

GENERALIZED TRIPLE PRODUCTS

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1. INTRODUCTION

For arbitrary integers a and b , Horadam [2] and [3] established the notation

$$W_n = W_n(a, b; p, q), \quad (1.1)$$

meaning that

$$W_n = pW_{n-1} - qW_{n-2}, \quad W_0 = a, \quad W_1 = b, \quad n \geq 2. \quad (1.2)$$

The sequence $\{W_n\}_{n=0}^{\infty}$ thus defined can be extended to negative integer subscripts by the use of (1.2), and with this understanding we write simply $\{W_n\}$. In this paper we assume that a , b , p , and q are arbitrary real numbers.

By using the generating functions of $\{F_{n+m}\}_{n=0}^{\infty}$ and $\{L_{n+m}\}_{n=0}^{\infty}$ Hansen [1] obtained expansions for $F_j F_k F_l$, $F_j F_k L_l$, $F_j L_k L_l$, and $L_j L_k L_l$. By following the same techniques, Serkland [5] produced similar expansions for the Pell and Pell-Lucas numbers defined by

$$\begin{cases} P_n = W_n(0, 1; 2, -1), \\ Q_n = W_n(2, 2; 2, -1). \end{cases} \quad (1.3)$$

Later Horadam [4] generalized the results of both these writers to the sequences

$$\begin{cases} U_n = W_n(0, 1; p, -1), \\ V_n = W_n(2, p; p, -1). \end{cases} \quad (1.4)$$

Define the sequences $\{W_n\}$ and $\{X_n\}$ by

$$\begin{cases} W_n = W_n(a, b; p, -1), \\ X_n = W_{n+1} + W_{n-1}. \end{cases} \quad (1.5)$$

Here we emphasize that W_n is as in (1.2) but with $q = -1$, and this is the case for the remainder of the paper. Since $\{W_n\}$ generalizes $\{U_n\}$, then $\{X_n\}$ generalizes $\{V_n\}$ by virtue of the fact that $V_n = U_{n+1} + U_{n-1}$. The object of this paper is to generalize the results of Horadam, and so also of Serkland and of Hansen, by incorporating terms from the sequences $\{W_n\}$ and $\{X_n\}$ into the products.

Since $\Delta = p^2 + 4 \neq 0$, the roots α and β of $x^2 - px - 1 = 0$ are distinct. Hence, the Binet form (see [2] and [3]) for W_n is

$$W_n = \frac{A\alpha^n - B\beta^n}{\alpha - \beta},$$

where $A = b - a\beta$ and $B = b - a\alpha$. It can also be shown that

$$X_n = A\alpha^n + B\beta^n.$$

2. SOME PRELIMINARY RESULTS

We shall need the following, each of which can be proved with the use of Binet forms:

$$(-1)^n U_{-n} = -U_n, \tag{2.1}$$

$$(-1)^n V_{-n} = V_n, \tag{2.2}$$

$$\Delta W_n = X_{n+1} + X_{n-1}, \tag{2.3}$$

$$\Delta U_{m+d} W_{n-d} - V_m X_n = (-1)^{m+1} V_d X_{n-m-d}, \tag{2.4}$$

$$W_{m+d} V_{n-d} - U_m X_n = (-1)^m W_d V_{n-m-d}, \tag{2.5}$$

$$W_n U_m + W_{n-1} U_{m-1} = W_{n+m-1}, \tag{2.6}$$

$$W_n V_m + W_{n-1} V_{m-1} = X_{n+m-1}, \tag{2.7}$$

$$U_n X_m + U_{n-1} X_{m-1} = X_{n+m-1}, \tag{2.8}$$

$$X_n V_m + X_{n-1} V_{m-1} = X_{n+m} + X_{n+m-2} = \Delta W_{n+m-1}. \tag{2.9}$$

3. THE MAIN RESULTS

Using the Binet form for W_n we have, for m an integer and $|x|$ small,

$$\begin{aligned} \sum_{n=0}^{\infty} W_{n+m} x^n &= \sum_{n=0}^{\infty} \frac{(A\alpha^{n+m} - B\beta^{n+m})x^n}{\alpha - \beta} = \frac{1}{\alpha - \beta} \left(\alpha^m \sum_{n=0}^{\infty} A\alpha^n x^n - \beta^m \sum_{n=0}^{\infty} B\beta^n x^n \right) \\ &= \frac{1}{\alpha - \beta} \left(\frac{A\alpha^m}{1 - \alpha x} - \frac{B\beta^m}{1 - \beta x} \right) = \frac{1}{\alpha - \beta} \left(\frac{(A\alpha^m - B\beta^m) - \alpha\beta(A\alpha^{m-1} - B\beta^{m-1})x}{(1 - \alpha x)(1 - \beta x)} \right). \end{aligned}$$

Then, putting $D = 1 - px - x^2$, we have

$$\sum_{n=0}^{\infty} W_{n+m} x^n = \frac{W_m + W_{m-1}x}{D}. \tag{3.1}$$

Of course, in (3.1), we can replace $\{W_n\}$ by any of the sequences in this paper. In particular, with $m = 1$ and $\{W_n\} = \{U_n\}$, (3.1) becomes

$$\sum_{n=0}^{\infty} U_{n+1} x^n = \frac{1}{D}. \tag{3.2}$$

The following result, which is essential for what follows, can be proved with partial fractions techniques:

$$\begin{aligned} \frac{(j+kx)}{D} \cdot \frac{(l+tx)}{D} &= \frac{j l + (j t + k l)x + k t x^2}{D^2} \\ &= \frac{-k t}{D} + \frac{(j l + k t) + (j t + k l - p k t)x}{D^2}. \end{aligned} \tag{3.3}$$

Now

$$\frac{U_m + U_{m-1}x}{D} \cdot \frac{X_s + X_{s-1}x}{D} = \sum_{n=0}^{\infty} U_{n+m} x^n \cdot \sum_{n=0}^{\infty} X_{n+s} x^n = \sum_{n=0}^{\infty} \sum_{i=0}^n U_{i+m} X_{n-i+s} x^n. \tag{3.4}$$

Alternatively, using (3.3), we have

$$\begin{aligned} & \frac{U_m + U_{m-1}x}{D} \cdot \frac{X_s + X_{s-1}x}{D} \\ &= \frac{-U_{m-1}X_{s-1}}{D} + \frac{(U_m X_s + U_{m-1}X_{s-1}) + (U_m X_{s-1} + U_{m-1}X_s - pU_{m-1}X_{s-1})x}{D^2} \end{aligned}$$

Then, by using (2.8) and the recurrence relation (1.2), this becomes

$$\begin{aligned} & \frac{-U_{m-1}X_{s-1}}{D} + \frac{X_{m+s-1} + (U_{m-1}X_s + U_{m-2}X_{s-1})x}{D^2} \\ &= \frac{-U_{m-1}X_{s-1}}{D} + \frac{X_{m+s-1} + X_{m+s-2}x}{D^2} = -(U_{m-1}X_{s-1}) \cdot \frac{1}{D} + \frac{X_{m+s-1} + X_{m+s-2}x}{D} \cdot \frac{1}{D} \end{aligned}$$

Now, by using (3.1) and (3.2), this in turn becomes

$$\begin{aligned} & -U_{m-1}X_{s-1} \sum_{n=0}^{\infty} U_{n+1}x^n + \sum_{n=0}^{\infty} X_{n+m+s-1}x^n \cdot \sum_{n=0}^{\infty} U_{n+1}x^n \\ &= \sum_{n=0}^{\infty} (-U_{n+1}U_{m-1}X_{s-1})x^n + \sum_{n=0}^{\infty} \sum_{i=0}^n U_{i+1}X_{n-i+m+s-1}x^n \\ &= \sum_{n=0}^{\infty} \left(-U_{n+1}U_{m-1}X_{s-1} + \sum_{i=0}^n U_{i+1}X_{n-i+m+s-1} \right) x^n \end{aligned}$$

By equating the coefficients of x^n in the last line and the right side of (3.4), we obtain

$$\sum_{i=0}^n U_{i+m}X_{n-i+s} = -U_{n+1}U_{m-1}X_{s-1} + \sum_{i=0}^n U_{i+1}X_{n-i+m+s-1}.$$

Finally, putting $j = m - 1$, $k = n + 1$, and $l = s - 1$, we get

$$U_j U_k X_l = \sum_{i=0}^{k-1} (U_{i+1} X_{j+k+l-i} - U_{j+i+1} X_{k+l-i}). \tag{3.5}$$

If we replace X by V , we see that this generalizes Horadam's Theorem 4, which contains a typographical error in one of the subscripts.

In exactly the same manner, taking the product of

$$\frac{U_m + U_{m-1}x}{D} \quad \text{and} \quad \frac{W_s + W_{s-1}x}{D}$$

and using (2.6), we obtain

$$W_j U_k U_l = \sum_{i=0}^{l-1} (W_{j+k+l-i} U_{i+1} - W_{j+i+1} U_{k+l-i}). \tag{3.6}$$

This generalizes Horadam's Theorem 5.

Again, taking the product of

$$\frac{V_m + V_{m-1}x}{D} \quad \text{and} \quad \frac{X_s + X_{s-1}x}{D}$$

and using (2.9) yields

$$U_j V_k X_l = \sum_{i=0}^{j-1} (\Delta U_{j-i} W_{k+l+i+1} - V_{k+i+1} X_{j+l-i}). \tag{3.7}$$

This generalizes Horadam's Theorem 6.

Further, taking the product of

$$\frac{W_m + W_{m-1}x}{D} \quad \text{and} \quad \frac{V_s + V_{s-1}x}{D}$$

and using (2.7) leads to

$$W_j U_k V_l = \sum_{i=0}^{k-1} (U_{i+1} X_{j+k+l-i} - W_{j+i+1} V_{k+l-i}). \tag{3.8}$$

Making use of (3.7), we have

$$\begin{aligned} V_j V_k X_l &= (U_{j+1} + U_{j-1}) V_k X_l = U_{j+1} V_k X_l + U_{j-1} V_k X_l \\ &= \sum_{i=0}^j (\Delta U_{j-i+1} W_{k+l+i+1} - V_{k+i+1} X_{j+l-i+1}) + \sum_{i=0}^{j-2} (\Delta U_{j-i-1} W_{k+l+i+1} - V_{k+i+1} X_{j+l-i-1}) \\ &= \left(\sum_{i=0}^{j-2} (\Delta W_{k+l+i+1} (U_{j-i+1} + U_{j-i-1}) - V_{k+i+1} (X_{j+l-i+1} + X_{j+l-i-1})) \right) \\ &\quad + (\Delta U_2 W_{k+l+j} - V_{k+j} X_{l+2}) + (\Delta U_1 W_{k+l+j+1} - V_{k+j+1} X_{l+1}). \end{aligned}$$

We now use (2.4) and (2.2) to simplify the last two terms on the right side. Finally, recalling that $U_{n+1} + U_{n-1} = V_n$ and using (2.3), we obtain

$$V_j V_k X_l = \left(\Delta \sum_{i=0}^{j-2} (W_{k+l+i+1} V_{j-i} - W_{j+l-i} V_{k+i+1}) \right) + p X_l V_{j+k-1}. \tag{3.9}$$

This generalizes Horadam's Theorem 7, and is more concisely written.

To obtain our final product, we write

$$W_j V_k V_l = W_j (U_{k+1} + U_{k-1}) V_l.$$

Then proceeding in the same manner we use (3.8) and (2.5) to obtain

$$W_j V_k V_l = \left(\Delta \sum_{i=0}^{k-2} (W_{j+k+l-i} U_{i+1} - W_{j+i+1} U_{k+l-i}) \right) + (-1)^{k+1} p W_j V_{l+1-k}. \tag{3.10}$$

Of course, in each summation identity, the parameter contained in the upper limit of summation must be chosen so that the sum is well defined. For example, in (3.10), we assume $k \geq 2$.

4. THE MAIN RESULTS SIMPLIFIED

We have chosen to present the results (3.5)-(3.10) in the given manner in order to facilitate comparison with the results of Horadam, Serkland, and Hansen. We now demonstrate that they can be simplified considerably.

By using Binet forms, it can be shown that

$$U_{i+1}X_{j+k+l-i} - U_{j+i+1}X_{k+l-i} = (-1)^i U_j X_{k+l-2i-1}, \tag{4.1}$$

$$W_{j+k+l-i}U_{i+1} - W_{j+i+1}U_{k+l-i} = (-1)^i W_j U_{k+l-2i-1}, \tag{4.2}$$

$$\Delta U_{j-i}W_{k+l+i+1} - V_{k+i+1}X_{j+l-i} = (-1)^{i+j+1} X_l V_{k+2i+1-j}, \tag{4.3}$$

$$U_{i+1}X_{j+k+l-i} - W_{j+i+1}V_{k+l-i} = (-1)^i W_j V_{k+l-2i-1}, \tag{4.4}$$

$$W_{k+l+i+1}V_{j-i} - W_{j+l-i}V_{k+i+1} = (-1)^{i+j} X_l U_{k+2i+1-j}, \tag{4.5}$$

$$W_{j+k+l-i}U_{i+1} - W_{j+i+1}U_{k+l-i} = (-1)^i W_j U_{k+l-2i-1}. \tag{4.6}$$

Now, if we substitute the left side of (4.1) into (3.5) and replace k by j and l by k , we obtain

$$U_j X_k = \sum_{i=0}^{j-1} (-1)^i X_{j+k-2i-1}. \tag{4.7}$$

In the same manner, we use (4.2)-(4.6) to simplify (3.6)-(3.10), which become, respectively,

$$U_j U_k = \sum_{i=0}^{k-1} (-1)^i U_{j+k-2i-1}, \tag{4.8}$$

$$U_j V_k = \sum_{i=0}^{j-1} (-1)^{i+j+1} V_{k+2i+1-j}, \tag{4.9}$$

$$U_j V_k = \sum_{i=0}^{j-1} (-1)^i V_{j+k-2i-1}, \tag{4.10}$$

$$V_j V_k = \left(\Delta \sum_{i=0}^{j-2} (-1)^{i+j} U_{k+2i+1-j} \right) + p V_{j+k-1}, \tag{4.11}$$

$$V_j V_k = \left(\Delta \sum_{i=0}^{j-2} (-1)^i U_{j+k-2i-1} \right) + (-1)^{j+1} p V_{k+1-j}. \tag{4.12}$$

By noting that $\sum_{i=0}^n f(i) = \sum_{i=0}^n f(n-i)$, we see that the right sides of (4.9) and (4.10) are identical. However, the right sides of (4.11) and (4.12) are different expressions which reduce to $V_j V_k$.

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