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ORIGINAL RESEARCH PAPER

Optimal annuitisation, housing and reverse mortgage in retirement in the presence of a means-tested public pension

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Abstract

In this paper we develop a model to find optimal decisions in retirement with respect to the consumption, risky asset allocation, access to annuities, reverse mortgage and the option to scale housing in the presence of a means-tested public pension. To solve the corresponding high-dimensional optimal stochastic control problem, we use the Least-Squares Monte Carlo simulation method. The model is applied in the context of the Australian retirement system. Few retirees in Australia utilise financial products in retirement, such as annuities or reverse mortgages. Since the government-provided means-tested Age Pension in Australia is an indirect annuity stream which is typically higher than the consumption floor, it can be argued that this could be the reason why many Australians do not annuitise. In addition, in Australia where assets allocated to the family home are not included in the means tests of Age Pension, the incentive to over-allocate wealth into housing assets is high. This raises the question whether a retiree is really better off over-allocating into the family home, while accessing home equity later on either via downsizing housing or by taking out a reverse mortgage. Our findings confirm that means-tested pension crowds out voluntary annuitisation in retirement, and that annuitisation is optimal sooner rather than later once retired. We find that it is never optimal to downscale housing when the means-tested pension and a reverse mortgage are available; only when there is no other way to access equity then downsizing is the only option.

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1 Introduction

Modelling the retirement phase using life cycle models is a complex task in many aspects. Retirees have many different options for managing and spending their life savings. Most life cycle models offer very limited choices, mainly due to the difficulties and computational limitations of solving such models. While there is a plethora of research on life cycle models in retirement, the majority of them only allow very few control, state or stochastic variables, thus limiting the practical applicability of their models.

In this paper we develop a retirement phase model based on the basic expected utility model proposed in Andréasson et al. [4], Andréasson and Shevchenko [2] and Ding et al. [22] for optimal consumption, housing and investment in a presence of means-tested public pension. Here, we extend the basic model with a stochastic interest rate, availability of an investment account with taxable earnings in addition to a tax-free pension account, and control variables for lifetime annuities, reverse mortgages and the option to scale housing. This allows to investigate optimal decisions in retirement with respect to the annuitization, reverse mortgage and house scaling in a presence of means-tested public pension. The model can be adapted to the retirement phase of people's lives in various countries but this requires a good knowledge of country specific retirement systems. Many countries offering public pension incorporate some kind of means test, e.g. 'Supplemental Security Income' in United States or 'Social Solidarity Benefit' in Greece.

The basic model developed in Andréasson et al. [4] was calibrated to the empirical data of consumption and housing in Australia and we apply the extended model in the context of the Australian retirement system too. We use the Australian pension system rules from 2017 as in the basic model analysed in Andréasson et al. [4] and Andréasson and Shevchenko [2] for consistency and comparison purposes. It is important to note that the pension system rules are reviewed regularly, e.g. the currently applied minimum regulatory withdrawal rates from pension accounts are different from the 2017 rates but will change back to the 2017 rates from July 2023. Thus, our numerical results cannot be used for direct analysis of the current pension rules in Australia. However, only the values of some rates and thresholds in pension rules are different from the 2017 rules while the rules are structurally the same. The model developed in this paper can be easily adjusted for new pension rules if needed.

The extended model is not feasible to solve numerically using deterministic methods due to too many state and control variables. To solve the corresponding high-dimensional optimal stochastic control problem, we use the Least-Squares Monte Carlo (LSMC) simulation method by utilising the control randomisation method introduced in Kharroubi et al. [31] to handle state variables affected by controls (endogenous variables) with additional improvements from Andréasson and Shevchenko [3] for the life cycle utility based models. The method is based on simulation of all state and control variables and does not require a deterministic grid for endogenous state variables as in the LSMC methods developed in Brandt et al. [11] and Koijen et al. [33] for life cycle portfolio choice problems; for more discussion see "Appendix A".

The government-provided Age Pension in Australia is means-tested to provide support for retirees with low wealth and/or low income. The Age Pension is an indirect annuity stream higher than the consumption floor of 'average' retiree[1] and thus should have a strong impact on annuitisation. Also, the assets allocated to the family home are not included in the means tests of Age Pension and the incentive to over-allocate wealth into housing assets is high. This raises the question whether a retiree is really better off over-allocating into the family home, while accessing home equity later on either via downsizing housing or by taking out a reverse mortgage. This motivates us to study the optimal annuitisation and housing decisions in a presence of means-tested pension.

The traditional fact in retirement modelling is that a risk averse retiree tends to be better off by annuitising part of his/her wealth [18, 38, 48]. As the means-tested Age Pension provides an income stream typically exceeding the consumption floor, the Age Pension becomes a possible substitute for a voluntary annuitisation. We therefore examine the optimal level of wealth allocation into a lifetime annuity, which in turn relates to the means tests. A lifetime annuity is a financial product that pays a guaranteed income and insures against outliving one's savings (longevity risk). By purchasing an annuity the retiree gives up wealth that could potentially earn a higher return and which could be used as bequest. Even after the mortality credit,[2] the payout rate is generally low but insures the retirees from outliving their savings. Risk averse agents,[3] however, discount the risk premium more and value a protected income over potentially higher future consumption, thus annuitising more wealth [29]. There are alternative annuities that address the negative aspects of a lifetime annuity, such as variable annuities with guaranteed minimum withdrawal and guaranteed minimum death benefits, which allow for equity growth and bequest motives respectively; see e.g. seminal works [34, 47] or more recent review in Shevchenko and Luo [43]. These products tend to be more expensive due to the additional benefits.

The retiree needs to find a balance between a guaranteed consumption and the possibility to leave a bequest. Yaari [48] showed that if no bequest motive is present, then a full annuitisation is optimal. If such bequest motive exists, however, annuitisation is still optimal but typically only partial [18, 24], which is also the case when a certain consumption floor is present. Lockwood [35] and Arandjelović et al. [5] also showed that bequest motives play a central role in limiting the demand for annuities and can eliminate annuitisation completely in the presence of annuity price loadings. Hubener et al. [26] studied a lifecycle consumption model with bequest motive and optimal portfolio choice for stocks, bonds, life insurance and annuities. Very few Australians annuitise any wealth [29, 32], which is consistent with retirees globally who receive other stable income streams [23, 28, 32]. The exception is Switzerland, where the

[1] For some households with, e.g. health complications, the Age Pension may not be larger than the consumption floor.

[2] Mortality credit refers to the discounting of future income streams based on survival probabilities. The value of the future income stream is weighted by the probability of being alive to receive this future income.

[3] This is true for rational investors only. Irrational investors, however, may value their current level of consumption too much and therefore defer annuitisation [37].

majority of retirees do annuitise [9, 10]. Results of our modelling confirm that means-tested pension crowds out voluntary annuitisation in retirement, and that annuitisation is optimal sooner rather than later once retired.

In the life cycle modelling, it is critical to include investment in housing because owner-occupied house is typically the most expensive asset for many households. Important research in this direction includes studies on the investment portfolio choice in the presence of housing [16, 49]. In our paper we study the optimal housing decisions in the presence of means-tested public pension. An important aspect of the means tests is the lenient treatment of the family home. Most Australian households do not convert housing assets into liquid assets in order to cover expenses in retirement, with the exception of certain events such as the death of a spouse, divorce, or moving to an aged care facility [6, 41]. However, by allocating more assets to the family home, the means-tested assets can be lowered which in turn results in more Age Pension received, and home equity can be accessed later in retirement if needed. As with annuities, this raises the question whether retirees should access home equity, either by selling the home or through home equity products, or if the means tests crowd out such products as well. Sun et al. [45] find that the reverse mortgage is a very risky asset, owing to the uncertainty of interest rates and housing markets.

However, the decision to access home equity cannot be made purely for financial reasons and needs to be set in the context of typical Australian retirement behaviour. Due to both financial benefits and attachment to their home, and especially neighbourhood, retirees tend to stay homeowners late in life [41]. The possibility to borrow money decreases with age, mainly due to having no labour income, and the retiree becomes increasingly locked into their home equity [40].

An increasingly popular solution is therefore a reverse mortgage, which allows the retiree to borrow against home equity, up to a certain loan-to-value ratio (LVR) threshold. The LVR threshold tends to increase with age. The initial principal limit generally starts with 20–25% at age 65 (subject to expected interest rate and property value), which translates to either the lump sum or the present value of future payments, and increases 1% per year. The house equity is used as collateral and allows the retiree to access housing equity while maintaining residence in the house. The retiree can typically choose between six repayment options: lump sum, line of credit (allowing flexible amounts and payment times), tenure (equal monthly payments), term (tenure but with a fixed time horizon) and combinations of line of credit with either tenure or term [14]. The loan is charged with either fixed or variable interest, but instead of requiring amortisation or interest payments they accumulate (although the retiree is free to make repayments at any time to reduce debt). The main benefits of such an arrangement are that it limits the risk as the loan repayments are capped at the house value, and allows the retiree to access more equity with age (contrary to traditional loans). However, interest rates are higher due to lending margins and insurance.

Chiang and Tsai [15] find that the desire for reverse mortgages is negatively correlated with the costs (application costs and insurance/spread added to the interest rate) as well as the income for a retiree, and according to Nakajima [40] the loans are very expensive for retirees. In addition, if a lump sum is received and allocated to what is considered an asset in the means tests, such as a risky investment or simply a bank account, it will affect the Age Pension. On the other hand, if the funds are spent right away they will not have an impact on the Age Pension received.

Previous research found that the Age Pension crowds out decisions that otherwise are optimal [12, 29]. In our paper we evaluate whether such findings are consistent in a more realistic framework. Asher et al. [6] finds evidence that few households use financial products to access home equity, such as reverse mortgages. For these reasons, we investigate whether the retiree is better off based on two additional control variables: borrowing against housing assets with a reverse mortgage or up/downsizing the housing in retirement. Since the family home is exempt from the means tests, it might be optimal to over-allocate in housing and then draw it down by a reverse mortgage. We find that it is never optimal to downscale housing with the means-tested Age Pension when a reverse mortgage is available; only when there is no other way to access equity then downsizing is the only option.

The paper is structured as follows. First, the benchmark stochastic model is defined in Sect. 2 which is the foundation used in this paper. In Sect. 3, additional optimal controls with respect to annuitisation decisions and home equity access are modelled individually. The results of each extended model are evaluated in Sect. 4. Finally, the paper is concluded in Sect. 5.

2 Benchmark model

We utilise the basic model developed in Andréasson et al. [4], with the same utility functions and parameters, but extend the model in several important aspects. First, a stochastic interest rate is introduced, which is important due to long time horizon of the retirement phase. Second, an investment account is now available in addition to the pension account, which is important since the pension account does not allow for deposits in retirement. This investment account allows financial investments, interest rate investments, and yearly withdrawals and deposits with no restrictions on size as long as the account balance is non-negative. Later, in Sects. 3.1 and 3.2, the model is extended to cover annuitisation (extension 1) and decisions on scaling housing and reverse mortgage (extension 2). These extensions are applied separately due to numerical complexity caused by too many state and control variables. Solving the model incorporating both extensions at the same time will be a subject of future research.

2.1 Stochastic model

We assume that the objective of the retiree is to maximise the expected utility generated from consumption, housing and bequest. Consider the retiree starting off with a total wealth $\mathbb{W} > 0$ (total wealth includes current house value if the retiree is a homeowner) at the age of retirement $t = t_0$ years; hereafter t is the age of the retiree. It is assumed that a proportion ϱ of this total wealth can be allocated to purchase an owner-occupied house $H_{t_0} = \varrho\mathbb{W} \in \{0, [\widetilde{H}_L, \mathbb{W}]\}$, where $\widetilde{H}_L$ is a minimum house price. This means that if the retiree is already a homeowner just before t_0, then the house up/down sizing may be required. The remaining wealth $W_{t_0} = \mathbb{W}(1-\varrho)$ is placed in a pension account, which is a special type of account that does not have a tax on investment earnings and is subject to the regulatory minimum withdrawal rates (Table 3) that depend on the age of the retiree.

In addition to the pension account W_t, the retiree has access to an investment account $\widetilde{W}_t$ (assumed to be zero at $t = t_0$) which is an account that holds a liquid wealth separate from the pension account. The earnings of this investment account are taxable and the account balance is included in the Age Pension means tests. The purpose of such account is that the retiree will be able to save part of the Age Pension and/or drawdowns from the pension account when minimum withdrawals are larger than what is optimal to consume. Such account is also necessary later in Sects. 3.1 and 3.2 when model is extended to include annuitization, housing scaling and reverse mortgage because pension accounts do not allow funds to be added to them after retirement.

2.1.1 Wealth account evolution

At the start of each year $t = t_0, t_0+1, \ldots, T-1$, the retiree will receive a means-tested Age Pension P_t and will decide what amount of saved liquid wealth from the pension account W_t and investment account $\widetilde{W}_t$ will be used for consumption C_t. Here, T is the maximum age of the agent beyond which survival is deemed impossible. The Age Pension P_t is a function of W_t and $\widetilde{W}_t$, and depends on the household being a couple/single and homeowner/non-homeowner; it will be fully defined in Sect. 2.3. It is assumed that the investment account (whose earnings are taxable) is invested in the same way as the pension account and thus it is always optimal to deplete the investment account before drawing from the pension account. Thus, the retiree has to draw from the pension account up to the minimum regulatory withdrawal rate $\nu_t \in (0, 1)$ each period, and in case the consumption exceeds this amount the difference is taken from the investment account (and if the investment account balance is not sufficient then the difference is taken from the pension account). This leads to the following evolution for the pension and investment accounts:
If $C_t \leq \widetilde{W}_t + P_t + \nu_t W_t$, then

$$\begin{aligned} W_t^+ &= W_t(1-\nu_t), \\ \widetilde{W}_t^+ &= \widetilde{W}_t + P_t + \nu_t W_t - C_t, \end{aligned} \tag{1}$$

otherwise

$$\begin{aligned} W_t^+ &= W_t + \widetilde{W}_t + P_t - C_t, \\ \widetilde{W}_t^+ &= 0. \end{aligned} \tag{2}$$

Here, W_t^+ and $\widetilde{W}_t^+$ are the pension and investment account balances immediately after the consumption withdrawal. Note that in this model setup, the pension P_t is received just before consumption C_t, and W_t and $\widetilde{W}_t$ correspond to the wealth account balances just before receiving pension. Also, the consumption should satisfy the following constraint to ensure the pension account is non-negative

$$\widetilde{W}_t + W_t + P_t - C_t \geq 0. \tag{3}$$

The remaining liquid wealth after drawdown is invested in a risky asset with real[4] stochastic annual log-return Z_t and a cash asset growing at the real rate $\tilde{r}_{t,t+1}$. We assume that stochastic log-returns of the risky asset Z_t, $t = t_0, t_0 + 1, \ldots$ are independent and identically distributed random variables from a normal distribution $\mathcal{N}(\mu, \sigma_Z^2)$ with mean μ defined in real terms and variance σ_Z^2. Denote the proportion of the wealth invested in the risky asset as $\delta_t \in [0, 1]$, then the evolution of the wealth accounts over $(t, t+1)$ is given by

$$\begin{aligned} W_{t+1} &= W_t^+ \left(\delta_t e^{Z_{t+1}} + (1-\delta_t) e^{\tilde{r}_{t,t+1}}\right), \\ \widetilde{W}_{t+1} &= \widetilde{W}_t^+ (\delta_t e^{Z_{t+1}} + (1-\delta_t) e^{\tilde{r}_{t,t+1}}) \\ &\quad - \Theta\left(\widetilde{W}_t^+ (\delta_t e^{Z_{t+1}} + (1-\delta_t) e^{\tilde{r}_{t,t+1}}) - \widetilde{W}_t^+\right), \end{aligned} \tag{4}$$

where function $\Theta(x)$ calculates the tax on the investment account earnings.

2.1.2 Stochastic interest rate

The cash asset annual growing rate is

$$\tilde{r}_{t,t+1} = \int_t^{t+1} r_u du,$$

where the short rate r_t is assumed to follow the Vasicek model[5]

$$dr_t = b(\bar{r} - r_t)dt + \sigma_R dB(t), \tag{5}$$

with $b > 0$ is the speed of reversion to the mean, $\bar{r}$ is the mean level the process reverts to, $\sigma_R > 0$ is the volatility and $B(t)$ is the standard Brownian motion. In this case the distribution of r_{t+1} and $\tilde{r}_{t,t+1}$ conditional on r_t can be found in closed form; see e.g. Shevchenko and Luo [44]. It is bivariate normal with

$$\begin{aligned} &\mathbb{E}[r_{t+1}] = \bar{r} + e^{-b}(r_t - \bar{r}),\ \operatorname{var}[r_{t+1}] = \frac{\sigma_R^2}{2b}(1 - e^{-2b}), \\ &\mathbb{E}[\tilde{r}_{t,t+1}] = \frac{1}{b}(1 - e^{-b})(r_t - \bar{r}) + \bar{r}, \\ &\operatorname{var}[\tilde{r}_{t,t+1}] = \frac{\sigma_R^2}{2b^3}(2b - 3 + 4e^{-b} - e^{-2b}), \\ &\operatorname{cov}[r_{t+1}, \tilde{r}_{t,t+1}] = \frac{\sigma_R^2}{2b^2}(1 - 2e^{-b} + e^{-2b}). \end{aligned} \tag{6}$$

[4] By defining the model in real terms (adjusted for inflation), time-dependent variables do not have to include inflation, which otherwise would be an additional stochastic variable.

[5] A single-factor short rate model commonly used in economics introduced in Vasicek [46].

Note that the Vasicek process allows for negative interest rates, which is suitable as the rate is defined in real terms. A negative interest rate would then indicate that inflation is higher than the nominal risk-free rate. To simplify, we could also assume that the cash account grows at the annual deposit rate derived from 1-year bond prices, or approximate $\tilde{r}_{t,t+1}$ by r_t, but it does not lead to a material difference in the results.

2.1.3 Mortality model

We consider couple and single retiree households (the Age Pension treats couples as a single entity) where possible life states of the household are modelled by a family-status random variable G_t defined as follows: $G_t = -1$ corresponds to the household already deceased at time $t-1$, $G_t = 0$ corresponds to the household deceased during $(t-1, t]$, and $G_t = 1$ and $G_t = 2$ correspond to the household being alive at time t in a single or couple state respectively.

We consider the households that do not change a single state to a couple in the retirement. That is, if the household starts the retirement as a couple and one of the spouses passes away, the household is treated as a single for the remaining years. Also, if the household starts the retirement as a single then it is assumed to stay in a single state. Correspondingly, we assume that evolution in time of the family state variable G_t is fully specified by transition probabilities $q(g_{t+1}, g_t) := \Pr[G_{t+1} = g_{t+1} \mid G_t = g_t]$:

$$
\begin{aligned}
&q(2,2) = p_t^{\mathrm{C}}, \quad q(1,2) = 1 - p_t^{\mathrm{C}},\\
&q(1,1) = p_t^{\mathrm{S}}, \quad q(0,1) = 1 - p_t^{\mathrm{S}},\\
&q(-1,0) = q(-1,-1) = 1,
\end{aligned}
\tag{7}
$$

where p_t^{C} and p_t^{S} are the probabilities of surviving for one more year as a couple and single respectively that can be easily estimated from the official Life Tables as in Andréasson et al. [4]. All other transition probabilities $q(g_{t+1}, g_t)$ are 0.

2.2 Utility model

Denote the vector of state variables as $X_t = (W_t, \widetilde{W}_t, G_t, r_t)$ and the value of family home as H_t at time t. The agent receives utility reward each period $t = t_0, \ldots, T-1$:

$$
R_t(X_t, C_t) = \begin{cases} U_C(C_t, G_t, t) + U_H(H_t, G_t), & \text{if } G_t = 1, 2,\\ U_B(W_t + \widetilde{W}_t + H_t), & \text{if } G_t = 0,\\ 0, & \text{if } G_t = -1 \end{cases}
\tag{8}
$$

and the terminal reward at $t = T$:

$$
\widetilde{R}(X_T) = \begin{cases} U_B(W_T + \widetilde{W}_T + H_T), & \text{if } G_T \geq 0,\\ 0, & \text{if } G_T = -1. \end{cases}
\tag{9}
$$

Here, $U_C(\cdot)$, $U_H(\cdot)$, $U_B(\cdot)$ are consumption, housing and bequest utility functions respectively. That is, if the household is alive, it receives reward (utility) based on consumption U_C and housing U_H; if the household died during the year $(t-1, t]$, the reward comes from the bequest U_B; and if the household died before or at $t-1$, there is no reward. Note that the reward received when the household is alive depends on whether the household family state is a couple or single due to different utility parameters and Age Pension thresholds.

We use the same definition of consumption, bequest and housing utility functions as in Andréasson et al. [4], where parameterization and interpretation are discussed in detail.

- Consumption utility function:

$$U_C(C_t, G_t, t) = \frac{1}{\psi^{t-t_0}\gamma_d}\left(\frac{C_t - \bar{c}_d}{\zeta_d}\right)^{\gamma_d}, \quad d = \begin{cases} \text{C}, & \text{if } G_t = 2 \quad \text{(couple)}, \\ \text{S}, & \text{if } G_t = 1 \quad \text{(single)}, \end{cases} \tag{10}$$

  where $\gamma_d \in (-\infty, 0)$ denotes the risk aversion, $\bar{c}_d$ is the consumption floor, ζ_d is an equivalence scale parameter that normalises utility of couple and single households. These parameters are subject to family state G_t. Finally, $\psi \in [1, \infty)$ is the parameter to model the effect of reducing utility gain from consumption as the retiree ages (note that it is not applied to housing and bequest utilities). This parameter was introduced in Andréasson et al. [4], Andréasson and Shevchenko [2] and Ding [21, pp. 43–33] to get better fit of the model to the consumption and housing data in Australia.
- Bequest utility function:

$$U_B(W_t + \widetilde{W}_t + H_t) = \left(\frac{\theta}{1-\theta}\right)^{1-\gamma_\text{S}} \frac{\left(\frac{\theta}{1-\theta}a + W_t + \widetilde{W}_t + H_t\right)^{\gamma_\text{S}}}{\gamma_\text{S}}, \tag{11}$$

  where W_t denotes the liquid assets available for bequest, γ_S denotes the risk aversion parameters of a singles household, $\theta \in [0, 1)$ the utility parameter for bequest preferences over consumption, and $a \in \mathbb{R}^+$ the threshold for luxury bequest.
- Housing utility function:

$$U_H(H_t, G_t) = \begin{cases} \frac{1}{\gamma_\text{H}}\left(\frac{\lambda_d H_t}{\zeta_d}\right)^{\gamma_\text{H}}, & \text{if } H_t > 0, \\ 0, & \text{if } H_t = 0, \end{cases} \tag{12}$$

  where γ_H is the risk aversion parameter for housing (different from risk aversion for consumption and bequest), H_t is the value of the family home and $\lambda_d \in (0, 1]$ is the housing preference defined as a proportion of the market value. In the benchmark model, H_t is assumed to be constant (in real terms) for all t. Note that some researchers choose to work with Cobb-Douglas utility function combining housing and consumption in a multiplicative way that leads to spending on consumption and housing in a fixed proportion, e.g. Yao and Zhang [49] and

Cocco [16]. This is somewhat inconsistent with an increasing ratio of housing services over consumption by age observed in the data, e.g. Jeske [30]. Thus some researchers, e.g. Chambers et al. [13], prefer to work with additive model by adding utility of housing to the utility of consumption like in our benchmark model.

The retiree has to find the decisions that maximise the total expected utility with respect to the consumption, investment and housing. This is defined as a stochastic control problem, where decisions (controls) at time t depend on the realisation of stochastic state variable vector X_t at time t with unknown future realisations. Then, the overall problem of maximization of expected utility is defined as:

$$\max_{\varrho}\left[\sup_{\pi}\mathbb{E}\left[\beta_{t_0,T}\widetilde{R}(X_T)+\sum_{t=t_0}^{T-1}\beta_{t_0,t}R_t(X_t,C_t)\,\bigg|\,X_{t_0}\right]\right], \tag{13}$$

where $\mathbb{E}[\cdot]$ is the expectation with respect to the state vector X_t at $t=t_0+1,\ldots,T$, conditional on the state variables at time $t=t_0$ and the use of control policy $\pi=(\pi_t)_{t=t_0,\ldots,t_{T-1}}$, $\pi_t=(C_t,\delta_t)$. The subjective discount factor $\beta_{t,t'}$ is a proxy for personal impatience between time t and t'.

This problem can be solved numerically with dynamic programming using backward in time recursion of the Bellman equation

$$V_t(X_t)=\sup_{\pi_t\in\mathcal{A}_t}\left\{R_t(X_t,C_t)+\mathbb{E}\left[\beta_{t,t+1}V_{t+1}(X_{t+1})\mid X_t\right]\right\}, \tag{14}$$

for $t=T-1,\ldots,0$, starting from the terminal condition $V_T(X_T)=\widetilde{R}(X_T)$ and optimal value of control is found as

$$\pi_t^*(X_t)=\arg\sup_{\pi_t\in\mathcal{A}_t}\left\{R_t(X_t,C_t)+\mathbb{E}\left[\beta_{t,t+1}V_{t+1}(X_{t+1})\mid X_t\right]\right\}, \tag{15}$$

where $\mathcal{A}_t$ is a space of possible actions π_t that may depend on X_t. Then, optimal housing decision control ϱ maximising $V_0(X_0)$ is calculated. Note that the death probabilities are not explicit in the objective function, but affect the evolution of the family status and, thus, are involved in the calculation of the conditional expectation. Later in Sect. 3.2 we will also consider housing decisions over time.

2.3 Model and age pension parameters

The utility model parameters are taken from Andréasson and Shevchenko [2]; for details of calibration to the Australian empirical retirement data and S&P/ASX 200 market prices, see Andréasson et al. [4]. The Age pension rule parameters are taken from Andréasson and Shevchenko [2], i.e. date back to 2017. All utility model parameter values are shown in Table 1, the risky asset annual log-return Z_t is from normal distribution with $\mathbb{E}[Z_t]=0.056$ and $\mathrm{var}[Z_t]=0.018$, the terminal age $T=100$, the minimum house price $\widetilde{H}_L=\$30,000$, and the time impatience discounting factor

Table 1 The utility model parameters

	γ_d	γ_{H}	θ	a	$\bar{c}_d$	ψ	λ	ζ_d
Single household	−1.98	−1.87	0.96	$27,200	$13,284	1.18	0.044	1.0
Couples household	−1.78	−1.87	0.96	$27,200	$20,607	1.18	0.044	1.3

$\beta_{t,t+1}$ is set to $\beta = 0.995$ for all time steps.[6] In addition, for simplicity, we assume 15% tax on the investment account earnings, i.e. $\Theta(x) = 0.15\max(x, 0)$ in (4).

In Australia, retirees aged 65.5[7] are entitled to Age Pension and can receive at most the full Age Pension, which decreases as assets and/or income increase and is determined by the income and asset tests. In the income test, the income streams from the pension accounts[8] and financial assets are based on the deemed income, which refers to a progressive assumed return from financial assets without reference to the actual returns on the assets held. Therefore, the income test can depend on both labour income (if any), deemed income from financial investments not held in the pension account and deemed income on pension accounts. Two different deeming rates may apply based on the value of the account: a lower rate ς_- for assets under the deeming threshold κ_d and a higher rate ς_+ for assets exceeding the threshold.

The Age Pension received depends on the current liquid assets, where the combined investment and pension account values are used for the asset test. The Age Pension function can be defined as

$$P_t := f(W_t + \widetilde{W}_t) = \max\left[0, \min\left[P^d_{\max}, \min\left[P_{\mathrm{A}}, P_{\mathrm{I}}\right]\right]\right], \tag{16}$$

where $P^d_{\max}$ is the full Age Pension, P_{A} is the asset test and P_{I} is the income test functions.

The P_{A} function is defined as

$$P_{\mathrm{A}} := P^d_{\max} - (W_t + \widetilde{W}_t - L^{d,h}_{\mathrm{A}})\varpi^d_{\mathrm{A}}, \tag{17}$$

where $L^{d,h}_{\mathrm{A}}$ is the threshold for the asset test and ϖ^d_{A} is the taper rate for assets exceeding the thresholds. Superscript $d \in \{\mathrm{S}, \mathrm{C}\}$ is the categorical index indicating couple or single household status. The variables are subject to whether it is a single or couple household, and the threshold for the asset test is also subject to whether the household is a homeowner or not ($h \in \{0, 1\}$).

6 In the model, the subjective discount factor $\beta_{t,t+1}$ can be time dependent and even stochastic if defined via the stochastic cash rate $\tilde{r}_{t,t+1}$, e.g. some modellers use financial discount factor $\beta_{t,t+1} = \exp(-\tilde{r}_{t,t+1})$ or it can be estimated as in Andersen et al. [1]. For numerical experiments it is set to $\beta = 0.995$ as was used in Andréasson et al. [4] where it was observed that changes in the discount rate have no significant impact on results if model is re-calibrated.

7 This is the retirement age as of July 2017, which is increased by 6 months every 2 years up to 67 from 1 July 2023 (https://www.humanservices.gov.au/individuals/enablers/age-rules-age-pension).

8 This applies to pension accounts opened after 1 January 2015 (http://guides.dss.gov.au/guide-social-security-law/3/9/3/31). Older accounts may have different rules which are not considered in this paper.

Table 2 Age pension rates published by Centrelink in June 2017 (www.humanservices.gov.au/customer/services/centrelink/age-pension, accessed June 5, 2017)

Single			Couple
$P^d_{\max}$	Full age pension per annum	\$22,721	\$34,252
	Income-test		
L^d_{I}	Threshold	\$4,264	\$7,592
ϖ^d_{I}	Rate of reduction	\$0.5	\$0.5
	Asset-test		
$L^{d,h=1}_{\mathrm{I}}$	Threshold: homeowners	\$250,000	\$450,000
$L^{d,h=0}_{\mathrm{I}}$	Threshold: non-homeowners	\$375,000	\$575,000
ϖ^d_{A}	Rate of reduction	\$0.078	\$0.078
	Deeming income		
κ^d	Deeming threshold	\$49,200	\$81,600
ς_-	Deeming rate below κ^d	1.75%	1.75%
ς_+	Deeming rate above κ^d	3.25%	3.25%

Table 3 Minimum regulatory withdrawal rates for pension accounts (https://www.ato.gov.au/rates/key-superannuation-rates-and-thresholds/?page=10, accessed June 5, 2017)

Age	≤64	65–74	75–79	80–84	85–89	90–94	≤95
Min. drawdown	4%	5%	6%	7%	9%	11%	14%

The function for the income test is defined as

$$P_{\mathrm{I}} := P^d_{\max} - (P_{\mathrm{D}}(W_t + \widetilde{W}_t) - L^d_{\mathrm{I}})\varpi^d_{\mathrm{I}}, \tag{18}$$

$$P_{\mathrm{D}}(W_t + \widetilde{W}_t) = \varsigma_- \min\left[W_t + \widetilde{W}_t, \kappa^d\right] + \varsigma_+ \max\left[0, W_t + \widetilde{W}_t - \kappa^d\right], \tag{19}$$

where L^d_{I} is the threshold for the income test and ϖ^d_{I} the taper rate for income exceeding the threshold. Function $P_{\mathrm{D}}(W_t + \widetilde{W}_t)$ calculates the deemed income, where κ_d is the deeming threshold, and ς_- and ς_+ are the deeming rates that apply to assets below and above the deeming threshold, respectively. The parameters for the Age Pension policy are presented in Table 2.

The Age Pension parameters from July 2017 are shown in Table 2, while the minimum withdrawal rates ν_t for pension accounts in 2017 are shown in Table 3 (note that these rates were halved during 2019–2023 and will reset back to values in Table 3 from 1 July 2023). Mortality probabilities are based on unisex data, and taken from Life Tables published by Australian Bureau of Statistics [8].

3 Model extensions

The model is now extended to include: annuitisation (extension 1) and scaling housing/reverse mortgage (extension 2). Note that extension 1 does not apply in extension 2 and vice versa—they are separate and independent extensions to isolate the impact of each extension.

3.1 Extension 1—annuitisation

The argument why the Australian market has shown such a lack of interest in annuities comes down to the fact that the Age Pension is indirectly an indexed life annuity which pays a known and increasing amount as wealth and income decrease, hence crowding out annuitisation [12, 29]. The Age Pension provides an implicit insurance against both longevity and financial risk, which otherwise is the main argument to annuitise. If annuities were exempt from the Age Pension means tests, then it would be reasonable to expect an increased interest in annuities. However, the annuity value as well as the annuity payment are included in the means tests. If the retiree annuitises, then the Age Pension decreases if any of the means tests are binding.

3.1.1 Model

The retiree can each year decide to annuitise part of the wealth and have the remaining wealth to be available for annuitisation later; see e.g. Milevsky and Young [38]. This introduces the possibility for the retiree to receive additional equity growth on the wealth yet to be annuitised, although with the risk associated but without requiring more complex annuity products.

In the context of our model, we assume that a retiree can at any time $t \in \{t_0, \ldots, T-1\}$ make a (non-reversible) decision to purchase an annuity for amount A_t that will provide annual life time payments y_t (constant in real terms) starting from $t+1$. This leads to a new state variable Y_t, which holds the information of the size of annuity payments each period evolving as

$$Y_{t+1} = Y_t + y_t, \; Y_{t_0} = 0. \tag{20}$$

The evolution of the pension W_t and investment $\widetilde{W}_t$ accounts (1,2) is updated as follows.
If $C_t + A_t \leq \widetilde{W}_t + P_t + \nu_t W_t + Y_t$, then

$$\begin{aligned} W_t^+ &= W_t(1-\nu_t), \\ \widetilde{W}_t^+ &= \widetilde{W}_t + P_t + \nu_t W_t - C_t + Y_t - A_t, \end{aligned} \tag{21}$$

otherwise

$$\begin{aligned} W_t^+ &= W_t + \widetilde{W}_t + P_t - C_t + Y_t - A_t, \\ \widetilde{W}_t^+ &= 0, \end{aligned} \tag{22}$$

and the evolution of accounts over $(t, t+1)$ is the same as before, i.e. given by (4). To ensure that the pension account W_t is nonnegative, the possible actions C_t and A_t should satisfy the constraint:

$$W_t + \widetilde{W}_t + P_t + Y_t - C_t - A_t \geq 0 \tag{23}$$

in addition to $A_t \geq 0,\ C_t > \bar{c}_d$. Then the optimal stochastic control problem (13–15) should be solved with the state vector extended to $X_t = (W_t, \widetilde{W}_t, G_t, r_t, Y_t)$ to find optimal value of $\pi_t = (C_t, \delta_t, A_t)$ from (15) for $t = t_0, \ldots, T-1$. This will be accomplished numerically using the LSMC method described in "Appendix A" and results will be discussed in Sect. 4.1. The budget constraint (23) and conditions $A_t \geq 0,\ C_t > \bar{c}_d$ fully specify space $\mathcal{A}_t$ of possible values of control π_t.

If the annuity purchased by the retiree provides lifetime annual payments constant in real terms, then its actuarial present value can be written as

$$a_t(y) := \sum_{i=t+1}^{T} {}_i p_t^{1-h} J(t, i, y), \tag{24}$$

where $J(t, i, y)$ represents the price of an inflation linked zero coupon bond at time t with maturity i and face value y (the constant real annuity payment, i.e. adjusted for inflation), ${}_i p_t$ is the probability of surviving from year t to i, and $h \geq 0$ is an annuity commercial price loading factor ($h = 0$ corresponds to the fairly priced annuity). In practice, the annuity price loading tends to exceed the transaction costs of other financial instruments [39]. The price of this kind of annuity is the sum of the mortality risk weighted bonds with maturities from $t+1$ up to T, adjusted with the annuity loading factor, which is set $h = 0.15$ as in Huang and Milevsky [25]. Note that this means that y_t in (20) should be calculated from the annuity price formula (24) by solving $A_t = a_t(y_t)$.

At time t, the price of a bond with maturity t' is

$$J(t, t', y) = y\mathbb{E}^{\widetilde{Q}}[e^{-\int_t^{t'} r_\tau d\tau}] := y e^{-r(t,t')(t'-t)}, \tag{25}$$

where $\widetilde{Q}$ is the risk-neutral probability measure for pricing interest rate derivatives and $r(t, t')$ is the zero rate (yield) from t to t'. The corresponding Vasicek risk-neutral process is

$$dr_t = [b(\bar{r} - r_t) - \lambda\sigma_R]dt + \sigma_R d\tilde{B}(t), \tag{26}$$

where λ is the market price of risk and $\widetilde{B}_t$ is the standard Brownian motion under $\widetilde{Q}$. The formulas for the bond price and corresponding zero rate can easily be calculated (see, e.g., Hull [27])

$$r(t, t') = \frac{-\ln A(t, t') + B(t, t')r_t}{t' - t}, \tag{27}$$

where

$$A(t,t') = \exp\left[(B(t,t') - t' + t)\left(\bar{r} - \frac{\lambda\sigma_R}{b} - \frac{\sigma_R^2}{2b^2} \right) - \frac{\sigma_R^2}{4b} B(t,t')^2 \right], \quad (28)$$

$$B(t,t') = \frac{1}{b}\left(1 - e^{-b(t'-t)} \right). \quad (29)$$

Equation (27) gives the full term structure of zero rates of different maturities. This means that $a_t(y)$ depends on r_t.

3.1.2 Calibration of Vasicek model

In order to estimate parameters of the interest rate Vasicek model (both real and risk-neutral processes), we follow a simple two-stage procedure outlined in Hull [27]. First, the real r_t process is estimated using spot interest rate data and then the market price of risk λ is estimated using term structure of zero coupon bonds.

The Australian cash rate adjusted for inflation is chosen to represent a real risk-free rate r_t which the retiree has access to, where the dataset[9] contains rates for 1990–2017 in quarterly intervals. Then parameters of the Vasicek model are estimated using the maximum likelihood method applied to the discretized version of the Vasicek model (5)

$$\max_{b,\bar{r},\sigma} \sum_{i=1}^{n} \left(-\frac{1}{2} \ln\left(\frac{\pi\sigma_R^2}{b}\left(1 - e^{-2b\Delta_t} \right) \right) - \frac{(r_i - \bar{r} - e^{-b\Delta_t}(r_{i-1} - \bar{r}))^2}{\frac{\sigma_R^2}{2b}(1 - e^{-2b\Delta_t})} \right), \quad (30)$$

where r_i is the observed real cash rate at time t_i and $\Delta_t = t_i - t_{i-1}$, $i = 1, \ldots, n$. The parameter estimates can be found in closed form and for the considered dataset are $\hat{b} = 0.120$, $\hat{\bar{r}} = 0.021$ and $\hat{\sigma}_R = 0.012$. The current real risk-free rate is set to $r_0 = -0.003$ that corresponds to the last available datapoint in the considered dataset (it is negative because inflation was higher than the cash rate).

Then, the market price of risk λ can be estimated by minimising the sum of squared difference between the observed term structure of the zero coupon market rates[10] and model predicted zero rates (27) over the trading dates t_i, $i = 1, \ldots, n$ and maturities T_j, $j = 1, \ldots, J$:

$$\min_{\lambda} \sum_i \sum_j \left(r(t_i, t_i + T_j) - r_{i,j}^{obs} \right)^2, \quad (31)$$

[9] Taken from https://www.quandl.com/data/RBA/F13-International-Official-Interest-Rates and https://tradingeconomics.com/australia/inflation-cpi

[10] Taken from https://www.quandl.com/data/RBA/F17_0-Zero-Coupon-Interest-Rates-Analytical-Series-Yields

where $r_{i,j}^{obs}$ represents the observed yield at time t_i with maturity T_j. The estimate comes out as $\hat{\lambda} = -0.050$, hence the risk-neutral parameter for the mean rate is $\hat{\bar{r}} - \hat{\lambda}\hat{\sigma}_R/\hat{b} = 0.026$.

3.1.3 Treatment of annuity in the age pension means tests

Annuities are included in the Age Pension means tests. Annuities are assessed based on the income they provide with a deduction for part of the annuity value [20]. The definition of annuity income for the income test is

$$y_t - \frac{a_{t_x}(y_t)}{e_x - t_x}, \tag{32}$$

where t_x is the annuity purchasing time and e_x is the life expectancy at time t_x. The assessment value in the income test is therefore the annuity payments received each year, adjusted for an income test deduction determined at the time of purchase. In the asset test, the value of the annuity is assumed to be equal to the original purchase price of the annuity with a linear yearly value decrease until the life expectancy age is reached, i.e.

$$\max\left(a_{t_x}(y_t) - \frac{a_{t_x}(y_t)}{e_x - t_x}(t - t_x), 0\right). \tag{33}$$

These rules cause some implications to our model, as it will require additional state variables in terms of annuity purchase price and annuity purchasing time (which complicates the problem definition further as it is allowed to add on to annuities later in retirement). Even if a numerical solution using LSMC method technically could handle the additional states, it is preferred to avoid this as the additional state variables will have a very minor impact on the value function but are prone to unnecessary regression errors.

To avoid this, the calculations in Eqs. (32) and (33) are approximated. The annuity income deduction for the income test is approximated with a constant proportion $\Upsilon = 0.9$ of the annuity payments, which tends to match the deduction amount in the income test very well over time as illustrated in Fig. 1. The annuity value in the asset test is approximated using Eq. (24) to re-value the annuity in the actuarially correct way at the current time given the known annuity payments, thus the asset test annuity assessment approximately equals $a_t(Y_t)$. This approximation is correct at the time of purchase, but overestimates the asset test annuity value after that. However, the asset test tends to impose less penalty on the Age Pension received compared with the income test, and only binds for lower levels of wealth [2]. The overestimation of the annuity value in the asset test therefore has a very minor impact on the Age Pension received, and does not have a material effect on the optimal annuitisation.

The means test Age Pension functions (17,18) now need to be updated. The function for the income test becomes

$$P_{\mathrm{I}} := P_{\max}^d - \left(P_{\mathrm{D}}(W_t + \widetilde{W}_t) + Y_t(1-\Upsilon) - L_{\mathrm{I}}^d\right)\varpi_{\mathrm{I}}^d, \tag{34}$$

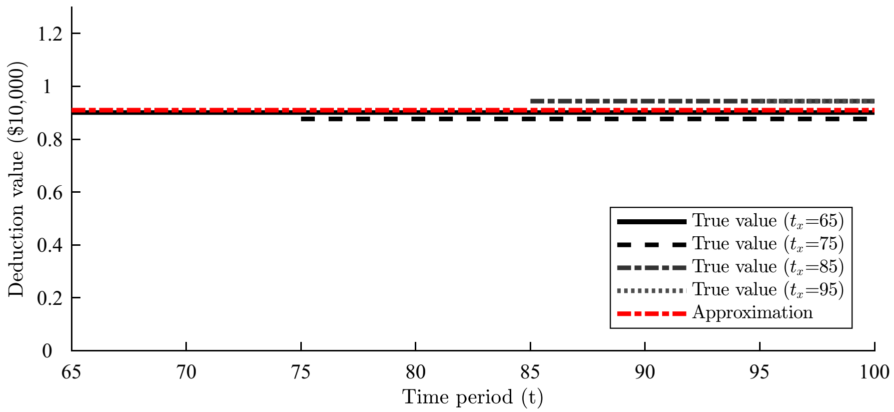

Fig. 1 The value of the annuity income deduction in the income test (for annuities purchased at different ages) compared to the approximation of the annuity income deduction by a constant proportion $\Upsilon = 0.9$ of annuity payments. The annual annuity payment is set to $10,000

and the function for the asset test is

$$P_{\mathrm{A}} := P^{d}_{\max} - \left(W_t + \widetilde{W}_t + a_t(Y_t) - L^{d,h}_{\mathrm{A}} \right) \varpi^{d}_{\mathrm{A}}. \tag{35}$$

3.2 Extension 2—scaling housing and reverse mortgages

The second main extension to the model allows the retiree to either scale the housing by selling the current home and acquiring a new one of a different size or standard. Although downsizing is more common in retirement, especially in the case of a spouse passing away [6, 41], in our model the retiree is allowed to both up- and downscale at any point in time by making a decision $\tau_t \in [-1, \infty)$ for $t = t_0 + 1, \ldots, T - 1$ so that the evolution of the house value state variable becomes

$$H_{t+1} = H_t(1 + \tau_t). \tag{36}$$

A positive value of τ_t represents the proportion of the current house value added to housing (upsizing from the current house) while negative corresponds to the house downscaling. The decision variable is therefore bounded below by the current house value, and the upper bound depends on wealth. Decision is made at the start of each period and any house scaling is assumed to be instantaneous (no delay between the decision, the sale of the house and buying a new on).

To capture the illiquid nature of housing assets, we will apply a proportional transaction cost. This will reflect the actual costs associated with a sale of the house, as well as avoiding the risk of the optimal decision being a gradual yearly change in the housing asset. The transaction cost only affects the sale of the house, as any transaction cost for a new purchase is assumed to be absorbed by the other party.

The retiree can also choose to take out a reverse mortgage on the house. The assumptions of the loan structure is based on Shao et al. [42], although not limited to a single payment at issuance. Define L_t to be the loan value at time t. The loan is based on a variable interest rate, where the outstanding loan amount accumulates over time. The retiree can at any time make the decision to loan a certain proportion $l_t \in [0, I(t)]$ of the house value, and it is possible to increase an existing loan at any time up to $\overline{L}_t$, which is given by

$$\overline{L}_t = H_t I(t), \tag{37}$$

where $I(t)$ is a maximum LVR ratio that changes with age, and is defined as

$$I(t) = 0.2 + 0.01(\min(85, t) - 65). \tag{38}$$

The maximum LVR therefore starts at 20% for age 65, which increases with 1% per year to a maximum of 40% at age 85.[11] The retiree is not liable to repay part of the loan if the maximum LVR is exceeded or if the loan value exceeds the house value due to accumulated interest(the so-called cross-over risk). If the retiree dies, or decides to sell the house, any remaining house value after loan repayments goes to the bequest wealth.

As Australian reverse mortgages include a 'no negative equity guarantee',[12] the retiree (or the beneficiaries) are not required to cover any remaining negative house asset if $L_t > H_t$ at time of death or if the house is sold.[13] From the lender's point of view, this results in two main risks: house price risk and longevity risk. If the house price decreases, or the retiree lives too long so that the loan value accumulates over the house value, the lender is liable for any losses unless these are forwarded to a third party via insurance. Increased interest rates can also speed up compounding of the loan, which increases cross-over risk. These risks are in practice covered with a mortgage insurance premium rate added to the loan, in addition to any lending margin required by the lender.

The loan value state variable therefore evolves as

$$L_{t+1} = (L_t \mathbb{I}_{\{\tau_t=0\}} + l_t H_t (1 + \tau_t))e^{\widetilde{r}_{t,t+1}+\varphi}, \tag{39}$$

where $\mathbb{I}_{\{\cdot\}}$ is the indicator symbol equals 1 if condition in brackets $\{\cdot\}$ is true and zero otherwise (i.e. the first term in the above equation is nonzero if no changes to the house asset are made), and φ represents the lending margin and mortgage insurance premium combined. In the case $\tau_t \neq 0$, any outstanding loan value must be repaid, hence the loan is reset and a new loan can be taken out subject to the new house value.

[11] The parameterisation follows 'Equity Unlock Loan for Seniors' offered by the Commonwealth Bank of Australia in 2017, but does not impose a minimum or maximum dollar value for the loans.

[12] The guarantee is still subject to default clauses which can negate the guarantee, such as not maintaining the property, malicious damage to the property by the owner, failure to pay council rates and failure to inform the provider that another person is living in the house.

[13] Even if the possibility exists, it will not be optimal to sell the house if the net house asset is negative as the retiree will give up 'free' housing utility and receive no extra wealth. The exception is a significant upsizing at old age, which is not very likely.

The costs of any decision (transaction cost, the difference in house assets in case of scaling and repayment of loan) is reflected in the wealth process. Define

$$b(l_t, \tau_t, L_t, H_t) := l_t H_t(1+\tau_t) - \mathbb{I}_{\tau_t \neq 0}\left(H_t(\tau_t + \eta) + L_t\right)$$

to represent all changes to the wealth from house scaling and reverse mortgage decisions, where η is the proportional transaction cost. The evolution of the pension W_t and investment $\widetilde{W}_t$ accounts (1,2) is updated as follows.
If $C_t \leq \widetilde{W}_t + P_t + \nu_t W_t + b(l_t, \tau_t, L_t, H_t)$, then

$$\begin{aligned} W_t^+ &= W_t(1-\nu_t), \\ \widetilde{W}_t^+ &= \widetilde{W}_t + P_t + \nu_t W_t + b(l_t, \tau_t, L_t, H_t) - C_t, \end{aligned} \tag{40}$$

otherwise

$$\begin{aligned} W_t^+ &= W_t + \widetilde{W}_t + P_t + b(l_t, \tau_t, L_t, H_t) - C_t, \\ \widetilde{W}_t^+ &= 0. \end{aligned} \tag{41}$$

In addition, the bequest function needs to include the house asset after any reverse mortgage has been repaid, and becomes $U_B(W_t + \widetilde{W}_t, \max(H_t - L_t, 0))$. Then the optimisation problem (13–15) should be solved with the state vector extended to $X_t = (W_t, \widetilde{W}_t, G_t, r_t, H_t, L_t)$ to find optimal value of $\pi_t = (C_t, \delta_t, \tau_t, l_t)$ for $t = t_0, \ldots, T-1$. This will be accomplished numerically using the LSMC method described in "Appendix A" and results will be discussed in Sect. 4.2.

Some constraints need to be imposed on the control variables. The option to take out (or add to) a reverse mortgage is bounded from above by the difference of any outstanding mortgage and the LVR, hence

$$l_t \leq \max\left(0, \frac{\overline{L}_t - L_t \mathbb{I}_{\tau_t = 0}}{H_t(1+\tau_t)}\right). \tag{42}$$

Note that if the control variable τ_t for scaling housing is not 0, any outstanding reverse mortgage must be paid back in full and a new reverse mortgage is available against the new house value. The max-condition in the formula is to ensure that the upper bound does not fall below the lower bound to ensure a feasible solution.

For the scaling of housing, an upper bound for how much the house asset can be increased is determined by the available wealth after costs associated with selling the current house (and repaying any outstanding reverse mortgage) and allocating additional wealth to the new house

$$\tau_t \leq \frac{W_t + \widetilde{W}_t - \mathbb{I}_{\tau \neq 0}\left(\eta H_t + L_t\right)}{H_t}. \tag{43}$$

The lower bound is simply -1, because the retiree cannot downscale further than selling the house and not buying a new one, and the cost associated with the sale is reflected in the transitions of the state variables. Finally, the budget constraint should be satisfied

$$b(l_t, \tau_t, L_t, H_t) + W_t + \widetilde{W}_t + P_t - C_t \geq 0. \quad (44)$$

The constraint ensures that the wealth is enough to cover consumption and scaling housing/reverse mortgage costs. The conditions (44), (43), (42) with $\tau_t \geq -1, l_t \geq 0$, $C_t > \bar{c}_d$ fully specify space $\mathcal{A}_t$ of possible values of control π_t.

In our numerical calculations in the next section, the transaction cost of house selling is set to $\eta = 6\%$ as in Nakajima [40] and Shao et al. [42]. The markup to the interest rate is set according to Chen et al. [14], $\varphi = 0.0242$, but does not require a starting cost to access the loan. In addition, it is assumed there is no current debt on the house and it is not used as security for other liabilities, and that there are no monthly fees in addition to φ.

4 Results

In this section we present results for optimal decisions for each model extension. Extension 1 is focused on the optimal annuitisation over time in retirement and is considered for a single status household. Extension 2 is focused on the optimal changes to the house asset in retirement subject to the age and total wealth. The numerical solution for both extensions utilises the LSMC method with control randomisation presented in "Appendix A" and implemented in Matlab. On a modern computer (Intel i7, 16GB RAM) using 10,000 sample paths the calculation takes approximately few days, subject to the number of control variables and extension type.

4.1 Extension 1: annuitisation

The optimal annuitisation is expected to differ from previous research due to a number of reasons. In both Iskhakov et al. [29] and Bütler et al. [12], the retirement is modelled with a starting wealth that is assumed to be fully consumed and cannot be bequeathed. This means that the level of annuitisation identified given a certain wealth, age and parameters is optimal on a relative basis compared to alternative investment options. Since the model utilised in this paper was calibrated to the behaviour of Australian retirees [4], where wealth appears in the bequest function, the annuitisation rate is expected to be lower. Similarly, as consumption declines with age,[14] any desired consumption above the consumption floor which can be covered with annuitising early in retirement is not as desirable at older age. In many cases this excess consumption is fully covered by the Age Pension payments. In addition to this, Iskhakov et al. [29] do not allow for a risk-free asset, hence the annuity is the only (non-reversible) option to access risk-free investments. As the extension model allows the retiree to choose a risk-free asset allocation, this option can decrease the annuitisation further.

[14] Optimal consumption calculated using our model is decreasing with age matching the empirical data; for details see Andréasson et al. [4] and references therein.

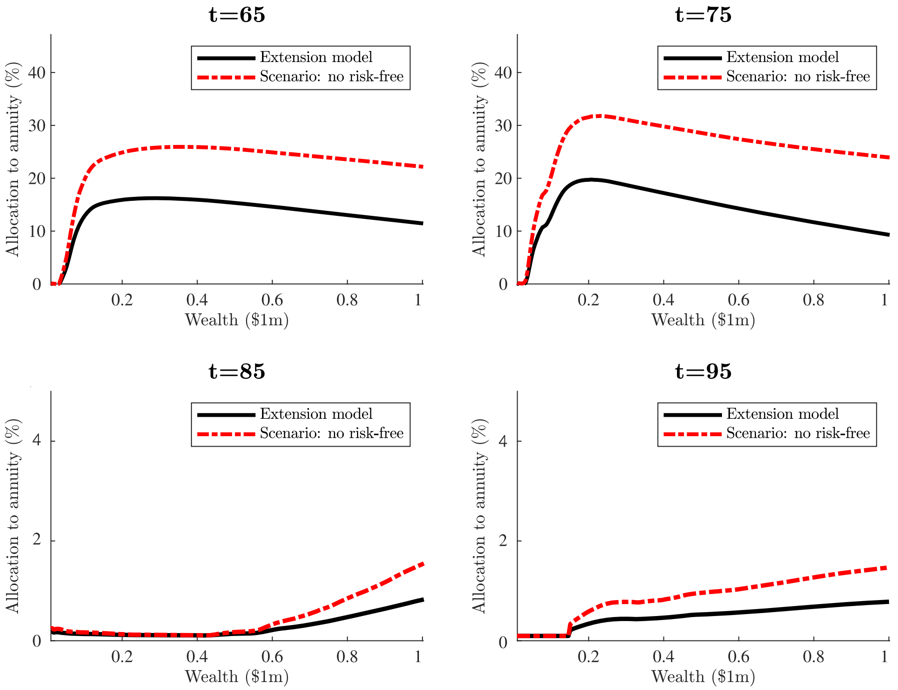

Fig. 2 Optimal annuitisation at age t years versus liquid wealth $W_t + \widetilde{W}_t$ assuming no prior annuitisation. The risky asset and interest rate are assumed to follow the expected paths. In addition to the default case ("extension model") where the risk-free asset is available, the scenarios when no risk-free asset is available are presented

Figure 2 presents the results for the optimal annuitisation at different ages for the cases when risk-free investment is available (default case) and is not available. The latter corresponds to setting $\delta_t = 1$ for $t = t_0, \ldots, T-1$. Each scenario assumes no prior annuitisation. The case where no risk-free asset is available is significantly higher than the default case (at least for early retirement age). Despite the presence of mortality credit in an annuity, the annuity price loading removes at least part of the incentive to annuitise at a higher rate. If the interest rates happen to be higher than normal, then allocation to annuities is slightly higher, even if the interest rate is expected to revert back to normal levels. The annuitisation level peaks around age 75 and quickly decreases with age, and is close to constant for higher levels of wealth. Already at age 85 the level of annuitisation is virtually non-existing and stays there. For low wealth levels, where full Age Pension is received, the optimal allocation quickly goes towards zero. A retiree with \$500,000 in liquid wealth at retirement optimally allocates 15% to annuities, which results in approximately \$4,800 in annual annuity payments. If the decision is deferred to age 75, the optimal annuitisation is approximately the same for the same wealth, but the resulting annual payments are higher at \$7,900. Although an Australian retiree has a lower desire for consumption at an older age, the mortality credit at this age is significant and the retiree can access

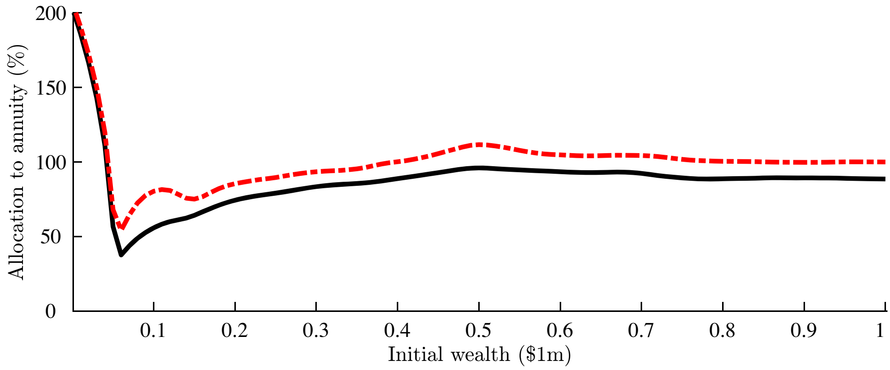

Fig. 3 Optimal total allocation to annuities over the life time in retirement relative to initial liquid wealth

a large boost in yearly consumption for a relatively small wealth sacrifice, resulting in higher overall utility.

To set this in relation to the previous research, the results from Iskhakov et al. [29] suggest on average a higher level of annuitisation, where the range of the authors' different risk preference and return parameters cover the ones calibrated in Andréasson et al. [4] and used in this paper. The suggested allocation in Iskhakov et al. [29] is expected to be higher, owing to the constraint that all wealth is to be consumed. That aside, the result confirms the general findings in Iskhakov et al. [29] and Bütler et al. [12]—annuitisation is crowded out by the Age Pension and annuitisation increases with wealth. Both papers find evidence that the means tests impact annuitisation, especially when binding. This can be seen as the decreasing annuitisation rate around $200,000 for early years in retirement in Fig. 2, which represents the transition from full to partial Age Pension. By annuitising at this (or lower) wealth level, no more Age Pension can be received by decreasing assets held, but the annuity payments lead to less Age Pension due to the income test. When a partial pension is received, however, any annuity payments are only partly assessed in the income test, hence annuitisation is still high until full Age Pension is received. The means-tested Age Pension thus effectively crowds out annuitisation at lower wealth, but not for wealthier households. There are no indications of high sensitivity to means-tested thresholds, other than decreasing annuitisation rate when the means tests bind.

Contrary to Iskhakov et al. [29] and Bütler et al. [12], but similar to Milevsky and Young [38], in our model the retiree is allowed to purchase annuities at any time, rather than only at time of retirement ($t = 65$). Figure 3 shows the total annuity allocation for a given initial liquid wealth during the retirement. In order to calculate this, it is assumed the retiree follows the optimal decision rules and that the wealth grows with the expected return (the risky asset and interest rate follow the expected paths). This gives a very different perspective of optimal annuitisation than what is seen in Fig. 2. Households with lower wealth now have a significant proportion of annuities. This is due to the effect of quickly decreasing consumption with age, hence Age Pension payments accumulate and wealth increases, which is then partly annuitised. It is sub-

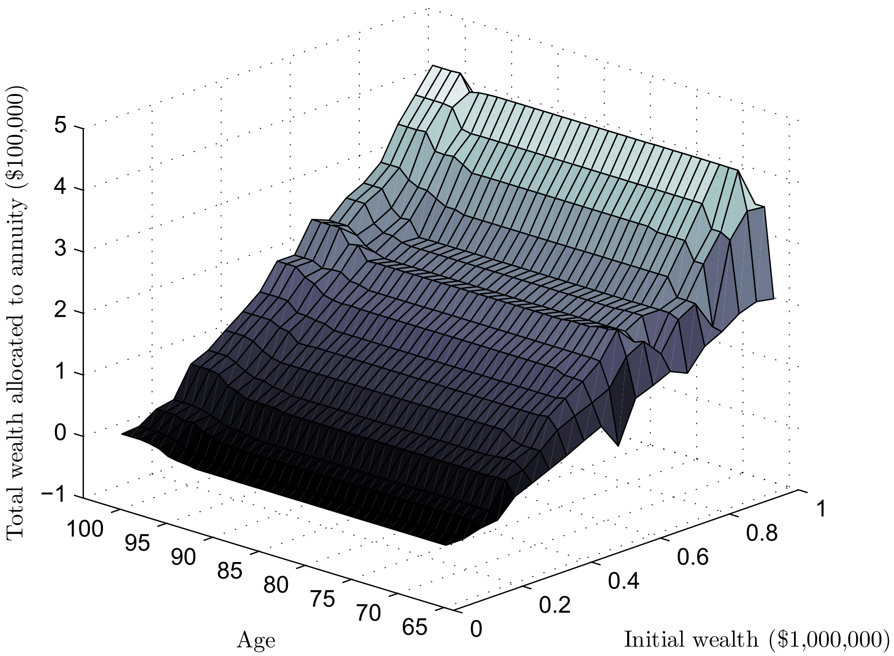

Fig. 4 Optimal allocation to annuities over time in retirement given initial liquid wealth $W_{t_0} + \widetilde{W}_{t_0}$, assuming no previous annuitisation at t_0

optimal for poor households to annuitise, but if their wealth grows it is optimal to annuitise at a later stage of retirement.

The calculations of total annuitisation in retirement can also be used to evaluate when in retirement annuitisation is optimal. The longer the retiree waits to annuitise, the larger the mortality credit will be in relation to price (due to the higher death probability), but on the other hand the desired excess consumption decreases towards the consumption floor. By deferring the choice to annuitise, the assets can instead be used to generate investment returns.

Figure 4 shows the cumulative wealth allocated to annuities with age. The majority of total annuitisation happens during the first year in retirement, and then increases slightly between ages 70 to 85. This supports the findings in Milevsky and Young [38] who showed that it is optimal to have immediate partial annuitisation, which also increases with wealth. Early annuitisation indicates that it is not optimal to delay annuities in order to get increased risky exposure. Iskhakov et al. [29] found that deferred annuities are more attractive to less wealthy retirees owing to the cheaper price. The extension model does not obtain the same result, due to the lack of additional mortality credit for immediate annuities compared to the deferred annuities which are purchased before the annuity payments start.

It should be noted that since wealthier households tend to live longer than less wealthy [19], the annuitisation is potentially underestimated for the wealthier households and overestimated for the less wealthy household. As the model does not include medical expenses at older age, nor aged care expenses, it can be argued that additional annuitisation is optimal when these costs are included. At the same time, since entering aged care (i.e. a retirement village) attracts rather large one-time costs, this can decrease the optimal level of annuitisation. The finding that annuitisation is optimal only early in retirement might also change in this case.

4.2 Extension 2: scaling housing and reverse mortgage

The purpose of this extension of the model is to evaluate whether scaling housing or accessing home equity is optimal in retirement. In order to test this, it is important that the retiree starts with the optimal house asset at the time of retirement. If not, then the solution might suggest scaling housing just to meet the initial optimal ratio of house assets to liquid wealth. This does not reflect whether it is optimal to scale housing in retirement, only that it is optimal with a certain level of housing assets in relation to wealth once retired. The retiree therefore starts with the optimal house asset at retirement for a given liquid wealth, and the wealth paths and optimal controls are then simulated until terminal time T.

Figure 5 shows the wealth, housing and reverse mortgage paths throughout retirement based on optimal decisions and assuming expected return on risky assets. Three different levels of total initial wealth at retirement are considered: \$0.5m, \$1 m and \$2 m where it is optimal to allocate approximately 84%, 80% and 77.5% respectively into housing for a single status household.[15] This corresponds to realistic house prices; e.g. the mean dwelling price in Australia is \$728,500 in December quarter 2020 as reported by Australian Bureau of Statistics.[16] As can be seen, it is not optimal to downscale housing in any of the cases, while all of them take advantage of the reverse mortgage to keep liquid wealth at a constant or higher level. The loan value is added on during retirement when required, but it also grows due to the interest accumulated.

The optimal reverse mortgage as a proportion of the house value decreases with wealth, and increases with the house value. Irrespective of house value, the loan proportion starts at the same value for households with no wealth. One might expect that the proportion would be less for a higher house value, as this would still access more wealth for the retiree, but this is not the case. However, the higher the house value, the more liquid wealth the retiree can have and still optimally takes out a reverse mortgage. This confirms the results in Chiang and Tsai [15], who found that as age increases, and the higher the initial wealth and house price are, the more the retiree is willing to use reverse mortgages.

Figure 6 shows the optimal loan proportion for different house values in relation to liquid wealth for single households, where the proportion in relation to wealth has a very linear relationship. A less wealthy household, which might need the wealth more than a wealthier household, generally should not take out a reverse mortgage unless the

15 These ratios of wealth allocation into housing are larger than approximately 50–60% ratio typically mentioned in the literature. It is not clear if this is due limitations of the dataset used in Andréasson et al. [4] to calibrate the base model or because retirees are not behaving optimally.

16 www.abs.gov.au/statistics/economy/price-indexes-and-inflation/residential-property-price-indexes-eight-capital-cities/dec-2020

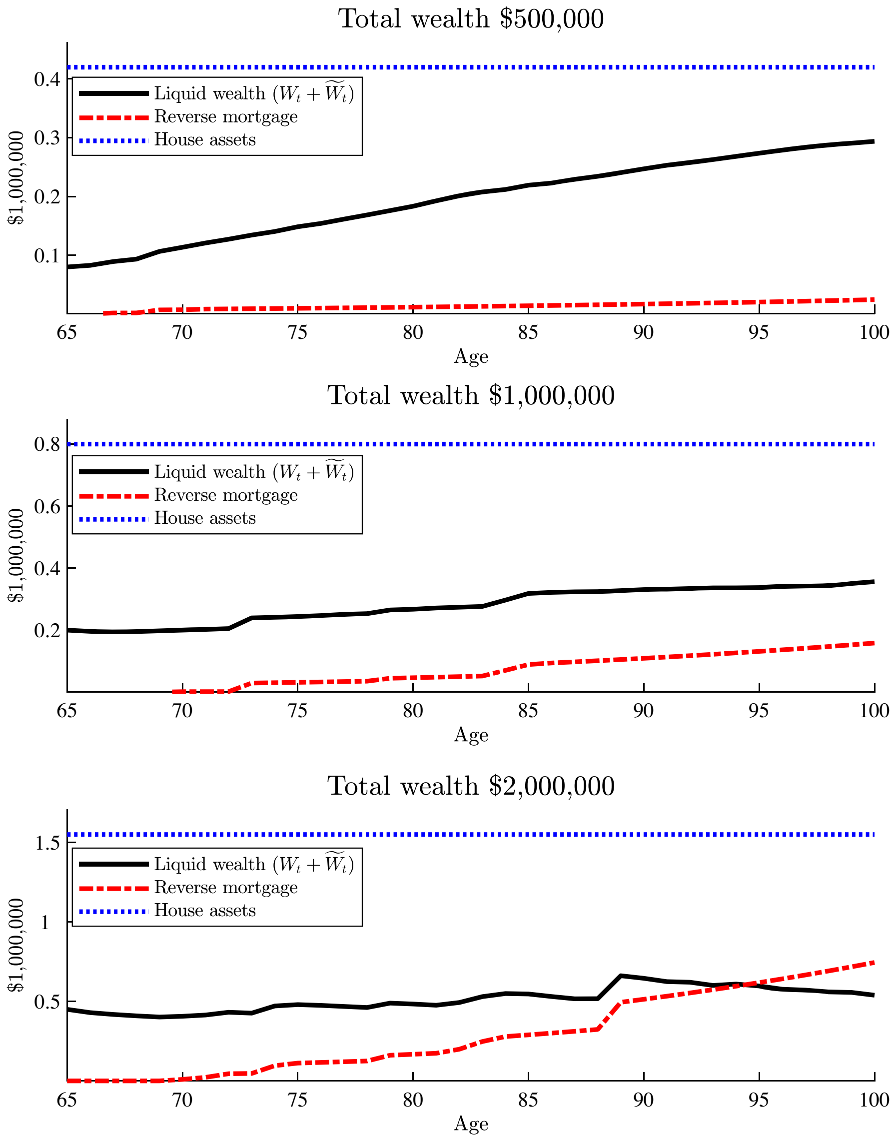

Fig. 5 Wealth, house and reverse mortgage paths in retirement given low, medium and high initial total wealth

house value is substantially higher than the liquid wealth. Each line in Fig. 6 reaches zero before it equals the optimal liquid wealth given the house value, hence a reverse mortgage is never optimal until wealth is drawn down enough to differ significantly from the house asset. The same relationship holds true for couple households, although at a slightly higher wealth level than for singles.

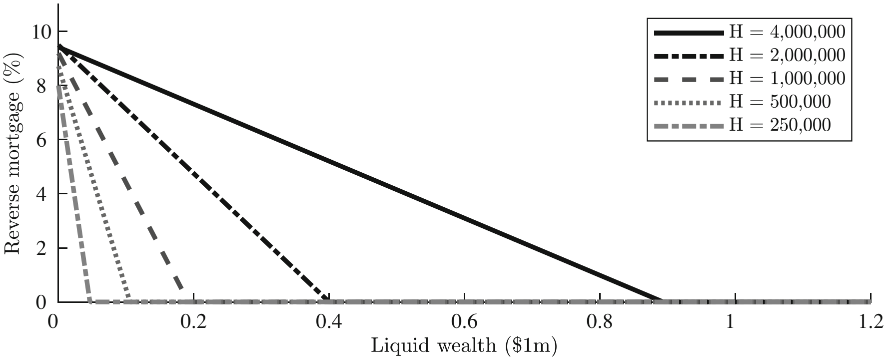

Fig. 6 Optimal proportion of reverse mortgage given housing wealth and liquid wealth at retirement age t_0 for a single household

When comparing the optimal loan proportion over time in retirement, the initial maximum level of approximately 10% increases yearly, but flattens out around year 80 and then remains constant at approximately 20%. The LVR threshold therefore never binds when a loan is created, given the calibrated parameters. It is reasonable to expect that if the risk aversion or preferences for bequest decreases, then the optimal loan value might increase. The optimal reverse mortgage in the solution is also an upper bound, as additional commercial loadings such as a fee to initiate the loan might apply in reality. However, a reverse mortgage could theoretically be refinanced if interest rates drop, thus any costs associated with the loan can be lowered that way.

If the retiree's housing asset is significantly less than optimal, then the solution will quickly suggest that the retiree should scale (or acquire) housing assets to get close to the optimal level. However, the opposite does not hold true. If the retiree starts with housing assets significantly larger than optimal, then it is not optimal to downscale, with the exception if the retiree has close to no wealth at all but significant wealth in the house asset. In general, it is therefore never optimal to downscale housing in retirement, not even when reverse mortgages are not available. Only in the case of an event which would incur a significant cost, such as a medical issue, it would be reasonable to downscale; this type of events is not considered in the model.

It is not optimal to upscale housing once retired either, with the exception of very low house assets ($\sim$\$100,000 or less) which only reflects the desire to get close to the initial optimal ratio rather than an actual upsizing decision.

The reason why downsizing housing is not optimal stems from a combination of the high cost associated with the sale of the house while housing is included in the bequest (hence wealth is given up by downsizing), and that the calibrated consumption floor is already covered by Age Pension payments. If the retiree wants to access just part of his/her house wealth, then downsizing the house will first incur a transaction cost on the full home value, even if the retiree only downscales slightly. To access 10% of the housing wealth, he/she needs to give up 6% of this equity in costs. It is therefore much more economical to take out a reverse mortgage. At the same time, housing utility is received based on the value of the house, even if there is an outstanding reverse mortgage. By utilising the reverse mortgage the retiree can therefore keep a high housing utility, while still accessing the equity. The retiree will give up bequeathable wealth as the loan value accumulates interest, but the funds received from the reverse mortgage can either be invested at a higher (although risky) return and the loss of utility in bequest is partly compensated with higher housing utility through retirement. The house and the pension act as fixed income but the retiree's risk preferences lead to a desire of a higher overall equity exposure; only the liquid wealth can be invested in equity and thus if there is not enough liquid wealth then retiree borrows money against the house in order to attain the desired equity exposure.

It should also be noted that unlike other jurisdictions, Australia does not tax the imputed rent of housing, further adding to the bias towards holding housing as an asset. A retiree can avoid having assets included in the means tests by over-allocating to housing assets, and therefore receive additional 'free' wealth from the Age Pension. As the liquid wealth is consumed, it can be replenished by taking out a reverse mortgage, while still accessing the Age Pension.

Even though the demand for the reverse mortgage products has increased in Australia since the 2008 global financial crisis (from $1.3 billion in 2008 to $2.5 billion in 2017), there are still negative perceptions against equity release products [7]. Here, we would like to note that the results reported in this section correspond to a very specific set of preference parameters representing an 'average' retiree, and market parameters such as transaction cost for selling house and lending margin; changing these parameters will have an impact on the results.

5 Conclusion

In this paper we developed a retirement phase model with the option to annuitise wealth, and option to scale housing and access the home equity with a reverse mortgage. It was applied in the context of the Australian retirement system and the following results were obtained.

In general, the optimal annuitisation in a realistic retirement model setup verifies previous research performed with more restricted models. The government provided means-tested Age Pension crowds out annuitisation, and allocation of wealth to a lifetime annuity is preferred over the risk-free asset. Even when a partial Age Pension is received it is optimal to have partial annuitisation, although the annuitisation decreases quickly when liquid wealth is close to the threshold of the full Age Pension. For wealthier households, the annuity payments are much higher than the partial Age Pension received, so even if 'free' wealth is given up the retiree is better off annuitising. The wealth allocation to annuities and allocation times during retirement are not known in advance and depend on the realisations of stochastic factors during time in retirement.

An annuity provides a significant discount in terms of mortality credits, where the additional utility is higher compared with the alternative to invest the funds in risky assets and annuitise at a later stage in retirement. As consumption decreases with age, this could make annuitisation less desirable, and the results indicate this to be true once the retiree passes age 85 even though the mortality credit is higher at older ages. It is optimal to annuitise sooner rather than later as it is cheaper to store wealth in an annuity rather than risk-free investments. In the Australian setting, it is not optimal to take a one-off decision to annuitise, but rather to gradually increase allocation in the first 10 years in retirement, and to annuitise additional wealth depending on the wealth evolution.

A retiree is in general better off utilising a reverse mortgage rather than downsizing the house, despite the accumulated interest of the loan. By keeping a house that is larger than optimal while drawing down the housing assets, the retiree still receives utility from living in the house, while it is still partly bequeathable. The additional utility from this outweighs the cost of an outstanding reverse mortgage. A reverse mortgage does therefore not necessarily benefit a retiree financially, unless the retiree can access additional Age Pension payments by 'hiding' assets in the family home, but it does help maximising the household utility throughout retirement. The optimal decisions are, however, subject to wealth levels and housing assets, where wealthier retirees with more housing assets optimally access a higher proportion reverse mortgage than less wealthier households.

In this paper, the extensions for annuitisation and housing decisions were applied separately due to numerical complexities. Solving the model incorporating both extensions at the same time will be a subject of future research. Then the marginal values of these extensions/financial instruments can be evaluated using certainty-equivalent consumptions as e.g. in Cocco et al. [17] or using the annual fee a household is willing to pay to have access to the financial instruments under these extensions as proposed in Koijen et al. [33]. Given the long term nature of the retirement phase, consideration of the regime switching model for stochastic interest rate instead of a simple Vasicek model is another avenue for future research.

The developed model can be easily adapted in the case of future changes to the Australia retirement system or extended to suit the retirement systems in other countries by adjusting the Age Pension function and necessary constraints. For example, in the case of the pension system in United States, the assumptions for the pension account need to be adjusted to match those of an 'Individual Retirement Account' or '401(k)' (defined-contribution retirement savings plan sponsored by the employer), and the Age Pension needs to be replaced with the 'Supplemental Security Income' and its associated means test function.

Appendix A Bias-corrected Least-Squares Monte Carlo

In stochastic control problems corresponding to the life cycle models there is always a choice to be made between the model complexity and the computational cost. Closed form solutions are limited to the problems with only few stochastic factors with restrictions on the dynamics and dimensions, otherwise one has to use the numerical methods. Dynamic programming solution via partial differential equation or direct integration methods suffers from the curse of dimensionality and simulation methods are required when the number of state variables and controls increases.

The Least-Squares Monte Carlo (LSMC) method received increasing interest among researchers due to its effectiveness in dealing with high dimensional problems, fewer restrictions on the constraints, and flexibility in the dynamics of the underlying stochastic processes. The idea is based on simulating the underlying stochastic variables over time and replacing the conditional expectation of the value function in the dynamic programming backward in time recursive solution of the stochastic control problem with an empirical least-squares regression estimate. The transition density of the underlying process is not even required to be known in closed form. The LSMC method was originally developed in Longstaff and Schwartz [36] for pricing American options with important extension (the so-called control randomisation method) to handle endogenous state variables (state variables affected by controls) developed in Kharroubi et al. [31]. When applied to the expected utility stochastic control problems, some further extensions are needed as proposed in Andréasson and Shevchenko [3] to achieve a stable and accurate solution.

Note that the LSMC methods developed in Brandt et al. [11] and Koijen et al. [33] for solving the life cycle portfolio choice problems require the deterministic grid for endogenous state variables after controls applied and thus are not practical for solving the model in our paper because there are three endogenous state variables for model extension 1 and four endogenous state variables for model extension 2.

The models considered in our paper involve six state variables and 3–4 control variables depending on extension. To solve these models numerically we implemented the LSMC algorithm with control randomisation adapted for expected utility optimal stochastic control problems described in Andréasson and Shevchenko [3]. The method is based on approximation of the conditional expectation of a value function in the Bellman equation using the ordinary least-squares regression applied on the transformed value function. In our implementation, the regression basis functions consist of the fourth order ordinary polynomials of the state and control variables. The exception is for model Extension 2, where the state variable covariate for the outstanding loan value L_t is replaced with the covariate $\max(0, H_t - L_t)$ as this is how it appears in the bequest function. To avoid the transformation bias in the regression estimate of the conditional expectation, the smearing estimate with controlled heteroskedasticity is used as proposed in Andréasson and Shevchenko [3]. The solution is run with 10,000 sample paths, and the optimal values of controls are estimated numerically at each time t with constraints corresponding to the space of possible values of controls $\mathcal{A}_t$. In particular, we used Matlab optimisation function `patternsearch` which is derivative free method that was checked to produce results consistent with a simple grid search method. This method is faster and more accurate than a grid search and does not require specification of the grid size. A brief description of the LSMC algorithm is provided below.

Let $t = 0, 1, \ldots, N$ correspond to equispaced points in the time interval $[0, T]$. Consider the standard discrete dynamic programming problem with the objective to maximise the expected value of the utility-based total reward function

$$V_0(x) = \sup_{\pi} \mathbb{E}\left[\beta^N R_N(X_N) + \sum_{t=0}^{N-1} \beta^t R_t(X_t, \pi_t) \;\middle|\; X_0 = x\right], \tag{45}$$

where $\pi = (\pi_t)_{t=0,\dots,N-1}$ is a control policy, $X = (X_t)_{t=0,\dots,N}$ is a controlled state variable, R_N and R_t are the reward functions and β is a time discount factor over a time step. Here, we assume that the evolution of the state variable is specified by a transition function $\mathcal{T}_t(\cdot)$ such that

$$X_{t+1} = \mathcal{T}_t\left(X_t, \pi_t, Z_{t+1}\right), \tag{46}$$

where $Z_1, \dots, Z_N$ are independent disturbance terms, i.e. the state of the next period depends on the state of the current period, the control decision and the realisation of the disturbance term. This type of problem can be solved with the backward recursion of the Bellman equation

$$\begin{aligned} V_t(x) &= \sup_{\pi_t \in \mathcal{A}_t} \left\{R_t(x, \pi_t) + \mathbb{E}\left[\beta V_{t+1}(X_{t+1}) \;\middle|\; X_t = x\right]\right\}, t = N-1, \dots, 0, \\ V_N(x) &= R_N(x) \end{aligned} \tag{47}$$

and optimal control is found as

$$\pi_t^*(x) = \arg \sup_{\pi_t \in \mathcal{A}_t} \left\{R_t(x, \pi_t) + \mathbb{E}\left[\beta V_{t+1}(X_{t+1}) \;\middle|\; X_t = x\right]\right\}.$$

Here, $\mathcal{A}_t$ denotes a space of possible values of π_t that may depend on x. It is not computationally feasible to use a quadrature based methods for evaluation of expectation in (47) when the number of state variables is more than three and simulation methods such as LSMC are favoured.

The idea behind utilitising the LSMC method is to approximate the conditional expectation in (47)

$$\Phi_t(X_t, \pi_t) = \mathbb{E}\left[\beta V_{t+1}(X_{t+1}) | X_t\right], \tag{48}$$

by a regression scheme with independent variables X_t and randomised π_t, and response variable $\beta V_{t+1}(X_{t+1})$. The approximation of the function is denoted as $\widehat{\Phi}_t$. The method is implemented in two stages. First, the random state, control and disturbance variables are simulated X_t^m, π_t^m, $m = 1, \dots, M$, $t = 0, \dots, T$ (forward in time simulation) using Algorithm 1, where π_t are sampled independent from X_t. Then, optimal stochastic control problem (45) is solved with the backward recursion (47) using Algorithm 2.

To avoid difficulties in the approximation of the value function due to the extreme curvature of utility functions, a transformation $H(x)$ that has a similar shape as the value function is required (in our implementation we use $H(x) = (e^x)^\gamma/\gamma, \gamma < 0$). At each time $t < T$, the value function is approximated using the ordinary least-squares regression

$$H^{-1}(\beta V_{t+1}(X^m_{t+1})) = \mathbf{\Lambda}'_t \mathbf{L}(X^m_t, \pi^m_t) + \epsilon^m_t, \quad m = 1, \ldots, M, \tag{49}$$

where ϵ^m_t, $m = 1, \ldots, M$ are zero mean and independent, $\mathbf{L}(X^m_t, \pi^m_t)$ is a vector of basis functions, Λ_t the regression coefficient vector and H^{-1} the inverse of the transformation function. Thus,

$$\Phi_t(X_t, \pi_t) = \int H(\mathbf{\Lambda}'_t \mathbf{L}(X_t, \pi_t) + \epsilon_t) dF_t(\epsilon_t). \tag{50}$$

Here, $F_t(\epsilon_t)$ is the distribution of disturbance term ϵ_t. The corresponding estimated regression coefficient vector is denoted $\widehat{\mathbf{\Lambda}}_t$, and the empirical distribution of residuals

$$\widehat{\epsilon}^m_t = H^{-1}(\beta V_{t+1}(X^m_{t+1})) - \widehat{\mathbf{\Lambda}}'_t \mathbf{L}(X^m_t, \pi^m_t) \tag{51}$$

can be used to perform this integration. To handle heteroscedasticity in residuals, the conditional variance is modelled as

$$\mathrm{var}[\epsilon_t | X_t, \pi_t] = (\Omega(\boldsymbol{\mathcal{L}}'_t \mathbf{C}(X_t, \pi_t)))^2, \tag{52}$$

where $\Omega(\cdot)$ is a positive function, $\boldsymbol{\mathcal{L}}_t$ is the vector of coefficients and $\mathbf{C}(X_t, \pi_t)$ is a vector of basis functions that can be estimated e.g. as described in Andréasson and Shevchenko [3]. Then, the estimate of $\Phi_t(X_t, \pi_t)$ is given by

$$\widehat{\Phi}_t(X_t, \pi_t) = \frac{1}{M} \sum_{m=1}^{M} H\left(\widehat{\mathbf{\Lambda}}'_t \mathbf{L}(X_t, \pi_t) + \Omega(\widehat{\boldsymbol{\mathcal{L}}}'_t \mathbf{C}(X_t, \pi_t)) \frac{\widehat{\epsilon}^m_t}{\Omega(\widehat{\boldsymbol{\mathcal{L}}}'_t \mathbf{C}(X^m_t, \pi^m_t))} \right). \tag{53}$$

The optimal control for each sample can now be calculated, and the value function needs to be updated with the optimal paths for $t, \ldots, T$ as the control at time t affect the future states. The algorithm is then iterated for all samples M backward in time to the starting time $t = 0$ as described in Algorithm 2. In general, finding optimal values $\pi^*_t(x)$ in step 8 of Algorithm 2 should be accomplished numerically.

Algorithm 1 Forward simulation

```
1: for t = 0 to N − 1 do
2:    for m = 1 to M do
         [Simulate random samples ]
3:       sample X_t^m in the domain of its possible values                 ▷ State
4:       sample π̃_t^m in the domain of its possible values A_t             ▷ Control
5:       sample Z_{t+1}^m from the distribution specified by the model     ▷ Disturbance
         [Compute the state variable after control]
6:       X̃_{t+1}^m := T_t(X_t^m, π̃_t^m, Z_{t+1}^m)                          ▷ Evolution of state
7:    end for
8: end for
```

Algorithm 2 Backward solution (Realised value)

for $t = N$ **to** 0 **do**
  if $t = N$ **then**
    $\widehat{V}_t(\widetilde{X}_t) := R_N(\widetilde{X}_t)$
  else if $t < N$ **then**
    [Regression of transformed value function]
    $\widehat{\mathbf{\Lambda}}_t := \arg\min_{\mathbf{\Lambda}_t} \sum_{m=1}^{M} \left[\mathbf{\Lambda}_t' \mathbf{L}(X_t^m, \widetilde{\pi}_t) - H^{-1}(\beta \widehat{V}_{t+1}(\widetilde{X}_{t+1}^m))\right]^2$
    Approximate conditional expectation $\widehat{\Phi}_t(X_t, \widetilde{\pi}_t)$ using eq. (53)
    for $m = 1$ **to** M **do**
      $\widehat{X}_t^m := \widetilde{X}_t^m$
      [Optimal control]
      $\pi_t^*(\widehat{X}_t^m) := \arg\sup_{\pi_t \in \mathcal{A}_t} \left\{R_t(\widehat{X}_t^m, \pi_t) + \widehat{\Phi}_t(\widehat{X}_t^m, \pi_t)\right\}$
      [Update value function with optimal paths]
      $\widehat{V}_t(\widehat{X}_t^m) := R_t(\widehat{X}_t^m, \pi_t^*(\widehat{X}_t^m))$
      $\widehat{X}_{t+1}^m := \mathcal{T}_t(\widehat{X}_t^m, \pi_t^*(\widehat{X}_t^m), Z_t^m)$
      for $\tau = t+1$ **to** $N-1$ **do**
        $\widehat{V}_t(\widehat{X}_t^m) := \widehat{V}_t(\widehat{X}_t^m) + \beta^{\tau-t} R_\tau(\widehat{X}_\tau^m, \pi_\tau^*(\widehat{X}_\tau^m))$
        $\widehat{X}_{\tau+1}^m := \mathcal{T}_t(\widehat{X}_\tau^m, \pi_\tau^*(\widehat{X}_\tau^m), Z_\tau^m)$
      end for
      $\widehat{V}_t(\widehat{X}_t^m) := \widehat{V}_t(\widehat{X}_t^m) + \beta^{N-t} R_N(\widehat{X}_N^m)$
    end for
  end if
end for

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