LAMBERT SERIES AND ELLIPTIC FUNCTIONS AND CERTAIN RECIPROCAL SUMS

R. S. Melham
School of Mathematical Sciences, University of Technology, Sydney
PO Box 123, Broadway, NSW 2007, Australia
(Submitted August 1997-Final Revision February 1998)

1. INTRODUCTION

For p a strictly positive real number define, for all integers n, the sequences

$$\begin{cases} U_n = pU_{n-1} + U_{n-2}, & U_0 = 0, \quad U_1 = 1, \\ V_n = pV_{n-1} + V_{n-2}, & V_0 = 2, \quad V_1 = p. \end{cases} \tag{1.1}$$

Then U_n and V_n generalize F_n and L_n, respectively. Their Binet forms are

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad V_n = \alpha^n + \beta^n,$$

where

$$\alpha = \frac{p + \sqrt{p^2+4}}{2} \quad \text{and} \quad \beta = \frac{p - \sqrt{p^2+4}}{2}.$$

We see that $\alpha\beta = -1$, $\alpha > 1$, and $-1 < \beta < 0$.

It is known that the infinite sums

$$\sum_{n=0}^{\infty} \frac{1}{F_{2n+1}} \quad \text{and} \quad \sum_{n=0}^{\infty} \frac{1}{L_{2n}}$$

can be found by using certain constants associated with Jacobian elliptic functions, while the sums

$$\sum_{n=1}^{\infty} \frac{1}{F_{2n}} \quad \text{and} \quad \sum_{n=0}^{\infty} \frac{1}{L_{2n+1}}$$

involve the Lambert series. For an introduction to these matters we recommend Horadam [6], which contains a wealth of references to original sources. Further excellent references are Bruckman [5], Almkvist [1], and Borwein and Borwein [3]. Other types of reciprocal sums which involve Lambert series can be found in André-Jeannin [2].

In the above four sums, the task of summation is shared equally between the Lambert series and the Jacobian elliptic functions. The purpose of this paper is to give further reciprocal sums in which the task of summation is similarly shared, thus exhibiting a pleasing symmetry of method.

While the results in Section 3 are believed to be new, they are variations and extensions of known results, and so their proofs contain nothing truly innovative. For this reason we simply state each result and indicate where in the literature a similar proof can be found. In Section 4 we obtain results the like of which we have not seen, and which involve Lambert series. Interestingly, certain special cases of these results have known "dual" results which involve the Jacobian elliptic functions, further highlighting our comments above.

2. NOTATION AND PRELIMINARY RESULTS

In the theory of Jacobian elliptic functions we have, in standard notation,

$$K=\int_0^{\pi/2}\frac{dt}{\sqrt{1-k^2\sin^2 t}} \quad\text{and}\quad K'=\int_0^{\pi/2}\frac{dt}{\sqrt{1+k'^2\sin^2 t}},$$

where $0<k,\ k'<1$, and $k^2+k'^2=1$. See, for example, [5] and [7]. Write $q=e^{-K'\pi/K}$ $(0<q<1)$. Then (see [7])

$$\frac{2K}{\pi}=1+\frac{4q}{1+q^2}+\frac{4q^2}{1+q^4}+\frac{4q^3}{1+q^6}+\cdots, \tag{2.1}$$

$$\frac{2kK}{\pi}=\frac{4\sqrt{q}}{1+q}+\frac{4\sqrt{q^3}}{1+q^3}+\frac{4\sqrt{q^5}}{1+q^5}+\cdots. \tag{2.2}$$

Thus, for a given q $(0<q<1)$, we are able to find the unique values of K, k, K', and k'.

The Lambert series is defined as

$$L(x)=\sum_{n=1}^{\infty}\frac{x^n}{1-x^n},\quad |x|<1.$$

For $|x|<1$ we require the following three results, which occur as Lemma 1 in [2]:

$$\sum_{n=0}^{\infty}\frac{x^{2n+1}}{1-x^{2n+1}}=L(x)-L(x^2); \tag{2.3}$$

$$\sum_{n=1}^{\infty}\frac{x^{n}}{1+x^{n}}=L(x)-2L(x^2); \tag{2.4}$$

$$\sum_{n=0}^{\infty}\frac{x^{2n+1}}{1+x^{2n+1}}=L(x)-3L(x^2)+2L(x^4). \tag{2.5}$$

Finally, we require the following lemma.

Lemma 1: Let m be a positive integer. Then

$$\frac{1}{(\alpha-\beta)}\left[\frac{1}{\alpha^{(2n+1)m}V_{(2n+1)m}}-\frac{1}{\alpha^{(2n+1)2m}V_{(2n+1)2m}}\right]=\frac{U_{(2n+1)m}}{V_{(2n+1)m}V_{(2n+1)2m}},\quad m\text{ even}; \tag{2.6}$$

$$\frac{1}{(\alpha-\beta)}\left[\frac{1}{\alpha^{(2n+1)m}V_{(2n+1)m}}+\frac{1}{\alpha^{(2n+1)2m}V_{(2n+1)2m}}\right]=\frac{U_{(2n+1)m}}{V_{(2n+1)m}V_{(2n+1)2m}},\quad m\text{ odd}; \tag{2.7}$$

$$\frac{1}{(\alpha-\beta)}\left[\frac{1}{\alpha^{nm}V_{nm}}-\frac{1}{\alpha^{2nm}V_{2nm}}\right]=\frac{U_{nm}}{V_{nm}V_{2nm}},\quad m\text{ even}. \tag{2.8}$$

Proof: We prove only (2.8) since the proofs of (2.6) and (2.7) are similar.

$$\frac{1}{(\alpha-\beta)}\left[\frac{1}{\alpha^{nm}V_{nm}}-\frac{1}{\alpha^{2nm}V_{2nm}}\right]$$

$$= \frac{1}{(\alpha-\beta)}\left[\frac{\alpha^{nm}V_{2nm}-V_{nm}}{\alpha^{2nm}V_{nm}V_{2nm}}\right]$$

$$= \frac{1}{(\alpha-\beta)}\left[\frac{\alpha^{nm}(\alpha^{2nm}+\beta^{2nm})-(\alpha^{nm}+\beta^{nm})}{\alpha^{2nm}V_{nm}V_{2nm}}\right]$$

$$= \frac{1}{(\alpha-\beta)}\left[\frac{\alpha^{3nm}-\alpha^{nm}}{\alpha^{2nm}V_{nm}V_{2nm}}\right] \text{ (since } m \text{ is even and } \alpha\beta=-1)$$

$$= \frac{1}{(\alpha-\beta)}\left[\frac{\alpha^{nm}-\beta^{nm}}{V_{nm}V_{2nm}}\right] = \frac{U_{nm}}{V_{nm}V_{2nm}}. \quad \square$$

3. RECIPROCAL SUMS I

Using the notation in Section 2, we now state the results of this section in the following theorem.

Theorem 1: Let m be a positive integer. Then

$$\sum_{n=1}^{\infty}\frac{1}{U_{2nm}} = (\alpha-\beta)[L(\beta^{2m})-L(\beta^{4m})], \tag{3.1}$$

$$\sum_{n=0}^{\infty}\frac{1}{V_{2nm}} = \frac{1}{4}\left[\frac{2K(\beta^{2m})}{\pi}+1\right], \tag{3.2}$$

$$\sum_{n=0}^{\infty}\frac{1}{U_{(2n+1)m}} = \begin{cases}(\alpha-\beta)[L(\beta^{m})-2L(\beta^{2m})+L(\beta^{4m})], & m \text{ even,}\\ \dfrac{(\alpha-\beta)k(\beta^{2m})K(\beta^{2m})}{2\pi}, & m \text{ odd,}\end{cases} \tag{3.3}$$

$$\sum_{n=0}^{\infty}\frac{1}{V_{(2n+1)m}} = \begin{cases}\dfrac{k(\beta^{2m})K(\beta^{2m})}{2\pi}, & m \text{ even,}\\ -L(\beta^{m})+2L(\beta^{2m})-L(\beta^{4m}), & m \text{ odd.}\end{cases} \tag{3.4}$$

For a special case of (3.1) concerning the Fibonacci numbers, see the paper of Brady [4], where there is an obvious misprint (for $2m\beta$ and $4m\beta$ read β^{2m} and β^{4m}, respectively). Also of interest is (2.1) in Shannon and Horadam [8]. The proof of (3.2) proceeds along the same lines as the proof of (3.12) in Horadam [6]. The proofs of the first part of (3.3) and the second part of (3.4) are similar to the proof of (4.12) in Horadam [6]. For the proofs, one uses the identity

$$\frac{x^{2n+1}}{1-x^{4n+2}} = \frac{x^{2n+1}}{1-x^{2n+1}} - \frac{x^{4n+2}}{1-x^{4n+2}}$$

together with (2.3). Finally, the proofs of the second part of (3.3) and the first part of (3.4) are similar to the proof on page 103 of the above-mentioned paper of Horadam.

4. RECIPROCAL SUMS II

The results of this section are contained in the following theorem.

Theorem 2: Let m be a positive integer. Then

$$\sum_{n=1}^{\infty}\frac{U_{nm}}{V_{nm}V_{2nm}}=\begin{cases}\dfrac{1}{(\alpha-\beta)}[L(\beta^{2m})-3L(\beta^{4m})+2L(\beta^{8m})], & m \text{ even},\\ \dfrac{1}{(\alpha-\beta)}[L(\beta^{2m})+L(\beta^{4m})-6L(\beta^{8m})+4L(\beta^{16m})], & m \text{ odd},\end{cases} \tag{4.1}$$

$$\sum_{n=0}^{\infty}\frac{U_{(2n+1)m}}{V_{(2n+1)m}V_{(2n+1)2m}}=\begin{cases}\dfrac{1}{(\alpha-\beta)}[L(\beta^{2m})-4L(\beta^{4m})+5L(\beta^{8m})-2L(\beta^{16m})], & m \text{ even},\\ \dfrac{1}{(\alpha-\beta)}[L(\beta^{2m})-3L(\beta^{8m})+2L(\beta^{16m})], & m \text{ odd}.\end{cases} \tag{4.2}$$

Proof: If m is even we have, from (2.8),

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{U_{nm}}{V_{nm}V_{2nm}} &= \frac{1}{(\alpha-\beta)}\left[\sum_{n=1}^{\infty}\frac{1}{\alpha^{nm}V_{nm}}-\sum_{n=1}^{\infty}\frac{1}{\alpha^{2nm}V_{2nm}}\right]\\ &= \frac{1}{(\alpha-\beta)}\left[\sum_{n=1}^{\infty}\frac{1}{\alpha^{2nm}+1}-\sum_{n=1}^{\infty}\frac{1}{\alpha^{4nm}+1}\right]\\ &= \frac{1}{(\alpha-\beta)}\left[\sum_{n=1}^{\infty}\frac{(\beta^{2m})^n}{1+(\beta^{2m})^n}-\sum_{n=1}^{\infty}\frac{(\beta^{4m})^n}{1+(\beta^{4m})^n}\right] \text{ (since } \alpha\beta=-1),\end{aligned}$$

and the first part of (4.1) follows from (2.4). To prove (4.2) we begin with (2.6) and (2.7) and proceed in the same manner, making use of (2.3) and (2.5). Now to the second part of (4.1). In the first part of (4.1), we replace m by $2m$ to obtain

$$\sum_{n=1}^{\infty}\frac{U_{2nm}}{V_{2nm}V_{4nm}}=\frac{1}{(\alpha-\beta)}[L(\beta^{4m})-3L(\beta^{8m})+2L(\beta^{16m})],$$

which is valid for all positive integers m. When we add this sum to the second sum in (4.2), we obtain the second sum in (4.1). This completes the proof. □

5. THE DUAL RESULTS

In the introduction we referred to known dual results of special cases of (4.1) and (4.2). To obtain these, we replace $U(V)$ by $V(U)$ in (4.1) and (4.2). Then, with the identity $U_{2n}=U_nV_n$, the summands become $1/U_{nm}^2$ and $1/U_{(2n+1)m}^2$. Now if we take $U_n=F_n$, then the sums

$$\sum_{n=1}^{\infty}\frac{1}{F_n^2},\quad \sum_{n=1}^{\infty}\frac{1}{F_{2n}^2},\quad \text{and}\quad \sum_{n=0}^{\infty}\frac{1}{F_{2n+1}^2}$$

are known. See, for example (44), (48), and (55) of Bruckman [5], where elliptic functions are used. See also (f) and (h) on page 320 of Almkvist [1], where theta functions are used.

We have not found the more general sums

$$\sum_{n=1}^{\infty}\frac{1}{U_{nm}^2} \quad \text{and} \quad \sum_{n=0}^{\infty}\frac{1}{U_{(2n+1)m}^2} \quad \text{(for the two parities of } m\text{)}$$

in the literature available to us, and we suspect that their determination is much more difficult.

REFERENCES

1. G. Almkvist. "A Solution to a Tantalizing Problem." *The Fibonacci Quarterly* **24.4** (1986): 316-22.
2. R. André-Jeannin. "Lambert Series and the Summation of Reciprocals in Certain Fibonacci-Lucas-Type Sequences." *The Fibonacci Quarterly* **28.3** (1990):223-26.
3. J. M. Borwein & P. B. Borwein. *Pi and the AGM.* New York: Wiley, 1987.
4. W. G. Brady. "The Lambert Function." *The Fibonacci Quarterly* **10.2** (1972):199-200.
5. P. S. Bruckman. "On the Evaluation of Certain Infinite Series by Elliptic Functions." *The Fibonacci Quarterly* **15.4** (1977):293-310.
6. A. F. Horadam. "Elliptic Functions and Lambert Series in the Summation of Reciprocals in Certain Recurrence-Generated Sequences." *The Fibonacci Quarterly* **26.2** (1988):98-114.
7. C. G. J. Jacobi. "Fundamenta Nova Theoriae Functionum Ellipticarum." *Gesammelte Werke* **1** (1881):159.
8. A. G. Shannon & A. F. Horadam. "Reciprocals of Generalized Fibonacci Numbers." *The Fibonacci Quarterly* **9.3** (1971):299-306, 312.

AMS Classification Numbers: 11B37, 11B39

❖❖❖

The Fibonacci Quarterly

Official Publication of The Fibonacci Association

Volume 37 **Number 3** **August 1999**

CONTENTS