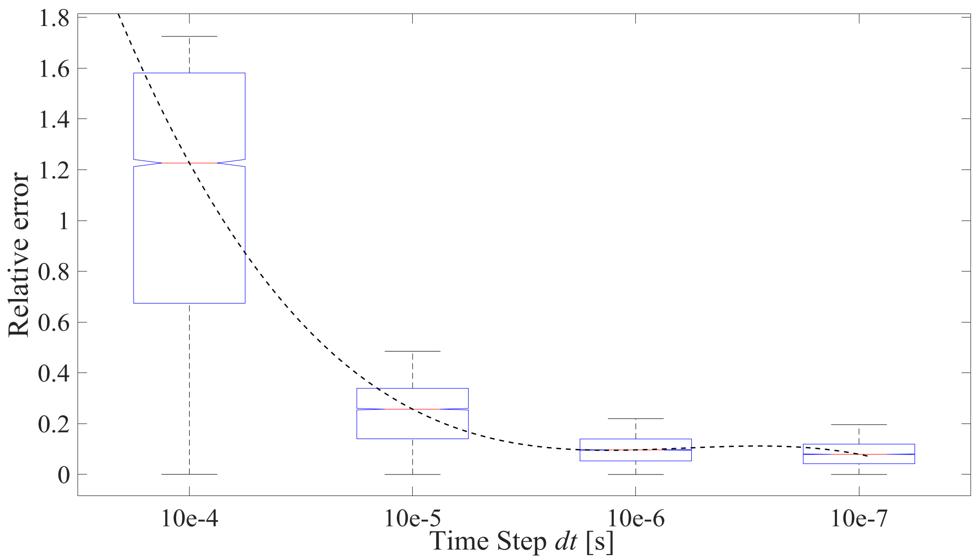

Figure 1. Time step convergence study using a box plot. The whiskers show the minimal and maximal values, the edges of the box indicate the 25 – and 75 – percentile the red line represents the median and the notches show the 95% confidence interval of the median. Non-overlapping notches show non-significant difference in median estimates of the relative error.

Global sensitivity analysis are generally variance-based methods that greatly rely on sampling data and input parameter distribution, which is a suitable tool for statistical analysis and determining the amount of input effects on the output responses of systems in most of the problems [13].

In the Sobol sensitivity analysis method [13], for a model defined by $Y = f(\vec{X})$ where Y is the output of the system and $\vec{X} = [x_1, x_2, ...x_n]$ is the vector of input parameters, $V(Y)$, the variance of the system can be defined by [21]:

$$V(Y) = \sum_{i=1}^{n} V_i + \sum_{i \le j \le n}^{n} V_{ij} + ... + V_{1,..,n}, \tag{9}$$

in which, V_i is the variance because of the first order effects for each parameter and V_{ij} to $V_{1...n}$ are second and higher order effects for each parameter which shows the variance because of the interaction between parameters. The first-order sensitivity coefficient for the i^{th} parameter can be defined by [13]:

$$S_i = \frac{V_i}{V(Y)}, \tag{10}$$

and the total sensitivity coefficient for the i^{th} parameter can be defined by

$$S_{Ti} = S_i + \sum_{i \ne j}^{n} S_{ij} + ..., \tag{11}$$

where, $S_{ij}, ...$ are the second and higher order sensitivity coefficient for each parameter.

For the sensitivity analysis, the model inputs $\vec{X} = [c_1, c_2, c_3, \omega_o]$ are set to be uniformly distributed. Assuming the following ranges for the test variables:

$c_1 = [6\ ,8] \times 10^4$ [-], $c_2 = [1.2\ ,1.8] \times 10^{-2}$ [s^{-2}], $c_3 = [8\ ,10] \times 10^4$ [s^{-2}], $\omega_0 = [10\ ,\ 25]$ [Hz],

the first order effects and the total effects of the parameters are shown in Fig. 2. It was found that the coefficient c_2 in Eq. (6) is the least sensitive parameter and c_3 is found to be the most sensitive parameter. In terms of total effects, the similar behavior was seen in Fig. 2. It is also seen that the total effects of c_2 are smaller than other parameters, i.e., there is not much interaction between c_2 and other parameters.

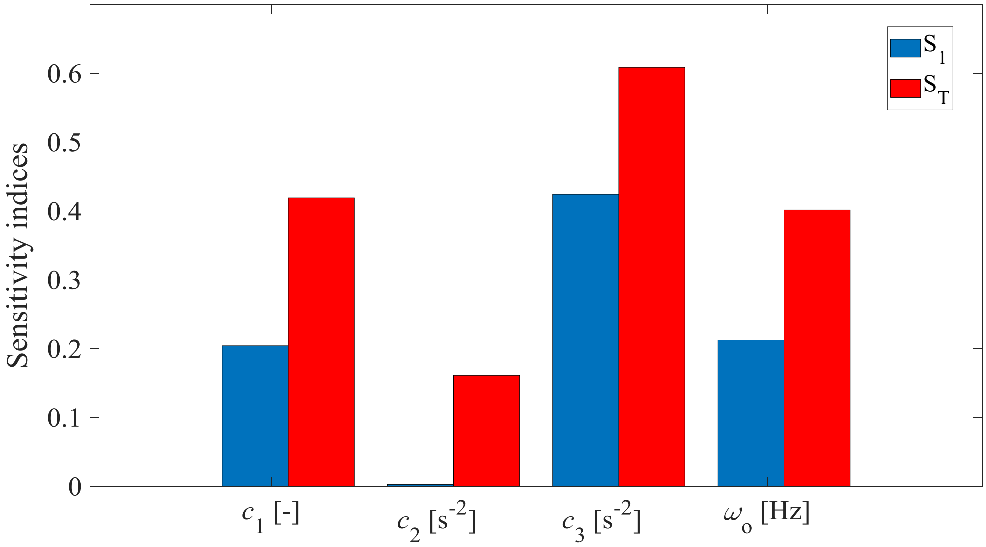

Figure 2: Results of the variance-based sensitivity analysis. Importance ranking of the coefficients in Eq. (6) by sensitivity analysis using uniformly distribution. The force has the strongest influence on the results followed by the parameters c_1 and ω_o ; c_2 is having a rather small influence on the dynamics.

Using result obtained by the sensitivity analysis the amplitude of disturbance (or second excitation) *A*, which influences c_3 in the Eq. (6), is selected as bifurcation parameters in the next section.

BIFURCATION DIAGRAM

Bifurcation diagram and its analysis is an efficient means to investigate the steady-state dynamics of nonlinear system over a wide range of so-called bifurcation parameters [18,19]. The bifurcation diagram in Fig. 3 obtained by using Eq. (6) by changing the external excitation amplitude from *A*=0 to *A*=1 or *kA*=0 to *kA*=0.2 in which four distinct regions can be observed. In the low amplitude, $A \le 0.05 \; [\mathrm{mm}]$ or $kA = 0.01$, the system shows chaotic (C) or quasi-periodic (QP) behaviour, then and by increasing the value of the bifurcation parameter jump (J) phenomenon can be observed. The discontinuity in the bifurcation diagram shows jumps and period-doubling bifurcation (PDB) and period adding bifurcations (PAB) are next different area in the bifurcation diagram.

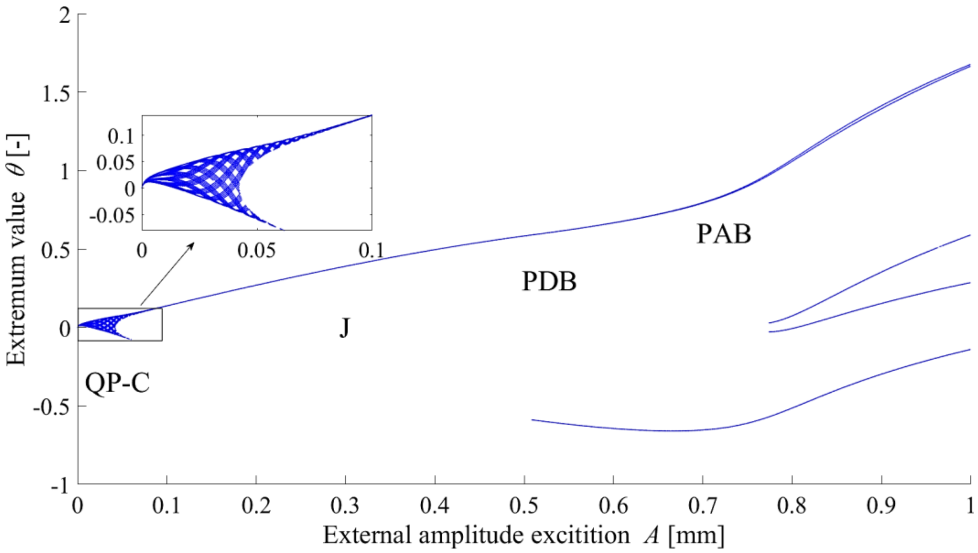

Figure 3: The bifurcation diagram for a particle trapped in an oscillating radiation force field using Eq. (6). The (QP-C) area indicates the physically relevant regime ranging from A~0 to A~0.05. The low amplitude regime shows intricate high-order periodic behaviour, the following bifurcation behaviour is indicated by jumps, and subsequent increase in periodic solutions at A~ 0.7 and A~0.8.

Fig. (3) shows that by increasing the value of the bifurcation parameter the period doubling bifurcation diagram is not valid anymore as predicted in [11], and period-2 changes to period-5. Previous studies in [11,16] proposed the Duffing equation of motion to describe the nonlinear dynamical behaviour of a particle trapped in an acoustic radiation force field. By making a comparison between the bifurcation diagram for Duffing equation with cubic nonlinearity term presented in [18] and present study, it can be observed that Duffing equation with cubic nonlinearity term in Eq. (7) cannot describe the nonlinear dynamical behaviour of the system and higher order terms should be considered.

CONCLUSIONS

Using Gor'kov's theory, a mathematical equation for modelling nonlinear vibration of a solid particle trapped in an acoustic radiation force field was developed. The governing equation of motion was solved using the Runge–Kutta order fourth and the appropriate time step was selected using a statistical observation and based on the variance of results shown by a box plot diagram. The sensitivity of the dynamic behaviour of the system was investigated by calculating the Sobol indices in relation to various system parameters, including coefficients related to particle and fluid properties, external vibration frequency and amplitude. The results show that the external vibration amplitude and frequency have a significant effect on the system response. The period-doubling bifurcation and high-order periodic behaviour were shown in the bifurcation diagram. The discontinuity in the bifurcation diagram showed the jump phenomenon. The result shows that the classical Duffing equation of motion does not describe the nonlinear dynamical system behaviour. However, the result indicate that more complex dynamics are possible and further investigation and experiments are needed. While these results are potentially important for practical object manipulation, only experiments, in highly nonlinear regimes, can verify our findings and their physical relevance.

REFERENCES

[1] H. Bruus, Acoustofluidics 7: The acoustic radiation force on small particles. *Lab Chip.* 12(6), 114–121, 2012.

[2] P. L. Marston, T. D. Daniel, A. R. Fortuner, Specular reflection contributions to dynamic radiation forces on highly reflecting spheres (L). *J. Acoust. Soc. Am.*, 150(1), 25–28, 2021.

[3] S. Sepehrirahnama, A. R. Mohapatra., S. Oberst, Y. K. Chiang, D. A. Powell, K. M. Lim, Acoustofluidics 24: theory and experimental measurements of acoustic interaction force. *Lab Chip.* Advance Article, 2022.

[4] Z. Ma, P. Fischer, Acoustic micro-manipulation and its biomedical applications. *Engineering*, In press, 2022.

[5] N. G. Durmus, H. C. Tekin, S. Guven, U. Demirci, Magnetic levitation of single cells. *Proc. Natl. Acad. Sci.*, 112 (28), 3661-3668, 2015.

[6] A. Agrawal, J. F. Douglas, M. Tirrell, A. Karim, Manipulation of coacervate droplets with an electric field. *Proc. Natl. Acad. Sci.*, 119 (32), 1-10, 2022.

[7] J. Park, S. Hong, Y. S. Lee, H. Lee, S. Kim, K. Dholakia, K. Oh, Optical manipulation of a dielectric particle along polygonal closed-loop geometries within a single water droplet. *Sci. Rep.*, 11, 12690, 2021.

[8] G. T. Silva, S. Chen, J. F. Greenleaf, M. Fatemi, Dynamic ultrasound radiation force in fluids. *Phys. Rev. E Stat. Nonlin. Soft Matter Phys*., 71(5), 056617–056617, 2005.

[9] S. Sepehrirahnama, S. Oberst, Y. K. Chiang, D. Powell, Acoustic radiation force and radiation torque beyond particles: Effects of nonspherical shape and Willis coupling. *Phys. Rev. E,* 104 (6), 2021.

[10] Sepehrirahnama, S. Oberst, Y. K. Chiang, D. Powell, Willis coupling-induced acoustic radiation force and torque reversal, arXiv:2110.01354.

[11] T. Fushimi, T. L. Hill, A. Marzo, B. W. Drinkwater, Nonlinear trapping stiffness of mid-air single-axis acoustic levitators. *Appl. Phys. Lett*., 113(3), 34102, 2018.

[12] L. P. Gor'kov, On the forces acting on a small particle in an acoustical field in an ideal fluid. *Sov. Phys.-Dokl.,* (6), 773-775, 1962.

[13] A. Saltelli, P. Annoni, I. Azzini, F. Campolongo, M. Ratto, S. Tarantola, Variance based sensitivity analysis of model output. Design and estimator for the total sensitivity index. *Comput. Phys. Commun*., 181,259–270, 2010.

[14] S. Gonga, S. Oberst, X. Wang, An experimentally validated rubber shear spring model for vibrating flip-flow screens. *Mech. Syst. Signal Process.*, 139, 106619, 2020.

[15] L. V. King, On the Acoustic Radiation Pressure on Spheres. *Proc. R. Soc. Lond: A Math. Phys. Sci.*, 147(861), 212–240,1934

[16] M. A. B., Andrade, T. S. Ramos, F. T. A. Okina, C. A. Julio, Nonlinear characterization of a single-axis acoustic levitator. *Rev. Sci. Instrum*. 85, 045125, 2014.

[17] S. L. Brunton, J. L. Proctor, J. N. Kutz, Discovering governing equations from data by sparse identification of nonlinear dynamical systems. *PNAS*, 113 (15), 3932-3937, 2016.

[18] S. Lu, S. Oberst, G. Zhang, Z. Luo, Bifurcation analysis of dynamic pricing processes with nonlinear external reference effects. *Commun. Nonlinear Sci. Numer. Simul*. 79, 104929, 2019.

[19] S. Oberst, J. C. S. Lai, Nonlinear transient and chaotic interactions in disc brake squeal. *J. Sound Vib*. 342, 272–289, 2015.