Sacred Geometry and Axial Symmetry in the Modern Handbell

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ABSTRACT

We demonstrate that a modern handbell, in addition to being axially symmetric, has an outer profile that is partly determined by aspects of Sacred Geometry. These include accurate golden rectangles, an approximate golden triangle, an accurate golden angle, and a circle whose radius and centre are fixed by the golden ratio. Because the triangle involved is only approximately golden, unlike other previously reported cases for both church and hand bells, the use of a golden triangle to fix the location of the shoulder is not appropriate for the case under investigation. Rather, its location can be determined by the crown circle and the golden angle from the centre of the mouth. We further show that the fit of the profile within a regular pentagon is less convincing than is the case for the modern church bell. We also summarise, based on group theory, the influence of axial symmetry on the degeneracy structure and the analytical forms of the bell's modal functions.

INTRODUCTION

The idea that there is a connection between visual beauty and aspects of "sacred" geometry goes back at least to Classical Greece. It has attracted the attention of savants down the centuries and has been well rehearsed in popular mathematical/scientific literature over recent decades [1-3]. Numerous topics have been considered but, of special interest to musical acousticians, has been its use in analysing violins of the Cremona school and other stringed musical instruments [4]. While it is not suggested here that modern Western bells can compete with Stradivarius violins in the visual beauty stakes, their forms are certainly aesthetically pleasing. Bells having hitherto escaped the attention of "golden" geometers. One of the present authors (RP), together with colleagues, investigated whether they contained any features associated with sacred geometry [5]. As only the outer surface of a bell is "on view", the analysis was restricted to outer profiles; founders would have no reason to make their inner profiles look beautiful.

The original study concentrated on modern church bells but also included some modern handbells. Most did indeed display a variety of golden features to unexpectedly high degrees of accuracy, with the handbells doing rather better. Since the computer-aided design (CAD) software available to analyse the profiles is now much improved, it has been decided to revisit the topic. Because the results for the handbells were slightly better in the original study, it was decided to begin with a modern handbell which had not been analysed in this way previously. A modern American C_5 Malmark handbell was arbitrarily selected as a reasonable representative bell in this study. This bell again showed numerous golden features, but in slightly different combinations to those previously found in both church bells and handbells.

Most bells are, to a good degree of approximation, axially symmetric. Consequently, it is convenient to describe them using cylindrical polar coordinates (r, θ, z) with the z-direction lying along the symmetry axis. If this were not the case, then it would be very difficult for them to display many golden features. Concentrating on one plane at fixed θ, any golden features located will be repeated at all other values of θ. It is therefore likely to be sufficient to examine the geometry in any one sample plane.

The presence of axial symmetry in bells has important consequences for their normal modes and for the degeneracy structures of their vibration spectra. These have been studied for church bells [6] whose results should apply equally well to other axially symmetric systems like handbells and some other percussive musical instruments. It is worthy of note that they have been applied with success to gamelan gongs [7]. Essentially, group theoretical arguments show that, for axially symmetric systems, the modal functions must vary like either sin(mθ) or cos(mθ) where m = 0, 1, 2, …. Those with m = 0 are singlets, either breathing modes or twisting modes, while all others are bending modes in degenerate pairs. It is the splitting of these pairs, due to small violations of axial symmetry, that results in the phenomenon of bell "warble".

BASIC SACRED GEOMETRY

"Sacred" geometry is a generic term applied to the golden ratio and various geometrical features based upon it. Sometimes it is called the "divine proportion" or the "extreme and mean ratio" and is usually given the symbol ϕ. It can be defined by reference to Figure 1 where the straight-line AB is divided internally at a point C chosen such that

$$AB/AC = AC/CB \qquad (1)$$

Figure 1. The golden ratio.

It is a simple matter to show [1, p.80] that this ratio can take either of two values ϕ and ϕ', given by

$$\phi = \frac{1+\sqrt{5}}{2} = 1.618\ldots \quad (2a)$$

$$\text{and } \phi' = \frac{1-\sqrt{5}}{2} = -0.618\ldots \quad (2b)$$

$$\text{so that } \phi + \phi' = 1 \quad (2c)$$

ϕ is an irrational number and is, in fact, equal to the asymptotic limit of the ratio of successive terms in the Fibonacci series [1, p.101]. Of the many remarkable properties of ϕ, the following two are especially "unusual" and can easily be proved by direct substitution:

$$\phi^2 = \phi + 1 \text{ and } \phi^{-1} = \phi - 1 \quad (3)$$

The geometrical figures with which the golden ratio is particularly associated are:

- The golden rectangle, which has long to short sides in the ratio of ϕ:1.
- The golden triangle, being (an isosceles triangle whose equal sides are in a ratio of ϕ:1 with the base).
- The regular pentagon, the pentagram and the two highest order Platonic solids (dodecahedron and icosahedron), all of which contain geometrical features associated with ϕ.

The earlier work with modern church bells was concerned only with the first two in this list plus the so-called "golden angle". However, we can now report that, due to its connection with the golden triangle, a regular pentagon is also associated with important features of church bells. We now show that this association does not extend to the C_5 Malmark handbell but is replaced by another involving a circle whose radius and centre are determined by the golden ratio.

MODERN C_5 MALMARK HANDBELL

Definitions and geometry

In Figure 2 we have taken a small modern C_5 Malmark handbell, stood it on a horizontal surface, taken a vertical cross-section containing its symmetry axis and then discarded the left half. Included is some terminology for readers unfamiliar with campanological jargon. The geometry of this bell had been measured previously with considerable accuracy for use in a finite-element model [8]. It should be noted that the inner and outer profiles are different although, in this paper, we are mainly concerned with the outer one.

The thickness of the wall has a minimum at the shoulder and increases slowly and monotonically as one moves down to the rim. This differs from church bells, which have a more complicated thickness variation. Likewise, the crown is much simpler here, consisting of circular arcs. Parameters of importance in describing the outer profiles are the mouth radius R, the shoulder radius r and the vertical heights above the mouth of the crown H and

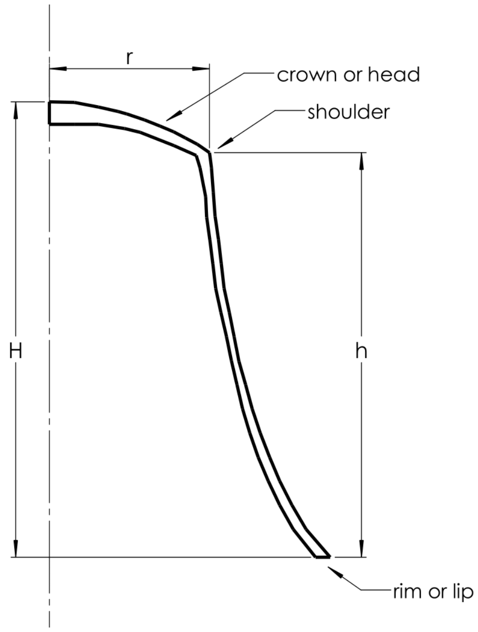

Figure 2. Half cross-section of C_5 Malmark handbell.

of the shoulder h. The radius of the crown circle proves to have the same value as that for the mouth, as we explain in the following section.

Some golden rectangles

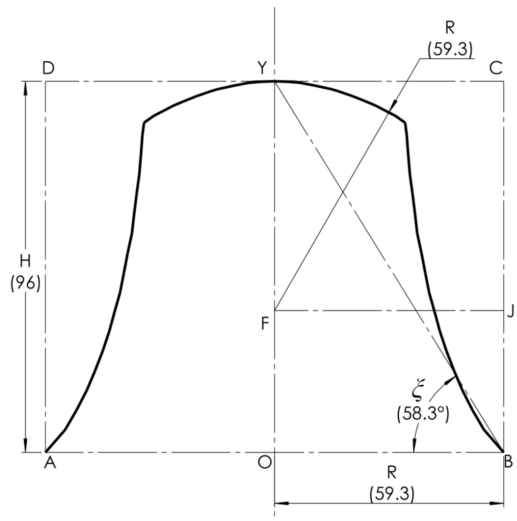

Figure 3. Outer profile of C_5 Malmark handbell showing golden rectangles.

Figure 3 shows the complete outer profile of the C_5 Malmark handbell. It has R = 59.3 mm and H = 96.0 mm so that H/R = 1.619 which is almost identical to the golden ratio of 1.618. Thus, the rectangle OBCY is almost perfectly golden. As a convenient alternative measure of its "goldenness" we join the rim at B to the centre of the crown at Y and measure the angle ξ which this makes with the mouth.

Clearly $\xi = \tan^{-1}(H/R)$ which gives $\xi = 58.3°$ in this case. This is to be compared with the golden value of $\xi_G = \tan^{-1}(\phi) = 58.28°$. The larger rectangle ABCD, containing the bell's entire outer profile, is built up of two identical golden rectangles lain side by side.

A well-known property of the golden rectangle [1, p.85] is that, if one constructs an internal square based on a shorter side, then the remaining portion is itself a (smaller) golden rectangle. In Fig. 3, the square FJCY based on the radius R of the mouth of the bell leaves a second golden rectangle OBJF. The point F, which divides the line OY by the golden ratio, is the centre of the circle defining the arc which forms the crown. The radius of this circle is equal to the radius of the mouth. Generating further, smaller, golden rectangles can be continued in this way ad infinitum but does not appear to yield any further significant results for this bell.

A golden triangle

Unlike the church bell, the shoulder and the rim of the handbell are very well defined. In Figure. 4 we again show the C_5 Malmark bell but now with the common tangent on the right extrapolated to cut the symmetry axis at E and the plane of the mouth at B. The tangent BE plus its mirror image AE in the symmetry axis AE and the baseline AOB form an isosceles triangle ABE. We are interested in the angle $\mathrm{OBE} \equiv \psi$ given by

$$\tan\psi = h/(R-r) \tag{4}$$

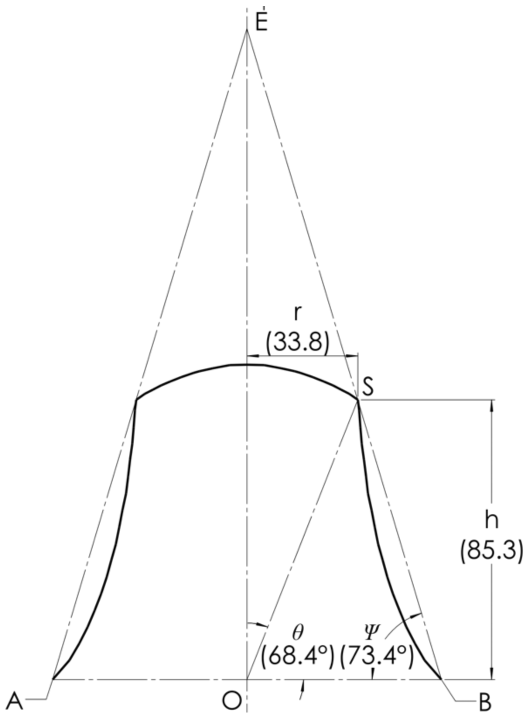

Figure 4. Outer profile of C_5 Malmark handbell showing fitting isosceles triangle and golden angle.

Using the above value for R, together with $h = 85.3$mm and $r = 33.8$mm gives $\psi = 73.6°$.

As set out in section 1, a golden triangle is an isosceles triangle in which the length of the equal sides is ϕ times that of the base. This gives a golden value for ψ of

$$\psi_G = \cos^{-1}[1/(2\phi)] = 72° \tag{5}$$

The triangle defined by the common tangents and base, having an angle of about 73.4°, is not very far from being golden. Clearly if we add further lines parallel to AB these will each form the base of a further approximately golden triangles with E as the apex. None of these seem to be of any particular significance. For the Taylor church bell, by contrast, the angle ψ came out to be 71.9° so the corresponding triangle was almost perfectly golden. A feature worthy of note here is that the point S divides EB in the golden ratio, with the length of ES being equal to the diameter of the bell's mouth. The reason for this is not immediately obvious.

A golden angle

It is usual to define "the "golden angle" as half a complete rotation divided by ϕ. This is 111.25°. However, it is more convenient for our purpose to work with the complementary angle of 68.75° which we designate as θ_G. Returning to Figure 4, we now draw an extra straight line from the centre of the mouth O to the shoulder S. The angle θ that OS makes with the mouth line OB helps to define the location of the shoulder. Clearly

$$\tan\theta = h/r \tag{6}$$

Substituting the dimensions of the C_5 Malmark bell gives $\theta = 68.4°$ which agrees remarkably well with the value of θ_G.

So, a sufficient method to specify the location of the shoulders in this bell is to use the intersection of the crown circle, with radius R and centre F (determined by the golden ratio), with a line from the centre of the mouth making a golden angle with the base line. It does not require the triangle ABE to be exactly golden. This does not work for church bells because their crowns are not simple circular arcs. In their case, the ABE triangles are much closer to being golden, as quoted above for the Taylor bell.

A regular pentagon

A text-book property of the regular pentagon is that, if one joins any corner to each of the ends of the opposite side, this produces an isosceles triangle with side-angles of exactly 72° (a golden triangle). If a bell's outer profile is partly defined by a golden triangle, then there must be a regular pentagon underlying the situation. Starting with the mouth of the bell as one side of the pentagon and using the known 72° angle, it is a straightforward matter to construct the complete pentagon. The result is shown in Figure 5 for the Taylor church bell and in Figure 6 for the C_5 Malmark handbell. In each of these figures we include the lines joining the centre of the pentagon to its vertices. For the church bell, where ψ is almost exactly 72°, the centre of the pentagon precisely defines the location of the bell's shoulders. However, in the case of the handbell, there is a noticeable deviation, possibly due to the difference of a modest 1.4° in ψ.

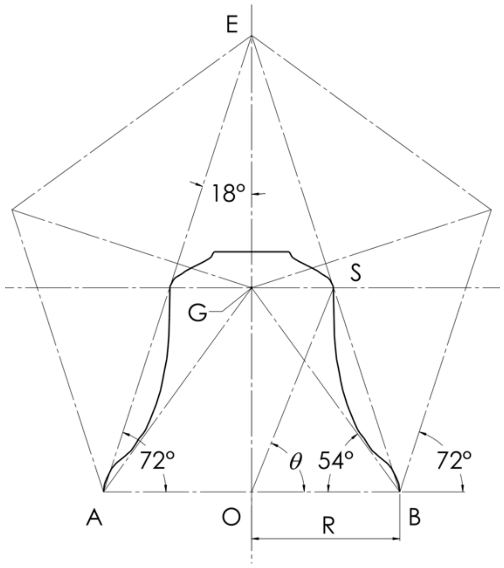

Figure 5. Outer profile of Taylor church bell showing accurate fit to both golden triangle and regular pentagon.

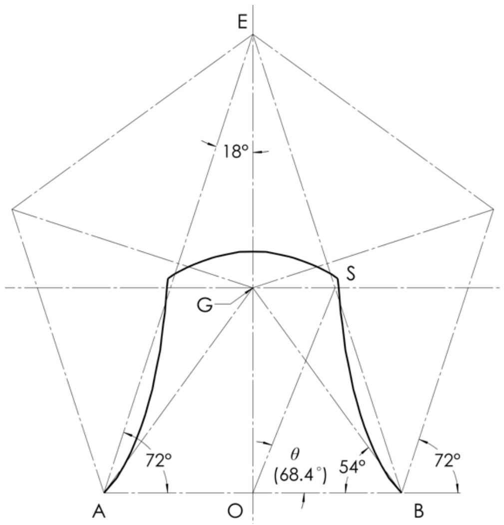

Figure 6. Outer profile of C_5 Malmark handbell showing less accurate fit to golden triangle and regular pentagon.

CONCLUSIONS

The use of modern software, more flexible than that used in earlier work, to analyse the outer profile of a C_5 Malmark handbell confirms that its design contains numerous features accurately described in terms of sacred geometry. The aspect ratio of crown height to mouth radius is exactly equal to the golden ratio. The crown is described by means of a circle whose radius and centre are fixed by the golden ratio. The shoulder is connected to the centre of the mouth by a line making a golden angle with the mouth line. These two factors together are sufficient to specify the shoulder location. A triangle enclosing the bell outer profile is approximately golden. However, possibly because of the small deviation from golden, its use to determine the shoulder location is problematic. This point is at odds with some previous results for handbells and suggests that further investigation of these other handbell results using the more recent software used herein may also be worthwhile. The same may also be true of church bells. Further work might also investigate the significance of both manufacturing tolerance differences in like bells and in the ability to measure the profiles.

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