GENERALIZATIONS OF SOME IDENTITIES OF LONG

Ray Melham
School of Mathematical Sciences, University of Technology, Sydney
PO Box 123, Broadway, NSW 2007, Australia
(*Submitted April 1997*)

1. INTRODUCTION

Long [4] considered the identity

$$L_n^2 - 5F_n^2 = 4(-1)^n \tag{1.1}$$

and noted that the left side consists of terms of the second degree. He gave numerous variations of (1.1) by varying the terms that make up the products and also the subscripts. For example, he obtained

$$L_nF_m - L_{n+d}F_{m+d} = \begin{cases} F_{-d}L_{m+n+d}, & d \text{ even,} \\ L_{-d}F_{m+n+d} - 2(-1)^nF_{m-n}, & d \text{ odd.} \end{cases} \tag{1.2}$$

Long noticed that the replacement of the minus sign on the left with a plus sign simply reversed the even and odd cases on the right side, so that a counterpart to (1.2) is

$$L_nF_m + L_{n+d}F_{m+d} = \begin{cases} L_{-d}F_{m+n+d} - 2(-1)^nF_{m-n}, & d \text{ even,} \\ F_{-d}L_{m+n+d}, & d \text{ odd.} \end{cases} \tag{1.3}$$

In this paper we generalize all the results of Long that focus on the difference of products, and we produce many more. A pleasing feature of the identities contained here is that while being more general than those of Long, they maintain the elegant properties which Long observed.

2. THE SEQUENCES

Define the sequences $\{U_n\}$, $\{V_n\}$, $\{W_n\}$, and $\{X_n\}$ for all integers n by

$$\begin{cases} U_n = pU_{n-1} - qU_{n-2}, \ U_0 = 0, \ U_1 = 1, \\ V_n = pV_{n-1} - qV_{n-2}, \ \ V_0 = 2, \ V_1 = p, \\ W_n = pW_{n-1} - qW_{n-2}, \ W_0 = a, \ W_1 = b, \\ X_n = W_{n+1} - qW_{n-1}. \end{cases} \tag{2.1}$$

Here a, b, p, and q are any complex numbers with $\Delta = p^2 - 4q \neq 0$. Then the roots α and β of $x^2 - px + q = 0$ are distinct. Hence, the Binet form (see [2] and [3]) for W_n is

$$W_n = \frac{A\alpha^n - B\beta^n}{\alpha - \beta},$$

where $A = b - a\beta$ and $B = b - a\alpha$. It can also be shown that $X_n = A\alpha^n + B\beta^n$. The sequences $\{U_n\}$ and $\{V_n\}$ generalize $\{F_n\}$ and $\{L_n\}$, respectively. Also, since $\{W_n\}$ generalizes $\{U_n\}$, then $\{X_n\}$ generalizes $\{V_n\}$ by virtue of the fact that $V_n = U_{n+1} - qU_{n-1}$, which can be proved using Binet forms.

We consider a second group of sequences obtained from (2.1) by putting $q=-1$. In the obvious order, we name these sequences $\{u_n\}$, $\{v_n\}$, $\{w_n\}$, and $\{x_n\}$. The sequences $\{u_n\}$ and $\{v_n\}$ also generalize $\{F_n\}$ and $\{L_n\}$, respectively. Furthermore, $\{w_n\}$ and $\{x_n\}$ generalize $\{u_n\}$ and $\{v_n\}$, respectively. We write $D=p^2+4$.

Finally, our third group of sequences is obtained from (2.1) by putting $q=1$. In order, we name these sequences $\{P_n\}$, $\{Q_n\}$, $\{R_n\}$, and $\{S_n\}$.

3. THE FIRST SET OF IDENTITIES

For the sequences $\{u_n\}$, $\{v_n\}$, $\{w_n\}$, and $\{x_n\}$, we have found the following:

$$x_n v_m - x_{n+d} v_{m+d} = \begin{cases} v_{-d} x_{m+n+d} + 2(-1)^m x_{n-m}, & d \text{ odd}, \\ D u_{-d} w_{m+n+d}, & d \text{ even}, \end{cases} \tag{3.1}$$

$$x_n v_m - x_{m+d} v_{n+d} = \begin{cases} v_{-d} x_{m+n+d} + (-1)^m x_0 v_{n-m}, & d \text{ odd}, \\ D(u_{-d} w_{m+n+d} + (-1)^m w_0 u_{n-m}), & d \text{ even}, \end{cases} \tag{3.2}$$

$$x_n u_m - x_{n+d} u_{m+d} = \begin{cases} v_{-d} w_{m+n+d} + 2(-1)^{m+1} w_{n-m}, & d \text{ odd}, \\ u_{-d} x_{m+n+d}, & d \text{ even}, \end{cases} \tag{3.3}$$

$$x_n u_m - x_{m+d} u_{n+d} = \begin{cases} v_{-d} w_{m+n+d} + (-1)^{m+1} w_0 v_{n-m}, & d \text{ odd}, \\ u_{-d} x_{m+n+d} + (-1)^{m+1} x_0 u_{n-m}, & d \text{ even}, \end{cases} \tag{3.4}$$

$$x_n u_m - v_{n+d} w_{m+d} = \begin{cases} v_{-d} w_{m+n+d} + (-1)^{m+1} x_0 u_{n-m}, & d \text{ odd}, \\ u_{-d} x_{m+n+d} + (-1)^{m+1} w_0 v_{n-m}, & d \text{ even}, \end{cases} \tag{3.5}$$

$$x_n u_m - v_{m+d} w_{n+d} = \begin{cases} v_{-d} w_{m+n+d}, & d \text{ odd}, \\ u_{-d} x_{m+n+d} + 2(-1)^{m+1} w_{n-m}, & d \text{ even}, \end{cases} \tag{3.6}$$

$$x_n v_m - D w_{n+d} u_{m+d} = \begin{cases} v_{-d} x_{m+n+d}, & d \text{ odd}, \\ D u_{-d} w_{m+n+d} + 2(-1)^m x_{n-m}, & d \text{ even}, \end{cases} \tag{3.7}$$

$$x_n v_m - D w_{m+d} u_{n+d} = \begin{cases} v_{-d} x_{m+n+d} + (-1)^m D w_0 u_{n-m}, & d \text{ odd}, \\ D u_{-d} w_{m+n+d} + (-1)^m x_0 v_{n-m}, & d \text{ even}, \end{cases} \tag{3.8}$$

$$w_n u_m - w_{n+d} u_{m+d} = \begin{cases} \frac{1}{D}(v_{-d} x_{m+n+d} + 2(-1)^{m+1} x_{n-m}), & d \text{ odd}, \\ u_{-d} w_{m+n+d}, & d \text{ even}, \end{cases} \tag{3.9}$$

$$w_n u_m - w_{m+d} u_{n+d} = \begin{cases} \frac{1}{D}(v_{-d} x_{m+n+d} + (-1)^{m+1} x_0 v_{n-m}), & d \text{ odd}, \\ u_{-d} w_{m+n+d} + (-1)^{m+1} w_0 u_{n-m}, & d \text{ even}, \end{cases} \tag{3.10}$$

$$w_n v_m - w_{n+d} v_{m+d} = \begin{cases} v_{-d} w_{m+n+d} + 2(-1)^m w_{n-m}, & d \text{ odd}, \\ u_{-d} x_{m+n+d}, & d \text{ even}, \end{cases} \tag{3.11}$$

$$w_n v_m - w_{m+d} v_{n+d} = \begin{cases} v_{-d} w_{m+n+d} + (-1)^m w_0 v_{n-m}, & d \text{ odd}, \\ u_{-d} x_{m+n+d} + (-1)^m x_0 u_{n-m}, & d \text{ even}. \end{cases} \tag{3.12}$$

If on the left side, in each case, we replace the minus sign with a plus sign, the identities are exactly as stated but with the even and odd cases reversed. This parallels the observations of Long for his Fibonacci-Lucas identities. For example, as a counterpart to (3.6), we have

$$x_n u_m + v_{m+d} w_{n+d} = \begin{cases} u_{-d} x_{m+n+d} + 2(-1)^{m+1} w_{n-m}, & d \text{ odd}, \\ v_{-d} w_{m+n+d}, & d \text{ even}. \end{cases} \tag{3.13}$$

The proofs of (3.1)-(3.12) and their counterparts with a plus sign on the left are similar. For the proofs, we require the following:

$$q^n U_{-n} = -U_n, \tag{3.14}$$

$$q^n V_{-n} = V_n, \tag{3.15}$$

$$W_{n+d} + q^d W_{n-d} = W_n V_d, \tag{3.16}$$

$$W_{n+d} - q^d W_{n-d} = X_n U_d, \tag{3.17}$$

$$X_{n+d} + q^d X_{n-d} = X_n V_d, \tag{3.18}$$

$$X_{n+d} - q^d X_{n-d} = \Delta W_n U_d. \tag{3.19}$$

Identities (3.14) and (3.15) can be proved using Binet forms, while (3.16)-(3.19) occur in Bergum and Hoggatt [1].

As an example, we prove (3.1).

Proof of (3.1): With $q = -1$ and using the Binet forms in Section 2, we have

$$\begin{aligned} x_n v_m - x_{n+d} v_{m+d} &= (A\alpha^n + B\beta^n)(\alpha^m + \beta^m) - (A\alpha^{n+d} + B\beta^{n+d})(\alpha^{m+d} + \beta^{m+d}) \\ &= (A\alpha^{m+n} + B\beta^{m+n}) - (A\alpha^{m+n+2d} + B\beta^{m+n+2d}) \\ &\quad + (A\alpha^n\beta^m + B\alpha^m\beta^n) - (A\alpha^{n+d}\beta^{m+d} + B\alpha^{m+d}\beta^{n+d}) \\ &= x_{m+n} - x_{m+n+2d} + (\alpha\beta)^m (A\alpha^{n-m} + B\beta^{n-m}) - (\alpha\beta)^{m+d}(A\alpha^{n-m} + B\beta^{n-m}) \\ &= -(x_{(m+n+d)+d} - x_{(m+n+d)-d}) + (1 - (\alpha\beta)^d)(\alpha\beta)^m x_{n-m}. \end{aligned}$$

Now $q = -1$ implies $\alpha\beta = -1$. From (3.18) and (3.19), the right side becomes

$$\begin{cases} -x_{m+n+d} v_d + 2(-1)^m x_{n-m}, & d \text{ odd}, \\ -D w_{m+n+d} u_d, & d \text{ even}, \end{cases}$$

and the use of (3.14) and (3.15) gives the result.

4. THE SECOND SET OF IDENTITIES

We now consider the sequences $\{P_n\}$, $\{Q_n\}$, $\{R_n\}$, and $\{S_n\}$. For these sequences, we have derived twenty-four identities that parallel (3.1)-(3.12). Twelve identities have a minus sign connecting the two products on the left, and twelve have a plus sign. Each can be obtained by looking at its counterpart in the list (3.1)-(3.12) and using the following rules:

(i) Replace u by P, v by Q, w by R, and x by S.
(ii) Replace any occurrence of $(-1)^m$ on the right side by 1.
(iii) Then the difference of the two products is equal to the even case, and the sum of the two products is equal to the odd case.

For example, using (3.3), we have

$$S_nP_m - S_{n+d}P_{m+d} = P_{-d}S_{m+n+d}, \tag{4.1}$$

$$S_nP_m + S_{n+d}P_{m+d} = Q_{-d}R_{m+n+d} - 2R_{n-m}. \tag{4.2}$$

These can be proved in the same manner shown previously, and because of the above rules of formation we refrain from listing the others.

5. THE THIRD SET OF IDENTITIES

The identities of Long [4, (18)-(24)] are generalizations and variations of Simson's identity

$$F_{n+1}F_{n-1} - F_n^2 = (-1)^n. \tag{5.1}$$

They share the common thread of being differences of products the sum of whose subscripts are equal. For example,

$$L_nF_m - L_{n-d}F_{m+d} = (-1)^n F_{-d}L_{m-n+d}. \tag{5.2}$$

We have found similar identities for mixtures of terms from the sequences $\{U_n\}$, $\{V_n\}$, $\{W_n\}$, and $\{X_n\}$. Following our notation in Section 2, we write $e = pab - qa^2 - b^2 = -AB$, which is essentially the notation of Horadam [2]. The first group of identities is:

$$X_nX_m - X_{n-d}X_{m+d} = -eq^m\Delta U_dU_{n-m-d}, \tag{5.3}$$

$$X_nV_m - X_{n-d}V_{m+d} = q^m\Delta U_dW_{n-m-d}, \tag{5.4}$$

$$X_nV_m - X_{m+d}V_{n-d} = q^m\Delta W_dU_{n-m-d}. \tag{5.5}$$

If we replace the minus sign connecting the two products on the left with a plus sign, then identity (5.3) does not have an interesting counterpart. But in (5.4) and (5.5) we modify the right side by replacing U with V, W with X, dividing by Δ, and then adding $2X_{m+n}$.

The second group is:

$$W_nW_m - W_{n-d}W_{m+d} = eq^mU_dU_{n-m-d}, \tag{5.6}$$

$$W_nU_m - W_{n-d}U_{m+d} = -q^mU_dW_{n-m-d}, \tag{5.7}$$

$$W_nU_m - W_{m+d}U_{n-d} = -q^mW_dU_{n-m-d}. \tag{5.8}$$

As before, if we replace the minus sign on the left with a plus sign, then (5.6) does not have an interesting counterpart. However, we change the right sides of (5.7) and (5.8) by replacing U with V, W with X, adding $2X_{m+n}$, and then dividing by Δ.

The third group is:

$$X_nX_m - \Delta W_{n-d}W_{m+d} = -eq^mV_dV_{n-m-d}, \tag{5.9}$$

$$X_nV_m - \Delta W_{n-d}U_{m+d} = q^mV_dX_{n-m-d}, \tag{5.10}$$

$$X_nV_m - \Delta W_{m+d}U_{n-d} = q^mX_dV_{n-m-d}. \tag{5.11}$$

Again, (5.9) yields nothing interesting after replacing the minus sign on the left with a plus sign. However, we change the right sides of (5.10) and (5.11) by replacing V with U, X with W, multiplying by Δ, and adding $2X_{m+n}$. This should be compared with the processes in the previous groups of identities.

The last group of identities is:

$$X_nW_m - X_{n-d}W_{m+d} = eq^mU_dV_{n-m-d}, \tag{5.12}$$

$$X_nW_m - X_{m+d}W_{n-d} = eq^mV_dU_{n-m-d}, \tag{5.13}$$

$$X_nU_m - X_{n-d}U_{m+d} = -q^mU_dX_{n-m-d}, \tag{5.14}$$

$$X_nU_m - X_{m+d}U_{n-d} = -q^mX_dU_{n-m-d}, \tag{5.15}$$

$$V_nW_m - V_{n-d}W_{m+d} = q^nU_{-d}X_{m-n+d}, \tag{5.16}$$

$$V_nW_m - V_{m+d}W_{n-d} = q^nX_{-d}U_{m-n+d}, \tag{5.17}$$

$$X_nU_m - V_{n-d}W_{m+d} = -q^mW_dV_{n-m-d}, \tag{5.18}$$

$$X_nU_m - V_{m+d}W_{n-d} = -q^mV_dW_{n-m-d}. \tag{5.19}$$

In (5.12) and (5.13), replacing the minus sign on the left with a plus sign yields identities which are not interesting. However, in (5.14)-(5.19), we change the right side by replacing $U(V)$ with $V(U)$ and $W(X)$ with $X(W)$ and adding $2W_{m+n}$.

We refrain from giving proofs of identities (5.3)-(5.19) and their counterparts because they are similar to the proof of (3.1) demonstrated earlier.

REFERENCES

1. G. E. Bergum & V. E. Hoggatt, Jr. "Sums and Products for Recurring Sequences." *The Fibonacci Quarterly* **13.2** (1975):115-20.
2. A. F. Horadam. "Basic Properties of a Certain Generalized Sequence of Numbers." *The Fibonacci Quarterly* **3.3** (1965):161-76.
3. A. F. Horadam. "Generating Functions for Powers of a Certain Generalized Sequence of Numbers." *Duke Math. J.* **32** (1965):437-46.
4. C. T. Long. "Discovering Fibonacci Identities." *The Fibonacci Quarterly* **24.2** (1986):160-67.

AMS Classification Numbers: 11B37, 11B39

❖❖❖

The Fibonacci Quarterly

Official Publication of The Fibonacci Association

Volume 37 Number 2 May 1999

CONTENTS