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Novel exact solutions for PDEs with mixed boundary conditions

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Abstract

We develop methods for the solution of inhomogeneous Robin-type boundary value problems (BVPs) that arise for certain linear parabolic partial differential equations (PDEs) on a half-line, as well as a second-order generalization. We are able to obtain nonstandard solutions to equations arising in a range of areas, including mathematical finance, stochastic analysis, hyperbolic geometry, and mathematical physics. Our approach uses the odd and even Hilbert transforms. The solutions we obtain and the method itself seem to be new.

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1 Introduction

There is a well-established theory for the solution of parabolic PDEs subject to the most common types of boundary conditions. The book by Friedman [1] provides a rigorous introduction to this topic. The classical method makes use of a fundamental solution of the PDE satisfying boundary conditions. The construction is straightforward, and we present an illustrative example in the Appendix.

However, the classical method can produce cumbersome representations of the solution, and the desired fundamental solution may not even be known. So we ask if it is possible to construct an analytical solution to a boundary value problem using only elementary solutions, without needing the fundamental solution required by the classical theory? This has potentially important practical implications, because there are many PDEs for which elementary solutions are readily obtainable but for which appropriate fundamental solutions are not known.

We will focus on parabolic PDEs on a half-line (b,∞) subject to the boundary condition

$$\alpha u(b,t) + \beta u_x(b,t) + \gamma u_{xx}(b,t) = g(t). \tag{1.1}$$

We will refer to this as a second-order Robin condition. If $\gamma = 0$, then this reduces to the usual Robin condition.[1] The Dirichlet condition $u(b,t) = 0$ and the Neumann condition $u_x(b,t) = 0$ are particular cases. We are also able to solve certain moving boundary problems. We give an example in the final section.

This particular nonclassical boundary value problem arises in a number of areas of applied mathematics. The earliest example that we know is a paper by Langer [3] in 1932, in which he studies the cooling of metal bars when one end is held in a liquid. The condition appears in the study of the boundary reaction of the diffusion of chemicals [4] and the modeling of transient flow pump experiments in a porous medium [5]. An investigation of the boundary condition for the heat equation with source term is presented in [6].

Here we develop a new method for the solution of these problems. Our technique relies on the odd and even Hilbert transforms and does not require a fundamental solution. We only need two elementary solutions. These can be found in a number of ways, such as separation of variables.

Suppose that we have a linear parabolic PDE $u_t = Lu$ on an interval (b,∞), where L may be time dependent. The essential idea is to look for solutions of the form

$$u(x,t) = \int_0^\infty \varphi(\xi) w_1(x,t;\xi)d\xi + \int_0^\infty \psi(\xi) w_2(x,t;\xi)d\xi, \tag{1.2}$$

where $w_{1,2}(x,t;\xi)$ satisfy the PDE for each ξ. Imposing (1.1) and an initial condition lead to a pair of integral equations for φ and ψ.

In general, there is no reason to suppose that these equations will be analytically tractable, though we might attempt to solve them numerically. This idea is reminiscent of the well-known boundary integral method used for higher-dimensional BVPs. See, for example, [7].

However, it turns out that for certain types of important problems, these equations admit explicit solutions and lead to representation of the solutions that differ from those produced by the classical method. In particular, we have a solution of the BVP that does not rely on knowing the fundamental solution.

In the current work, we focus on problems where w_1 and w_2 involve the sine and cosine functions. This will be made explicit in Sect. 3. We develop this new technique and present interesting examples. The outline of the paper is as follows. Theorem 5.1 is a general result covering a range of PDEs. We will present solutions to a number of problems that do not seem to be solved in the literature. For example, we solve the classical Robin problem for the Kolmogorov backward equation arising from a five-dimensional squared Bessel process.

We begin with a general discussion of boundary value problems and the representation of their solutions. We mention some recent work, particularly, that of Fokas.

Following this, Sect. 2, we turn to the solution of the problem that motivated this study, namely the second-order Robin problem for the Black–Scholes equation. We reduce the solution of this problem to the inversion of the Fourier sine transform. Our result appears to be new. We also briefly give the fundamental solution for the classical Robin problem, which can be solved by the same method.

[1] There is a historical curiosity here. The boundary condition with $\gamma = 0$ is named after Gustave Robin. However, according to [2], Robin never stated or studied this type of boundary conditions. It appears nowhere in his collected works. Why this condition was named after Robin is apparently a mystery.

Since the fundamental solution in the second-order case is cumbersome, we ask if it is possible to solve the Robin problem without the fundamental solution? The answer is positive, and we develop our new theory for the explicit case of the Black–Scholes equation. The method relies on the odd and even Hilbert transforms, and we begin by presenting the properties of these transforms we need. See Sect. 3.1. This work begins in Sect. 3 with the classical Robin problem. We then proceed to the second-order Robin problem in Sect. 4. We follow this with an example. Then in Sect. 5, we turn to the second-order Robin problem for a larger class of PDEs. For a class of second-order Robin problems, Theorem 5.1 gives explicit solutions involving the sine and cosine problems. The techniques we developed for the Black–Scholes example make the proof easier. This new method is the main contribution of our paper. Then we present examples of PDEs and the families of elementary solutions our method requires. In Sect. 5.1, we turn to the question of PDEs uncovered by Theorem 5.1. In Sect. 5.2, we briefly mention PDEs with time-dependent coefficients and then solve two further problems; first, for the harmonic oscillator, which requires a different set of elementary solutions. In Sect. 6, we solve the Robin problem for a five-dimensional squared Bessel process. This has the feature of using elementary solutions that involve sums of sines and cosines. In both cases, our method is effective.

In summary, our major contribution is the development of a method for solving a wide variety of BVPs without requiring a fundamental solution, in contrast with the classical method. We can solve BVPs using only elementary solutions. The connection between the Hilbert type transforms that we use and the Fourier sine and cosine transforms means that our methods can be potentially applied using fast Fourier transform techniques.

1.1 Boundary value problems

The literature on boundary value problems is enormous, but Sagan's book provides a good elementary introduction [8]. Boundary integral methods play an important role in the study of multidimensional BVPs. See, for example, McLean's book [9]. For a study of boundary conditions in the theory of diffusions, see [10]. We also mention the work of Fokas [11], which presents a novel integral transform method for the solution of BVPs on a half-line. There are obviously thousands of references, and we could not attempt to provide an exhaustive list. However, for Robin problems, we mention some recent work. Abels and Moser [12] investigated a nonlinear Robin boundary condition in a bounded smooth domain. Geng and Zhuge [13] studied a family of second-order elliptic systems subject to a periodically oscillating Robin boundary condition. In [14] the authors studied the diffusion equation with a stochastic boundary that randomly switches between Dirichlet and Neumann conditions, proving that the mean of the solution satisfies a new type of Robin condition.

It is worth mentioning also the paper [15], where the authors consider the eigenvalues of the Robin boundary value problem for the Laplacian. Finally, Bondurant and Fulling [16] introduce a map between Dirichlet and Robin boundary conditions for linear constant-coefficient equations.

The current work arose from the study of barrier options within the Black–Scholes (BS) framework. For so-called *knock out barrier options*, we have absorbing boundaries. Reflecting boundaries occur where the option is *knocked in*. Although many formulae exist for the pricing of barrier options, the solution of the Robin problem for the BS equation does not seem to have appeared in the literature. In fact, we solve the second-order Robin problem for the BS equation.

The fundamental solution we obtain appears to be new. A glance at the Appendix shows that this solution is extremely complex. This leads to the question, which is the main part of our study: Is it possible to solve the BVP without the fundamental solution? Although any well-posed BVP has a unique solution, there is in general no unique *representation* of that solution. This is not a trivial fact. For example, consider the BVP

$$\begin{aligned} u_t = u_{xx},\ x > 0, t > 0, u(x,0) = f(x), x > 0, f \in L^1(\mathbb{R}^+), \\ u(0,t) - \gamma u_x(0,t) = g(t),\ t > 0. \end{aligned}$$

Its solution can be written in terms of the classical heat kernel. This is well known. See Cannon's book [4] for details.

However, another representation of the solution to this problem was obtained by Fokas. We quote the result from [11]. The solution can be written as

$$\begin{aligned} u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-k^2t}\widehat{f}(k)dk - \frac{1}{2\pi}\int_{\partial D^+} e^{ikx-k^2t}\left[\frac{2k}{k+i\gamma}G_R(k^2)\right. \\ \left. -k\frac{k-i\gamma}{k+i\gamma}\widehat{f}(-k)\right]dk + \Theta(-\gamma)2\gamma e^{\gamma x+\gamma^2 t}[G_R(-\gamma^2) - \widehat{f}(i\gamma)], \end{aligned}$$

where Θ is the Heaviside step function, $G_R(k) = \int_0^T e^{ks}g(s)ds$, $k \in \mathbb{C}$, D^+ is the wedge in the upper half-plane making an angle of $\pi/4$ with the real axis on both sides of the origin (see Fig. 2 of [11]), and $\widehat{f}(k) = \int_0^\infty f(x)e^{-ikx}dx$.

A discussion of the advantages of this alternative way of obtaining the solution is beyond our scope, though we remark that Fokas-type representations can often be evaluated numerically with considerable efficiency. However, this is a large subject, and for brevity, we refer the reader to the aforementioned book by Fokas. We also mention the work of Donaldson [17] on obtaining alternative representations for the solutions of BVPs. There is a considerable literature on this topic.

We will use a combination of separation-of-variables and classical transform methods. The Hilbert transform has been used to solve integral equations for over a hundred years. See [18] for a lengthy discussion with examples. We in fact use the related *odd* and *even* Hilbert transforms. Our method has the elegant feature that it turns a pair of integral equations into a pair of simultaneous equations.

Some of our results rely upon the inversion of the Laplace transform. However, there are thousands of Laplace transform pairs known, and more can be constructed by standard methods; see [19]. For polynomial data, inversion produces Dirac delta functions and their derivatives, and these are easy to handle. If $g(t) = 0$, then there is no Laplace transform to invert.

Remark 1.1 We make an important comment here. There are many PDEs on the line that can be mapped to the heat equation. Some of our examples have this property, though not all. (For example, equation (6.3) cannot be reduced to the heat equation). We might think that it would be more efficient to reduce a suitable PDE to the heat equation, solve the resulting BVP, and then map back. However this ignores the question of what happens to the boundary conditions under the change of variables? This is a crucial question. The resulting boundary value problem may not have a known solution.

To illustrate, suppose that we wish to solve $u_t = u_{xx} + Axu$, $u(x,0) = f(x)$, $u(0,t) = 0$. The PDE can be mapped to the heat equation. However, the result of the mapping is a formidable *moving boundary problem*. For complete details, see [20]. Thus mapping to the heat equation in this case makes the problem much harder.

A similar phenomenon occurs with the harmonic oscillator. Reducing even the simple problem $u_t = u_{xx} - x^2u$, $u(x,0) = f(x)$, $u(0,t) = 0$ to the heat equation produces another moving boundary problem, which is much harder than the original one. There are numerous examples of this phenomenon.

There are also many PDEs that can theoretically be mapped to the heat equation, but the change of variables is itself impossible to compute. Consider the equation $u_t = \sigma(x)u_{xx} + f(x)u_x$, $x \in \Omega \subset \mathbb{R}$. Suppose that it can be reduced to the heat equation. To do so, we first let $y = \int_{x_0}^{x} (\sigma(z))^{-1/2} dz$. This makes the coefficient of the second derivative term equal to 1. For arbitrary σ, there is no reason why this integral should be computable. Inverting the change of variables to write x in terms of y may also be impossible.

Thus methods that produce solutions of BVPs without the need for a change of variables are important. Our results yield novel solutions for a wide variety of problems without having to make a change of variables.

2 The second-order Robin problem for the Black–Scholes equation

For the theory of option pricing and stochastic calculus, we refer the reader to a standard reference [21]. The conventional method for studying the Black–Scholes equation is to reduce to the heat equation, but we work in the original variables. For our purposes, the second-order Robin problem can be written as

$$w_t = \frac{1}{2}\sigma^2 S^2 w_{SS} + rSw_S, \tag{2.1}$$

$$w(S,0) = f(S), \tag{2.2}$$

$$\alpha w(b,t) + \beta w_S(b,t) + \gamma w_{SS}(b,t) = g(t), \tag{2.3}$$

where $S > b > 0$.

We will assume that the solution is nonnegative and satisfies a bound of the form $w(S,t) \leq MS^\theta$ for some positive constants M and θ. This is the usual type of behavior as $S \to \infty$ that appears in the literature. See, for example, the discussion of boundary conditions for the Black–Scholes equation in [21].

We will construct a fundamental solution for this problem in the case $g(t) = 0$. We first obtain separable solutions that satisfy the boundary condition. We then use these to construct a solution that also satisfies the initial condition. This second problem can be reduced to the solution of a tractable integral equation.

We make the ansatz $w(S,t) = e^{\lambda t}v(S)$. Then $\frac{1}{2}\sigma^2 S^2 v''(S) + rSv'(S) - \lambda v(S) = 0$ with the constant λ and the function v to be determined. Assuming that $v(S) = S^\delta$, we obtain the condition

$$\frac{1}{2}\sigma^2 S^2 \delta(\delta - 1)S^{\delta-2} + rS\delta S^{\delta-1} - \lambda S^\delta = 0.$$

This means that we must have $\frac{1}{2}\sigma^2\delta(\delta-1)+r\delta-\lambda=0$, which gives

$$\delta=\frac{-\left(r-\frac{\sigma^2}{2}\right)\pm\sqrt{\left(r-\frac{\sigma^2}{2}\right)^2+2\sigma^2\lambda}}{\sigma^2}.$$

We now set $\left(r-\frac{\sigma^2}{2}\right)^2+2\sigma^2\lambda=-\xi^2\sigma^4$. So we have

$$\lambda=-\frac{\xi^2\sigma^2}{2}-\frac{1}{2\sigma^2}\left(r-\frac{\sigma^2}{2}\right)^2. \tag{2.4}$$

Hence we can write $\delta=\mu\pm i\xi$, where $\mu=\frac{\sigma^2-2r}{2\sigma^2}$.

This gives us the solution, which can be easily checked to be valid for all $\xi>0$:

$$w(S,t,\lambda)=e^{\lambda t}\left(c_1S^{\mu+i\xi}+c_2S^{\mu-i\xi}\right).$$

We now wish to construct a solution of the PDE that satisfies the boundary condition. Observe that

$$S^{\mu\pm i\xi}=S^{\mu}\left(\cos(\xi\ln S)\pm i\sin(\xi\ln S)\right).$$

The real and imaginary parts must both satisfy the equation. So we obtain a solution that can be written as

$$\begin{aligned} w^{\xi}(S,t)&=e^{\lambda t}S^{\mu}\left(A\cos(\xi\ln S)+B\sin(\xi\ln S)\right)\\ &=e^{-\frac{\xi^2\sigma^2}{2}t+ct}S^{\mu}\left(A\cos(\xi\ln S)+B\sin(\xi\ln S)\right), \end{aligned} \tag{2.5}$$

where $c=-\frac{1}{2\sigma^2}\left(r-\frac{\sigma^2}{2}\right)^2$, and A,B are constants. We choose A and B by requiring that the solution satisfies the second-order Robin boundary condition with $g(t)=0$. Substituting the solution into the PDE and imposing the boundary condition, we obtain

$$A=\sin(\xi\log(b))z_1+\xi z_2\cos(\xi\log(b)), \tag{2.6}$$

$$B=-\cos(\xi\log(b))z_1+\xi z_2\sin(\xi\log(b)), \tag{2.7}$$

where $z_1=\left(\alpha b^2+\beta b\mu+\gamma(\mu-1)\mu-\gamma\xi^2\right)$ and $z_2=(b\beta+\gamma(2\mu-1))$.

The reader can check that for this choice of the coefficients A and B, the function w^{ξ} solves the PDE and also satisfies the homogeneous form of the second-order Robin boundary condition (2.1). These choices are not unique. However, every other choice in fact leads to the same solution. This is not hard to check, but it is somewhat tedious.

We now have to obtain a solution that also satisfies the initial data and boundary condition. To do this, we will construct a solution of the form

$$w(S,t)=\int_0^{\infty}\varphi(\xi)w^{\xi}(S,t)d\xi. \tag{2.8}$$

If the function φ has sufficient decay, then it is easy to show that $w(S,t)$ is also a solution of our PDE. See [22] for more on this idea. We note that $w^{\xi}(S,t)$ is locally integrable in ξ, since it is continuous in ξ for all $S > 0$ and has Gaussian decay for all $t > 0$.

Moreover, by construction, (2.8) satisfies the boundary condition. Our task now is to choose the function φ so that the initial condition is also satisfied. Taking $t = 0$, we get the integral equation

$$f(S) = \int_0^\infty \varphi(\xi) w^{\xi}(S,0)d\xi,$$

that is,

$$\int_0^\infty \varphi(\xi)\left(A\cos(\xi \ln S) + B\sin(\xi \ln S)\right)d\xi = S^{-\mu} f(S). \tag{2.9}$$

This can be reduced to a Fourier sine transform. To see this, notice that using some elementary trigonometric identities, we can rewrite (2.9) as

$$\int_0^\infty \varphi(\xi)\left(\tilde{\beta}\xi \cos\left(\xi \ln\left(\frac{S}{b}\right)\right) + \left(\gamma\xi^2 - \tilde{\alpha}\right)\sin\left(\xi \ln\left(\frac{S}{b}\right)\right)\right)d\xi = F(S),$$

where $F(S) = S^{-\mu} f(S)$, and

$$\tilde{\alpha} = \alpha b^2 + \beta b\mu + \gamma\mu(\mu - 1), \tag{2.10}$$

$$\tilde{\beta} = b\beta + \gamma(2\mu - 1). \tag{2.11}$$

We now reduce this integral equation to a Fourier sine transform via the solution of a second-order ordinary differential equation.

Let $\zeta(x) = \int_0^\infty \varphi(\xi)\sin(\xi x)d\xi$. After the change of variable $x = \ln\left(\frac{S}{b}\right)$, differentiating twice, we see that ζ must satisfy the equation

$$-\gamma\zeta''(x) + \tilde{\beta}\zeta'(x) - \tilde{\alpha}\zeta(x) = b^{-\mu}e^{-\mu x} f(be^x) \tag{2.12}$$

with initial conditions $\zeta(0) = 0$ and $\zeta''(0) = 0$. We will assume that $\Delta = \tilde{\beta}^2 - 4\gamma\tilde{\alpha} > 0$. For the particular cases where $\Delta \leq 0$, the differential equation for ζ has different solutions. However, we can proceed as we do here, and we obtain fundamental solutions valid for those particular choices of the parameter. We will omit the details for brevity.

Using variation of parameters, we see that the solution of (2.12) is given by

$$\zeta(x) = \frac{2b^{-\mu}}{\tilde{\beta}\sqrt{\Delta}}\left(\gamma f(b)e^{\frac{\tilde{\beta}x}{2\gamma}}\sinh\left(\frac{x\sqrt{\Delta}}{2\gamma}\right) - \tilde{\beta}\int_0^x f\left(be^z\right)\mathcal{K}(x,z)dz\right),$$

where $\mathcal{K}(x,z) = e^{-\mu z + (x-z)\frac{\tilde{\beta}}{2\gamma}}\sinh\left(\frac{(x-z)\sqrt{\Delta}}{2\gamma}\right)$.

Recall that the inverse Fourier sine transform of $\widehat{f} \in L^1([0,\infty))$ is

$$f(x) = \frac{2}{\pi}\int_0^\infty \widehat{f}(y)\sin(yx)dy; \tag{2.13}$$

see [23].

Inverting the Fourier sine transform in (2.8) then allows us to write down the solution of the BVP:

$$\begin{aligned} w(S,t) &= \int_0^\infty \varphi(\xi) w^\xi(S,t)d\xi \\ &= \int_0^\infty \frac{2}{\pi}\int_0^\infty \zeta(\eta)\sin(\xi\eta)\, w^\xi(S,t)d\eta d\xi \\ &= \int_0^\infty \int_0^\infty \frac{2}{\pi}\zeta(\eta)\sin(\xi\eta) w^\xi(S,t)d\xi d\eta \\ &= \int_0^\infty \zeta(\eta)\mathcal{G}(S,t,\eta)d\eta, \end{aligned} \tag{2.14}$$

where on the second line, we introduced the inversion integral for the Fourier sine transform. We have

$$\begin{aligned} \mathcal{G}(S,\eta,t) &= \frac{2}{\pi}\int_0^\infty \sin(\xi\eta) w^\xi(S,t)d\xi \\ &= \frac{e^{ct-\frac{\left(\eta-\ln\left(\frac{S}{b}\right)\right)^2}{2\sigma^2 t}} S^\mu}{\sqrt{2\pi}\left(\sigma^2 t\right)^{5/2}} k(S,\eta,t). \end{aligned}$$

Let $L(S,t) = \gamma \ln\left(\frac{S}{b}\right) + \tilde{\beta}\sigma^2 t$. Then

$$\begin{aligned} k(S,\eta,t) = e^{-\frac{2\eta\ln\left(\frac{S}{b}\right)}{\sigma^2 t}} &\left(\tilde{\alpha}\sigma^4 t^2 + \ln\left(\frac{S}{b}\right)(2\gamma\eta + L(S,t)) + \sigma^2 t\left(\eta\tilde{\beta} - \gamma\right)\right. \\ &\left. + \gamma\eta^2\right) - \tilde{\alpha}\sigma^4 t^2 + \ln\left(\frac{S}{b}\right)(2\gamma\eta - L(S,t)) + \sigma^2 t\left(\eta\tilde{\beta} + \gamma\right) - \gamma\eta^2 \end{aligned}$$

with $\tilde{\alpha}$ and $\tilde{\beta}$ given by (2.10) and (2.11). We would like to write the solution in the form

$$w(S,t) = \int_0^\infty f(y)p(S,y,t)dy,$$

where f is the initial data, and $p(S,y,t)$ is a fundamental solution. To do this, we use Fubini's Theorem, which leads us to an explicit expression for our fundamental solution subject to the homogeneous second-order Robin boundary conditions. In fact, we can rewrite the solution as

$$\begin{aligned} w(S,t) &= \int_0^\infty \zeta(\eta)\mathcal{G}(S,t,\eta)d\eta \\ &= \int_0^\infty \frac{2b^{-\mu}}{\tilde{\beta}\sqrt{\Delta}}\left(\gamma f(b)e^{\frac{\tilde{\beta}\eta}{2\gamma}}\sinh\left(\frac{\eta\sqrt{\Delta}}{2\gamma}\right) - \tilde{\beta}\int_0^\eta f\left(be^z\right)\mathcal{K}(\eta,z)dz\right) \\ &\quad \times \mathcal{G}(S,t,\eta)d\eta \\ &= \int_0^\infty \frac{2b^{-\mu}}{\tilde{\beta}\sqrt{\Delta}}\gamma f(b)e^{\frac{\tilde{\beta}\eta}{2\gamma}}\sinh\left(\frac{\eta\sqrt{\Delta}}{2\gamma}\right)\mathcal{G}(S,t,\eta)d\eta \\ &\quad - \int_0^\infty \frac{2b^{-\mu}}{\tilde{\beta}\sqrt{\Delta}}\left(\int_0^\eta f\left(be^z\right)\mathcal{K}(\eta,z)dz\right)\mathcal{G}(S,t,\eta)d\eta \end{aligned}$$

$$= \ell(S,t) - \int_0^\infty f\left(be^z\right) \int_z^\infty \frac{2b^{-\mu}}{\tilde{\beta}\sqrt{\Delta}} \mathcal{K}(\eta,z)\mathcal{G}(S,t,\eta)d\eta dz, \tag{2.15}$$

where

$$\ell(S,t) := \int_0^\infty \frac{2b^{-\mu}}{\tilde{\beta}\sqrt{\Delta}} \gamma f(b) e^{\frac{\tilde{\beta}\eta}{2\gamma}} \sinh\left(\frac{\eta\sqrt{\Delta}}{2\gamma}\right) \mathcal{G}(S,t,\eta)d\eta. \tag{2.16}$$

If we set

$$\bar{p}(S,z,t) := \int_z^\infty \frac{2b^{-\mu}}{\tilde{\beta}\sqrt{\Delta}} \mathcal{K}(\eta,z)\mathcal{G}(S,t,\eta)d\eta, \tag{2.17}$$

then we can express $w(S,t)$ as

$$\begin{aligned} w(S,t) &= \ell(S,t) - \int_0^\infty f\left(be^z\right) \bar{p}(S,z,t)dz \\ &= \ell(S,t) - \int_b^\infty f\left(y\right) \tilde{p}(S,y,t)dy, \end{aligned} \tag{2.18}$$

where

$$\tilde{p}(S,y,t) := \frac{1}{y} \bar{p}\left(S, \ln\left(\frac{y}{b}\right), t\right). \tag{2.19}$$

Lastly, formula (2.18) can be rewritten as

$$w(S,t) = \int_b^\infty f\left(y\right) p(S,y,t)dy, \tag{2.20}$$

where

$$p(S,y,t) = \frac{\ell(S,t)}{f(b)} \delta^{(b)}(y) - \tilde{p}(S,y,t), \tag{2.21}$$

with $\delta^{(b)}$ denoting the Dirac delta centered in $b \in \mathbb{R}$. The kernel in (2.19) can be computed explicitly in Mathematica in terms of Gaussians and error functions. It is complicated, so we present it in the Appendix.

Remark 2.1 We have solved the second-order Robin problem in the case $g(t) = 0$. We can also solve the standard Robin problem by this method. We sketch the calculation. We begin with (2.8) and apply the boundary condition. The analysis is very similar to the second-order case, though somewhat easier. We find that a solution satisfying the boundary condition is

$$w^\xi(S,t) = e^{-\frac{\xi^2\sigma^2}{2}t + ct} S^\mu \left(A\cos(\xi \ln S) + B\sin(\xi \ln S)\right), \tag{2.22}$$

where $c = -\frac{1}{2\sigma^2}\left(r - \frac{\sigma^2}{2}\right)^2$, and

$$A = \beta\xi \cos(\xi \ln b) + b\alpha \sin(\xi \ln b) + \beta\mu \sin(\xi \ln b), \tag{2.23}$$

$$B = -b\alpha \cos(\xi \ln b) - \beta\mu \cos(\xi \ln b) + \beta\xi \sin(\xi \ln b). \tag{2.24}$$

Setting $u(S,t) = \int_0^\infty \varphi(\xi) w^\xi(S,t)d\xi$ and imposing the initial condition lead to an integral equation, which also reduces to a Fourier sine transform. The difference is that in place of the second-order equation (2.12) for ζ, we have $\beta\zeta'(x) + (b\alpha + \beta\mu)\zeta(x) = b^{-\mu}e^{\mu x}f(be^{-x})$ with $\zeta(0) = 0$, which is of first order. Obtaining ζ, we proceed exactly as in the secon-order case and obtain the fundamental solution

$$p(S,y,t) = -\frac{e^{ct+\frac{\sigma^2 t\tilde{A}^2}{2\beta^2}} S^{\mu-\frac{\tilde{A}}{\beta}}}{\beta y^{\mu+1+\frac{\tilde{A}}{\beta}}} \tilde{A} b^{\frac{2\tilde{A}}{\beta}} \operatorname{erfc}\left(\frac{\sigma^2 t\tilde{A} + \beta\left(\ln\frac{b}{y} + \ln\left(\frac{b}{S}\right)\right)}{\sqrt{2\beta^2\sigma^2 t}}\right)$$
$$+ \frac{e^{ct}S^\mu y^{-\mu-1}}{\sqrt{2\pi\sigma^2 t}}\left(b^{\frac{2\ln\frac{b}{y}+\ln S}{\sigma^2 t}} + S^{\frac{2\ln\frac{b}{y}+\ln b}{\sigma^2 t}}\right) e^{\left(-\frac{\left(\ln\frac{b}{y}+\ln S\right)^2 + 2\ln b\ln\frac{b}{y} + \ln^2 b}{2\sigma^2 t}\right)}.$$

The classical approach to the solution of the problem with $g \neq 0$ is found in Friedman's book [1]. For completeness, we present it in the Appendix. Our purpose now is to develop a new method for solution of these problems that does not require any knowledge of the fundamental solution.

3 A novel representation using Hilbert transform methods

In this section, we construct a solution to

$$w_t = \frac{1}{2}\sigma^2 S^2 w_{SS} + rSw_S, \quad S > b > 0, \tag{3.1}$$

$$w(S,0) = f(S), \tag{3.2}$$

$$\alpha w(b,t) + \beta w_S(b,t) = e^{-\frac{1}{2}\sigma^2\mu^2 t} g(t), \tag{3.3}$$

which does not require a fundamental solution. We will then extend our method to include second-order Robin conditions and a significantly wider class of PDEs. We assume that $\alpha, \beta \neq 0$ for the remainder of the paper, unless stated otherwise. Choosing one of these constants to be zero reduces the BVP to one of either Dirichlet or Neumann type.

Although explicit solutions can be obtained by our method for many interesting problems, in most cases the representations we obtain will require numerical evaluation. However, this is true for every representation and is beyond the scope of our study. Note that there is a considerable literature on the numerical evaluation of Hilbert transforms. We refer the reader to Chap. 14 of [24] for an introduction to this topic.

Our approach to the problem uses the *even* and *odd* Hilbert transforms. For an exhaustive treatment of the Hilbert transform, we refer to King's two-volume treatise [18, 24].

3.1 The odd and even Hilbert transforms

For the reader's convenience, we introduce here the material we require. The odd and even Hilbert transforms have some very useful properties, which we will exploit. Although these are important operators in their own right, they arose originally as particular cases of the classical Hilbert transform.

Various Hilbert transforms are given by integrals defined in the principal value sense.

Definition 3.1 We will define the *even Hilbert transform* of a function $f : [0,\infty) \to \mathbb{R}$ by the principal value integral

$$(\mathcal{H}_e f)(x) = \frac{2x}{\pi}\mathrm{P}\int_0^\infty \frac{f(y)}{x^2 - y^2}dy, \tag{3.4}$$

assuming that it exists. Similarly, we define the *odd Hilbert transform* of a function $f : [0,\infty) \to \mathbb{R}$ by the principal value integral

$$(\mathcal{H}_o f)(x) = \frac{2}{\pi}\mathrm{P}\int_0^\infty \frac{yf(y)}{x^2 - y^2}dy, \tag{3.5}$$

again assuming the convergence in the principal value sense.

It is sufficient that $f \in L^2[0,\infty)$ for these transforms to exist. If $f \in L^2[0,\infty)$, then $\mathcal{H}_e f \in L^2[0,\infty)$ and $\mathcal{H}_o f \in L^2[0,\infty)$. We can extend the operators to other function spaces, but we will avoid a discussion and simply refer the reader to [24].

Remark 3.2 The Hilbert transform of a suitable function f is defined by

$$(\mathcal{H}f)(x) = \frac{1}{\pi}\mathrm{P}\int_{-\infty}^\infty \frac{f(s)}{x - s}ds. \tag{3.6}$$

Again, if $f \in L^2(\mathbb{R})$, then the Hilbert transform exists, and $\mathcal{H}f \in L^2(\mathbb{R})$.

The Hilbert transform has many useful properties, most of which can be found in King's books. An extensive table of transform pairs can be found in [18]. The inverse of the Hilbert transform is simply $-\mathcal{H}$, that is, $\mathcal{H}^2 = -I$, where I is the identity operator.

Now suppose that f is even. Then a simple change of variables in the integral yields

$$(\mathcal{H}f)(x) = \frac{2x}{\pi}\mathrm{P}\int_0^\infty \frac{f(y)}{x^2 - y^2}dy = (\mathcal{H}_e f)(x). \tag{3.7}$$

Conversely, if f is odd, then we obtain

$$(\mathcal{H}f)(x) = \frac{2}{\pi}\mathrm{P}\int_0^\infty \frac{yf(y)}{x^2 - y^2}dy = (\mathcal{H}_o f)(x). \tag{3.8}$$

So the odd and even Hilbert transforms can be regarded as particular cases of the usual Hilbert transform. This is extremely useful for obtaining properties of $\mathcal{H}_e$ and $\mathcal{H}_o$.

The fundamental relationship between the even and odd transforms is

$$\mathcal{H}_e\mathcal{H}_o = \mathcal{H}_o\mathcal{H}_e = -I.$$

See [24, p. 261]. Note also that if $k(x) = xh(x)$, then

$$(\mathcal{H}_e k)(x) = \frac{2x}{\pi}\mathrm{P}\int_0^\infty \frac{yh(y)}{x^2 - y^2}dy = x(\mathcal{H}_o h)(x). \tag{3.9}$$

This fact will be of importance below. For the second-order problem, we will require the even Hilbert transform of $y^2(\mathcal{H}_o f)(y)$, but this is best presented in context. See equation (4.6).

The odd and even Hilbert transforms arise because of their connection with the Fourier cosine and sine transforms. If $\mathcal{F}_c$ and $\mathcal{F}_s$ are the cosine and sine transforms, respectively, then we have $\mathcal{F}_s^{-1}\mathcal{F}_c = \mathcal{H}_e$ and $\mathcal{F}_c^{-1}\mathcal{F}_s = -\mathcal{H}_o$. A proof can be found in [24, p. 259].[2]

3.2 The solution of the Robin problem

We will construct a solution of problem (3.1)–(3.3). We will assume that $\alpha + \frac{\mu}{\beta} \neq 0$. The case where $\alpha + \frac{\mu}{\beta} = 0$ can be handled by a modification of our method.

We will use two linearly independent solutions of the PDE to solve our modified problem. These are

$$h_1^\xi(S,t) = \left(\frac{S}{b}\right)^\mu e^{-\frac{1}{2}\sigma^2 t(\xi^2+\mu^2)} \cos\left(\xi \ln\left(\frac{S}{b}\right)\right), \tag{3.10}$$

$$h_2^\xi(S,t) = \left(\frac{S}{b}\right)^\mu e^{-\frac{1}{2}\sigma^2 t(\xi^2+\mu^2)} \sin\left(\xi \ln\left(\frac{S}{b}\right)\right). \tag{3.11}$$

Notice that our BVP has a slightly different form to that used previously. We pose the problem in this way to avoid a technicality involving the Laplace transform. This arises in taking the inverse Laplace transform of $e^{as}F(s), a > 0$. If $a < 0$, then the inverse Laplace transform is simply $f(t+a)H(t+a)$, where f is the inverse Laplace transform of F, and H is the Heaviside function.

However, if a is positive, then we have to decide what the inverse transform will be. A natural choice is to insist that f be zero to the left of the origin, in which case the inverse Laplace transform will be $f(t+a)$. However, the issue requires a discussion of the Laplace transform as a distribution. So we solve the modified problem and refer the reader to Schwartz's treatment of the Laplace transform within the theory of distributions in the book [25].

We construct a solution of the PDE of the form

$$w(S,t) = \int_0^\infty \varphi(\xi) h_1^\xi(S,t)d\xi + \int_0^\infty \psi(\xi) h_2^\xi(S,t)d\xi. \tag{3.12}$$

Our aim is to choose φ and ψ so that the solution satisfies the boundary and initial conditions. Hence we must have

$$w(S,0) = \left(\frac{S}{b}\right)^\mu \left[\widehat{\varphi}_c\left(\ln\left(\frac{S}{b}\right)\right) + \widehat{\psi}_s\left(\ln\left(\frac{S}{b}\right)\right)\right] = f(S), \tag{3.13}$$

where $\widehat{\varphi}_c$ and $\widehat{\psi}_s$ are the cosine transform of φ and the sine transform of ψ, respectively. Setting $y = \ln\left(\frac{S}{b}\right)$, we obtain

$$\widehat{\varphi}_c(y) + \widehat{\psi}_s(y) = e^{-\mu y} f\left(be^y\right). \tag{3.14}$$

[2] Note: King uses a slightly different definition of the sine and cosine transforms by including a multiplicative factor of $\sqrt{\frac{2}{\pi}}$. Our statements are equivalent to his.

Applying the inverse Fourier cosine transform, we obtain the relation

$$\begin{aligned}\varphi(\xi) &= \frac{2}{\pi}\int_0^\infty e^{-\mu y} f\left(be^y\right)\cos(\xi y)dy - (\mathcal{F}_c^{-1}\mathcal{F}_s\psi)(\xi)\\ &= \frac{2}{\pi}\int_0^\infty e^{-\mu y} f\left(be^y\right)\cos(\xi y)dy + (\mathcal{H}_o\psi)(\xi).\end{aligned} \tag{3.15}$$

Set $F(\xi) = \frac{2}{\pi}\int_0^\infty e^{-\mu y} f(be^y)\cos(\xi y)dy$. Then $\varphi(\xi) = F(\xi) + (\mathcal{H}_o\psi)(\xi)$.

It is easy to see that

$$w(b,t) = \int_0^\infty \varphi(\xi) e^{-\frac{1}{2}\sigma^2 t(\xi^2+\mu^2)} d\xi, \tag{3.16}$$

and after some straightforward calculations, we obtain

$$\begin{aligned}\alpha w(b,t) + \beta w_S(b,t) &= \left(\alpha + \frac{\mu\beta}{b}\right)\int_0^\infty \varphi(\xi) e^{-\frac{1}{2}\sigma^2 t(\xi^2+\mu^2)} d\xi\\ &\quad + \frac{\beta}{b}\int_0^\infty \xi\psi(\xi) e^{-\frac{1}{2}\sigma^2 t(\xi^2+\mu^2)} d\xi = e^{-\frac{1}{2}\sigma^2\mu^2 t} g(t).\end{aligned}$$

Canceling the factor of $e^{-\frac{1}{2}\sigma^2\mu^2 t}$ and using the substitutions $z = \xi^2$ and $s = \frac{1}{2}\sigma^2 t$ gives us

$$\left(\alpha + \frac{\mu\beta}{b}\right)\int_0^\infty \frac{\varphi(\sqrt{z})}{2\sqrt{z}} e^{-zs} dz + \frac{\beta}{b}\int_0^\infty \frac{1}{2}\psi(\sqrt{z}) e^{-zs} dz = g\left(\frac{2s}{\sigma^2}\right).$$

We assume that $\left(\alpha + \frac{\mu\beta}{b}\right) \neq 0$. The case where $\left(\alpha + \frac{\mu\beta}{b}\right) = 0$ reduces to the inversion of a single Laplace transform for ψ. From this and the relation $\varphi(\xi) = F(\xi) + (\mathcal{H}_o\psi)(\xi)$ we immediately obtain φ, and we have our solution.

Assuming that $\left(\alpha + \frac{\mu\beta}{b}\right) \neq 0$, taking the inverse Laplace transform, and letting $z = \xi^2$, we obtain

$$\left(\alpha + \frac{\mu\beta}{b}\right)\varphi(\xi) = 2\xi\mathcal{L}^{-1}\left[g\left(\frac{2s}{\sigma^2}\right); \xi^2\right] - \frac{\xi\beta}{b}\psi(\xi). \tag{3.17}$$

Set $K(\xi) = 2\xi\mathcal{L}^{-1}\left[g\left(\frac{2s}{\sigma^2}\right); \xi^2\right]$. Obviously, in the homogeneous case, $K(\xi) = 0$, so there is no Laplace transform inversion necessary. Using (3.15), we have the relation

$$\left(\alpha + \frac{\mu\beta}{b}\right)(F(\xi) + (\mathcal{H}_o\psi)(\xi)) = K(\xi) - \frac{\xi\beta}{b}\psi(\xi). \tag{3.18}$$

We now take the even Hilbert transform of both sides to obtain

$$\left(\alpha + \frac{\mu\beta}{b}\right)[(\mathcal{H}_e F)(\xi) - \psi(\xi)] = (\mathcal{H}_e K)(\xi) - \frac{\beta}{b}\xi(\mathcal{H}_o\psi)(\xi), \tag{3.19}$$

where we used relation (3.9). So we have obtained a pair of simultaneous equations for $\mathcal{H}_o\psi$ and ψ. Clearly, equation (3.18) gives us

$$(\mathcal{H}_o\psi)(\xi) = \left(\alpha + \frac{\mu\beta}{b}\right)^{-1}\left[K(\xi) - \frac{\xi\beta}{b}\psi(\xi)\right] - F(\xi), \tag{3.20}$$

so that

$$(\mathcal{H}_e F)(\xi) - \psi(\xi) = \frac{(\mathcal{H}_e K)(\xi) - \frac{\beta}{b}\xi\left(\frac{\left[K(\xi) - \frac{\xi\beta}{b}\psi(\xi)\right]}{\left(\alpha + \frac{\mu\beta}{b}\right)} - F(\xi)\right)}{\left(\alpha + \frac{\mu\beta}{b}\right)}. \tag{3.21}$$

Rearranging this gives

$$\psi(\xi) = \frac{\nu\left[\frac{b\alpha+\mu\beta}{b}(\mathcal{H}_e F)(\xi) - (\mathcal{H}_e K)(\xi) + \frac{\beta\xi}{b}\left(\frac{b}{\alpha b+\mu\beta}K(\xi) - F(\xi)\right)\right]}{(\alpha b + \mu\beta)^2 + \beta^2\xi^2},$$

where $\nu = b(\alpha b + \mu\beta)$. From this we find φ and obtain a potential solution to the boundary value problem. This leads us to the following result.

Theorem 3.3 *Let $f \in L^1\left((b,\infty), \frac{d\eta}{\eta^{\mu+1}}\right)$ and $g(t) = \int_0^\infty G(x)e^{-xt}dx$ where G, $G'(x)$, and G'' are integrable. Let h_1^ξ and h_2^ξ be given by (3.10) and (3.11), respectively. Then if $\alpha + \frac{\mu\beta}{b} \neq 0$ and $\nu = b(\alpha b + \mu\beta)$, then problem (3.1)–(3.3) has a solution*

$$w(S,t) = \int_0^\infty \varphi(\xi)h_1^\xi(S,t)d\xi + \int_0^\infty \psi(\xi)h_2^\xi(S,t)d\xi, \tag{3.22}$$

where

$$\psi(\xi) = \frac{\nu\left[\frac{b\alpha+\mu\beta}{b}(\mathcal{H}_e F)(\xi) - (\mathcal{H}_e K)(\xi) + \frac{\beta\xi}{b}\left(\frac{b}{\alpha b+\mu\beta}K(\xi) - F(\xi)\right)\right]}{(\alpha b + \mu\beta)^2 + \beta^2\xi^2},$$

and

$$\varphi(\xi) = \left(\alpha + \frac{\mu\beta}{b}\right)^{-1}\left[\sigma^2\xi G\left(\frac{\sigma^2\xi^2}{2}\right) - \frac{\beta\xi}{b}\psi(\xi)\right],$$

where $F(\xi) = \frac{2}{\pi}\int_0^\infty e^{-\mu y}f(be^y)\cos(\xi y)dy$ and $K(\xi) = \sigma^2\xi G\left(\frac{\sigma^2\xi^2}{2}\right)$.

Proof To complete the proof, we must establish sufficient conditions to guarantee that we do in fact have a solution. To this end, we make the following observations. Since $\mathcal{H}_e = \mathcal{F}_s^{-1}\mathcal{F}_c$, we easily obtain

$$\begin{aligned}(\mathcal{H}_e F)(\xi) &= \mathcal{F}_s^{-1}\mathcal{F}_c\frac{2}{\pi}\int_0^\infty e^{-\mu y}f\left(be^y\right)\cos(\xi y)dy \\ &= \frac{2}{\pi}\int_0^\infty e^{-\mu y}f(be^y)\sin(\xi y)dy \\ &= \frac{2}{\pi}\int_b^\infty \left(\frac{\eta}{b}\right)^{-\mu}f(\eta)\sin\left(\xi\ln\left(\frac{\eta}{b}\right)\right)\frac{d\eta}{\eta}.\end{aligned}$$

Let $f \in L^1((b,\infty), \eta^{-\mu-1}d\eta)$ and denote $\|k\|_b = \int_b^\infty |k(x)|x^{\mu-1}dx$. Then we obtain the inequality

$$|\mathcal{H}_e F| \leq \frac{2}{\pi b^\mu}\|f\|_b. \tag{3.23}$$

It is also easy to see that $|F| \leq \frac{2}{\pi b^{\mu}} \|f\|_b$. From this we can conclude that

$$\frac{\partial^j}{\partial S^j}\left(\frac{-b(\alpha b+\mu\beta)}{(\alpha b+\mu\beta)^2+\beta^2\xi^2}(\mathcal{H}_e F)(\xi)h_k^{\xi}(S,t)\right) \tag{3.24}$$

is integrable for $k = 1,2$ and $j = 0,1,2$. Similarly,

$$\frac{\partial}{\partial t}\left(\frac{-b(\alpha b+\mu\beta)}{(\alpha b+\mu\beta)^2+\beta^2\xi^2}(\mathcal{H}_e F)(\xi)h_k^{\xi}(S,t)\right) \tag{3.25}$$

is integrable. Both facts follow from the Gaussian decay of the solutions $h_k^{\xi}(S,t)$. Specifically, we can bound (3.24) in ξ by

$$\left|\frac{\partial^j}{\partial S^j}\left(\frac{-b(\alpha b+\mu\beta)}{(\alpha b+\mu\beta)^2+\beta^2\xi^2}(\mathcal{H}_e F)(\xi)h_k^{\xi}(S,t)\right)\right| \leq C\xi^j e^{-\gamma\xi^2} \tag{3.26}$$

for some positive constants γ and C, which depend on S. Similarly for (3.25).

The same argument shows that

$$\frac{\partial^j}{\partial S^j}\left(\frac{b\beta\xi}{(\alpha b+\mu\beta)^2+\beta^2\xi^2}F(\xi)h_k^{\xi}(S,t)\right)$$

and

$$\frac{\partial}{\partial t}\left(\frac{-b\beta\xi}{(\alpha b+\mu\beta)^2+\beta^2\xi^2}F(\xi)h_k^{\xi}(S,t)\right)$$

are also integrable for $k = 1,2$ and $j = 0,1,2$.

Next, we suppose that $g(t) = \int_0^\infty G(x)e^{-xt}dx$, and for simplicity, we will suppose that $\|G\|_{L^1} = \int_0^\infty |G(x)|dx < \infty$. In fact, this assumption can be relaxed to allow $\int_0^\infty |e^{rx}G(x)|dx < \infty$ for some $r > 0$. We can also allow G to be a distribution. We will not go into these technicalities here, but below we will present an example where G is a distribution.

It immediately follows from our assumption that

$$K(\xi) = 2\xi\mathcal{L}^{-1}\left[g\left(\frac{2s}{\sigma^2}\right);\xi^2\right] = \sigma^2\xi G\left(\frac{\sigma^2\xi^2}{2}\right).$$

Consequently, a simple change of variables gives $\|K\|_{L^1} = \|G\|_{L^1} < \infty$. Arguing as previously, we see that

$$\frac{\partial^j}{\partial S^j}\left(\frac{b\beta\xi}{(\alpha b+\mu\beta)^2+\beta^2\xi^2}K(\xi)h_k^{\xi}(S,t)\right)$$

are integrable for $k = 1,2$ and $j = 0,1,2$, as is

$$\frac{\partial}{\partial t}\left(\frac{b\beta\xi}{(\alpha b+\mu\beta)^2+\beta^2\xi^2}K(\xi)h_k^{\xi}(S,t)\right).$$

Turning to $\mathcal{H}_e K$, we observe that if k is twice differentiable and k' and k'' are integrable, then the Fourier cosine transform $\mathcal{F}_c k$ is also integrable. This is a famous result and follows via integration by parts; see [26].

The observation that $|\mathcal{F}_c k| \leq \|k\|_{L^1}$ is elementary. So we have

$$|\mathcal{F}_s^{-1}(\mathcal{F}_c k)| \leq \frac{2}{\pi}|\mathcal{F}_c k| \leq \frac{2}{\pi}\|k\|_{L^1}.$$

Hence we can conclude that if G' and G'' are integrable, then $|\mathcal{H}_e K| \leq \frac{2}{\pi}\|G\|_{L^1}$.

This implies that for all $k = 1, 2$ and $j = 0, 1, 2$,

$$\frac{\partial^j}{\partial S^j}\left(\frac{(\alpha b + \mu\beta)^2}{(\alpha b + \mu\beta)^2 + \beta^2\xi^2}(\mathcal{H}_e K)(\xi)h_k^\xi(S,t)\right),$$

is integrable, as is

$$\frac{\partial}{\partial t}\left(\frac{(\alpha b + \mu\beta)^2}{(\alpha b + \mu\beta)^2 + \beta^2\xi^2}(\mathcal{H}_e K)(\xi)h_k^\xi(S,t)\right).$$

From this we see that if

$$w(S,t) = \int_0^\infty \varphi(\xi)h_1^\xi(S,t)d\xi + \int_0^\infty \psi(\xi)h_2^\xi(S,t)d\xi, \tag{3.27}$$

then three applications of the dominated convergence theorem allows us to differentiate under the integral sign with respect to t and twice with respect to S. Since $h_1^\xi(S,t)$ and $h_2^\xi(S,t)$ satisfy the PDE, it follows that (3.22) also solves the PDE. By construction it also satisfies the boundary conditions and initial data. □

Remark 3.4 Thus we have constructed a solution of the BVP that does not require us to know a fundamental solution for the problem. An alternative representation for the solution of a BVP can be very useful. It may be more tractable or more computationally efficient than the classical method. However, a discussion of this issue is beyond the scope of the current work. We content ourselves with an illustrative example.

Example 3.1 Let $g(t) = t$ and $f(S) = \frac{1}{\sigma^2}\left(\alpha + \frac{\mu\beta}{b}\right)^{-1}\left(\frac{S}{b}\right)^\mu\left(\ln\left(\frac{S}{b}\right)\right)^2$, and assume that $\alpha + \frac{\mu\beta}{b} \neq 0$. Now recall that if δ is the Dirac delta function, then by definition on $[0,\infty)$,

$$\int_0^\infty \delta'(x)f(x)dx = -\lim_{h\to 0^+} f'(h). \tag{3.28}$$

Hence $\int_0^\infty \delta'(x)e^{-sx}dx = s$. After some calculations, it turns out that we can take $\psi = 0$ and

$$\varphi(\xi) = \left(\alpha + \frac{\mu\beta}{b}\right)^{-1}\frac{4\xi}{\sigma^2}\delta'(\xi^2). \tag{3.29}$$

Consequently, a solution of our problem with $g(t) = t$ is

$$\begin{aligned} w(S,t) &= \int_0^\infty \frac{4\xi}{\sigma^2\left(\alpha + \frac{\mu\beta}{b}\right)}\delta'(\xi^2)\left(\frac{S}{b}\right)^\mu e^{-\frac{1}{2}\sigma^2 t(\xi^2+\mu^2)}\cos\left(\xi\ln\frac{S}{b}\right)d\xi \\ &= \frac{2\left(\frac{S}{b}\right)^\mu}{\sigma^2\left(\alpha + \frac{\mu\beta}{b}\right)}e^{-\frac{1}{2}\sigma^2\mu^2 t}\int_0^\infty \delta'(z)e^{-\frac{1}{2}\sigma^2 zt}\cos\left(\sqrt{z}\ln\frac{S}{b}\right)dz \end{aligned}$$

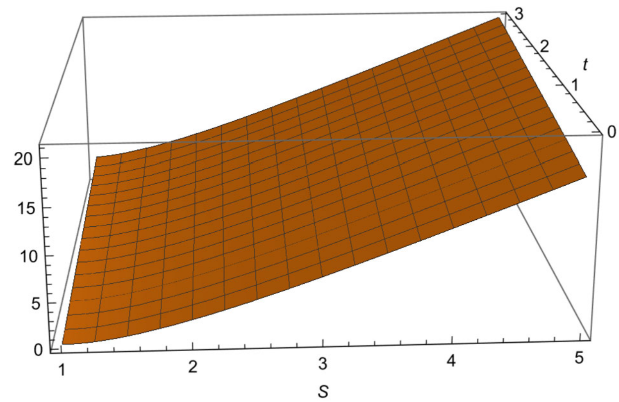

Figure 1 Plot of (3.30) for $S \in (1,5)$ and $t \in (0,3)$, which corresponds to the solution of the BVP (3.1)–(3.3) with $b = 1$, $f(S) = \frac{1}{\sigma^2}\left(\alpha + \frac{\mu\beta}{b}\right)^{-1}\left(\frac{S}{b}\right)^{\mu}\left(\ln\left(\frac{S}{b}\right)\right)^2$, $g(t) = t$, $\sigma = 0.5$, $r = 0.05$, $\alpha = 1$, $\beta = 1$, and $\mu = 1$

$$= \frac{2\left(\frac{S}{b}\right)^{\mu}}{\sigma^2\left(\alpha + \frac{\mu\beta}{b}\right)} e^{-\frac{1}{2}\sigma^2\mu^2 t}\left[\frac{\sigma^2 t}{2} + \frac{1}{2}\left(\ln\left(\frac{S}{b}\right)\right)^2\right]. \tag{3.30}$$

The apparent singularity at $z = 0$ is removable. We used a Taylor expansion to evaluate the integral. Specifically,

$$\begin{aligned}
&\int_0^\infty \delta'(z) e^{-\frac{1}{2}\sigma^2 zt}\cos\left(\sqrt{z}\ln\left(\frac{S}{b}\right)\right) dz = -\int_0^\infty \delta(z) \\
&\times\left(e^{-\frac{1}{2}\sigma^2 zt}\left(-\frac{\sigma^2 t}{2}\cos\left(\sqrt{z}\ln\left(\frac{S}{b}\right)\right) - \ln\left(\frac{S}{b}\right)\frac{\sin\left(\sqrt{z}\ln\left(\frac{S}{b}\right)\right)}{2\sqrt{z}}\right)\right) dz \\
&= \frac{\sigma^2 t}{2} + \ln\left(\frac{S}{b}\right)\int_0^\infty \delta(z)\left[\frac{1}{2}e^{-\frac{1}{2}\sigma^2 zt}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\left(\ln\left(\frac{S}{b}\right)\right)^{2n+1} z^n\right] dz \\
&= \frac{\sigma^2 t}{2} + \frac{1}{2}\left(\ln\left(\frac{S}{b}\right)\right)^2.
\end{aligned}$$

Note that all terms in the infinite series are zero after $n = 0$. This gives the result. It is not hard to see that

$$w(S,0) = \frac{1}{\sigma^2}\left(\alpha + \frac{\mu\beta}{b}\right)^{-1}\left(\frac{S}{b}\right)^{\mu}\left(\ln\left(\frac{S}{b}\right)\right)^2.$$

So the initial condition is satisfied. Using Mathematica, it is easy to see that the boundary conditions are satisfied. Choosing g to be a polynomial leads to the appearance of derivatives of the Dirac delta, and integrals involving these distributions particularly easy to evaluate.

In Fig. 1, we plot the solution (3.30) for $S \in (1,5)$ and $t \in (0,3)$.

4 Hilbert transform methods for second-order boundary conditions

The result of the previous section give rise to a number of questions. The first is whether the Hilbert transform approach can be extended to the second-order Robin problem? A more important question is whether the method can be applied to boundary value problems for other PDEs? The answer to both questions is positive.

We start with the extension to the second-order Robin problem and then turn to the second question. We will solve

$$w_t = \frac{1}{2}\sigma^2 S^2 w_{SS} + rSw_S, \tag{4.1}$$

$$w(S,0) = f(S), \tag{4.2}$$

$$\alpha w(b,t) + \beta w_S(b,t) + \gamma w_{SS}(b,t) = e^{-\frac{1}{2}\sigma^2\mu^2 t} g(t), \tag{4.3}$$

where $S > b > 0$.

In fact, the same method can be used to solve this problem with an extra step. Our solution will again be of the form (3.12) as in the previous section. It is straightforward to show that $w(b,0) = \int_0^\infty \varphi(\xi)d\xi = f(b)$. This fact will be useful.

As in the regular Robin condition case, we deduce that $\varphi(\xi) = F(\xi) + (\mathcal{H}_o\psi)(\xi)$, where F is as before. Here we make the observation that

$$\int_0^\infty (\mathcal{H}_o\psi)(\xi)d\xi = \int_0^\infty (\varphi(\xi) - F(\xi))d\xi = f(b) - \int_0^\infty F(\xi)d\xi.$$

Here we assume that F is integrable. This can be guaranteed by imposing mild conditions on f. We will give a sufficient condition for integrability below.

Introduce the constants $A = \alpha + \frac{1}{b^2}\gamma\mu(\mu-1) + \frac{\mu\beta}{b}$ and $B = \frac{\beta}{b} + \frac{\gamma}{b^2}(2\mu - 1)$. Then the boundary condition yields the equation

$$\int_0^\infty (A - \frac{\gamma}{b^2}\xi^2)\varphi(\xi)e^{-\frac{1}{2}\sigma^2 t\xi^2}d\xi + B\int_0^\infty \xi\psi(\xi)e^{-\frac{1}{2}\sigma^2 t\xi^2}d\xi = g(t). \tag{4.4}$$

Converting the integrals to Laplace transforms and inverting as previously, we obtain

$$(A - \frac{\gamma}{b^2}\xi^2)\varphi(\xi) + B\xi\psi(\xi) = 2\xi\mathcal{L}^{-1}\left[g\left(\frac{2s}{\sigma^2}\right);\xi^2\right].$$

We will insist that A and γ have opposite signs to ensure that φ is nonsingular. Let $G(\xi) = 2\xi\mathcal{L}^{-1}\left[g\left(\frac{2s}{\sigma^2}\right);\xi^2\right]$. Then we can rewrite the equation as

$$(A - \frac{\gamma}{b^2}\xi^2)[F(\xi) + (\mathcal{H}_0\psi)(\xi)] + B\xi\psi(\xi) = G(\xi).$$

or, equivalently,

$$A(\mathcal{H}_0\psi)(\xi) - \frac{\gamma}{b^2}\xi^2(\mathcal{H}_0\psi)(\xi) + B\xi\psi(\xi) = G(\xi) - (A - \frac{\gamma}{b^2}\xi^2)F(\xi). \tag{4.5}$$

To proceed, we need to take the even Hilbert transform of both sides. We make the observation that

$$\begin{aligned}\mathcal{H}_e(y^2(\mathcal{H}_0\psi)(y))(x) &= \frac{2x}{\pi}\mathrm{P}\int_0^\infty \frac{y^2(\mathcal{H}_0\psi)(y)}{x^2 - y^2}dy \\ &= \frac{2x}{\pi}\mathrm{P}\int_0^\infty \frac{(y^2 - x^2 + x^2)(\mathcal{H}_0\psi)(y)}{x^2 - y^2}dy\end{aligned}$$

$$= -\frac{2x}{\pi}\mathrm{P}\int_0^\infty (\mathcal{H}_0\psi)(y)dy + x^2(\mathcal{H}_e\mathcal{H}_o\psi)(x)$$
$$= -\frac{2x}{\pi}\left(f(b) - \mathrm{P}\int_0^\infty F(y)dy\right) - x^2\psi(x)$$
$$= Cx - x^2\psi(x), \tag{4.6}$$

where $C = -\frac{2}{\pi}\left(f(b) - \int_0^\infty F(y)dy\right)$. We dropped the principal value because we are assuming suitable integrability. We can of course weaken this assumption and reintroduce the principal values of the integrals.

Taking the even Hilbert transform of (4.5) gives us the relation

$$-(A - \frac{\gamma}{b^2}\xi^2)\psi(\xi) - \frac{\gamma}{b^2}C\xi + B\xi(\mathcal{H}_0\psi)(\xi) = \mathcal{H}_e[G(\xi) - (A - \frac{\gamma}{b^2}\xi^2)F(\xi)]. \tag{4.7}$$

Now we easily see that

$$(\mathcal{H}_e x^2 F(x))(\xi) = -\frac{2\xi}{\pi}\mathrm{P}\int_0^\infty F(x)dx + \xi^2(\mathcal{H}_e F)(\xi).$$

So our assumptions on F and the assumptions of the previous section are sufficient to guarantee that the right-hand side of (4.7) exists.

We know that $\varphi(\xi) = \frac{G(\xi)-B\xi\psi(\xi)}{A-\frac{\gamma}{b^2}\xi^2}$, so that

$$(\mathcal{H}_0\psi)(\xi) = \frac{G(\xi) - B\xi\psi(\xi)}{A - \frac{\gamma}{b^2}\xi^2} - F(\xi).$$

We thus have

$$-A\psi(\xi) - \frac{\gamma}{b^2}(C\xi - \xi^2\psi(\xi)) + B\xi\left(\frac{G(\xi) - B\xi\psi(\xi)}{A - \frac{\gamma}{b^2}\xi^2} - F(\xi)\right) = M(\xi)$$

with $M(\xi) = \mathcal{H}_e[G(\xi) - (A - \frac{\gamma}{b^2}\xi^2)F(\xi)]$.

This gives

$$-\left(A - \frac{\gamma}{b^2}\xi^2 + \frac{B^2\xi^2}{A - \frac{\gamma}{b^2}\xi^2}\right)\psi = C\frac{\gamma}{b^2}\xi + M(\xi) + B\xi F(\xi) - B\xi\frac{G(\xi)}{A - \frac{\gamma}{b^2}\xi^2}.$$

Let

$$N(\xi) = C\frac{\gamma}{b^2}\xi + M(\xi) + B\xi F(\xi) - B\xi\frac{G(\xi)}{A - \frac{\gamma}{b^2}\xi^2}. \tag{4.8}$$

Then we have

$$\psi(\xi) = \frac{-N(\xi)(A - \frac{\gamma}{b^2}\xi^2)}{(A - \frac{\gamma}{b^2}\xi^2)^2 + B^2\xi^2}. \tag{4.9}$$

Combining this, we have the following result.

Theorem 4.1 *Suppose that f and g satisfy the conditions in Theorem 3.3. Suppose further that $\eta f'(\eta), \eta^2 f''(\eta) \in L^1\left((b,\infty), \frac{d\eta}{\eta^{\mu+1}}\right)$. Let F, G be as in Theorem 3.3. Then Problem 4.1 with $\alpha, \beta, \gamma \neq 0$ has a solution given by*

$$w(S,t) = \int_0^\infty \varphi(\xi) h_1^\xi(S,t)d\xi + \int_0^\infty \psi(\xi) h_2^\xi(S,t)d\xi, \tag{4.10}$$

where $h_{1,2}^\xi(S,t)$ are as in Theorem 3.3,

$$\psi(\xi) = \frac{-N(\xi)(A - \frac{\gamma}{b^2}\xi^2)}{(A - \frac{\gamma}{b^2}\xi^2)^2 + B^2\xi^2}, \tag{4.11}$$

and

$$\varphi(\xi) = \frac{G(\xi) - B\xi\psi(\xi)}{A - \frac{\gamma}{b^2}\xi^2},$$

where $A = \alpha + \frac{1}{b^2}\gamma\mu(\mu-1) + \frac{\mu\beta}{b}$, $B = \frac{\beta}{b} + \frac{\gamma}{b^2}(2\mu - 1)$, N is as defined in (4.8), and A and γ have opposite signs.

Proof The proof follows the lines of the proof of Theorem 3. So here we only consider the integrability of F. We recall that

$$F(\xi) = \frac{2}{\pi}\int_0^\infty e^{-\mu y} f\left(be^y\right)\cos(\xi y)dy.$$

Obviously, we can write

$$\int_0^\infty F(\xi)d\xi = \int_0^1 F(\xi)d\xi + \int_1^\infty F(\xi)d\xi.$$

A simple application of Fubini's theorem shows that $|\int_0^1 F(\xi)d\xi| < \infty$, provided that

$$\int_0^1 \left|e^{-\mu y} f\left(be^y\right)\right| \frac{dy}{y} < \infty.$$

The obvious change of variables gives us

$$\int_0^1 \left|e^{-\mu y} f\left(be^y\right)\right| \frac{dy}{y} = b^\mu \int_b^{be} |f(\eta)| \frac{d\eta}{\eta^{\mu+1}}.$$

Thus $f \in L^1((b,\infty), \frac{d\eta}{\eta^{\mu+1}})$ guarantees that $|\int_0^1 F(\xi)d\xi| < \infty$.

Now we assume that f is twice differentiable. Integration by parts gives us

$$\begin{aligned}\int_1^\infty \int_0^\infty e^{-\mu y} f\left(be^y\right)\cos(\xi y)dy &= \int_1^\infty \int_0^\infty \frac{d}{dy}\left(e^{-\mu y} f\left(be^y\right)\right)\frac{\sin(\xi y)}{-\xi}dyd\xi \\ &= \int_1^\infty \left(\left[\frac{d}{dy}(e^{-\mu y} f\left(be^y\right))\frac{\cos(\xi y)}{\xi^2}\right]_0^\infty\right.\end{aligned}$$

$$- \int_0^\infty \frac{d^2}{dy^2}(e^{-\mu y} f\left(be^y\right)) \frac{\cos(\xi y)}{\xi^2} dy \Big) d\xi$$
$$= \int_1^\infty \frac{\mu f(b) - bf'(b)}{\xi^2} d\xi - I,$$

where $I = \int_1^\infty \int_0^\infty \frac{d^2}{dy^2}(e^{-\mu y} f(be^y)) \frac{\cos(\xi y)}{\xi^2} dy d\xi$. Clearly,

$$\int_1^\infty \frac{\mu f(b) - bf'(b)}{\xi^2} d\xi = \mu f(b) - bf'(b).$$

Further,

$$|I| = \left| \int_1^\infty \int_0^\infty \frac{d^2}{dy^2}(e^{-\mu y} f\left(be^y\right)) \frac{\cos(\xi y)}{\xi^2} dy d\xi \right|$$
$$\leq \int_1^\infty \int_0^\infty \left| \frac{d^2}{dy^2}(e^{-\mu y} f\left(be^y\right)) \frac{\cos(\xi y)}{\xi^2} \right| dy d\xi$$
$$\leq \int_0^\infty \left| \frac{d^2}{dy^2}(e^{-\mu y} f\left(be^y\right)) \right| dy < \infty,$$

where we have employed Fubini's theorem on the assumption that the final integral is finite. Thus the integrability of $\frac{d^2}{dy^2}(e^{-\mu y} f(be^y))$ is sufficient to guarantee that $|\int_0^\infty F(\xi) d\xi| < \infty$. Since

$$\int_0^\infty \left| \frac{d^2}{dy^2}(e^{-\mu y} f\left(be^y\right)) \right| dy = b^\mu \int_b^\infty \left| (Df)(\xi) \right| \frac{d\eta}{\eta^{\mu+1}},$$

where $(Df)(\xi) = \mu^2 f(\eta) + b(1-2\mu)\eta f'(\eta) + \mu^2 \eta^2 f''(\eta)$, it is sufficient to require $f(\eta), \eta f'(\eta), \eta^2 f''(\eta) \in L^1((b,\infty), \frac{d\eta}{\eta^{\mu+1}})$ for the integral of F to exist. This also justifies our use of Fubini's theorem.

The proof of the integrability of $\varphi(\xi) h_1^\xi(S,t)$ and $\psi(\xi) h_2^\xi(S,t)$ and the differentiability of the integrals defining the solution proceeds along the lines of the proof of Theorem 3.3. □

Again, we have a representation of the solution of the second-order Robin problem that does not rely upon any knowledge of a fundamental solution. Given the complexity of the fundamental solution in the second-order case, this may be preferred. The problem of efficient evaluation of the solutions will be treated elsewhere. However, we remark that the even and odd Hilbert transforms can be written in terms of the Fourier sine and cosine transforms. Hence techniques for the evaluation of these transforms can be employed to calculate the even and odd Hilbert transforms. In particular, we has access to fast Fourier transform methods.

5 A general result for second-order Robin problems

It is possible to construct general formulae for different types of boundary conditions. We present one particular case for second-order Robin conditions. In the simplest version of the theory, we need a family of solutions of the form

$$w_1(x,t;\xi) = \rho(x) \cos(\xi \mu(x)) e^{-\xi^2 t}, \tag{5.1}$$

$$w_2(x,t;\xi) = \rho(x)\sin(\xi\mu(x))e^{-\xi^2 t}, \tag{5.2}$$

where μ and ρ are assumed to be twice continuously differentiable. There are many variations on this, and we will present some examples below. By linearity, if w_1 and w_2 are solutions of a linear PDE, then so are

$$\bar{w}_1(x,t;\xi) = \rho(x)\cos(\xi(\mu(x)-\mu(a)))e^{-\xi^2 t}, \tag{5.3}$$

$$\bar{w}_2(x,t;\xi) = \rho(x)\sin(\xi(\mu(x)-\mu(a)))e^{-\xi^2 t}. \tag{5.4}$$

In fact, many PDEs on the line have solutions of this form. A simple characterization can be obtained as follows.

If w_1 is a solution of the PDE $u_t = P(x)u_{xx} + Q(x)u_x + R(x)u$, where P, Q, R are smooth functions, then we must have

$$\begin{aligned}
&P(x)(\rho''(x)\cos(\xi\mu(x)) - 2\xi\rho'(x)\sin(\xi\mu(x)) - \xi^2\rho(x)\cos(\xi\mu(x)))\\
&+ Q(x)(\rho'(x)\cos(\xi\mu(x)) - \xi\rho(x)\mu'(x)\sin(\xi\mu(x))\\
&+ R(x)(\rho(x)\cos(\xi\mu(x)) + \xi^2) = 0.
\end{aligned}$$

Substitution into the PDE shows that for this to be a solution, we require P, Q, R, μ, and ρ to satisfy the following system of equations:

$$P(x)\rho''(x) + Q(x)\rho(x) + R(x)\rho(x) = 0, \tag{5.5}$$

$$1 - P(x)(\mu'(x))^2 = 0, \tag{5.6}$$

$$Q(x)\rho(x)\mu'(x) + 2P(x)\mu'(x)\rho'(x) + P(x)\rho(x)\mu''(x) = 0. \tag{5.7}$$

This implies that $\mu'(x) = \pm(P(x))^{-1/2}$.

Obviously, ρ must be a time-independent solution of the PDE. It is not hard to show that $w_2(x,t;\xi)$ will be a solution of the PDE whenever w_1 is and vice versa. Most PDEs will not possess solutions of this form, but there are numerous PDEs that do. These defining equations allow us to generate examples, and we will present some below.

Now we present a general result for the solution of second-order Robin problems, where the PDE has solutions of this form. We observe that by taking the limit as $\gamma \to 0$ we obtain the solution of the standard Robin problem. There are various particular cases that we will not cover. They can all be handled by obvious modifications of our arguments. We make several simplifying assumptions to make the proof easier. More precise conditions are possible, but we will not present a full analysis here. The actual calculations are essentially the same as in the Black–Scholes case, so we leave these to the reader.

Theorem 5.1 *Suppose that the PDE*

$$u_t = P(x)u_{xx} + Q(x)u_x + R(x)u,\ x > b, t > 0, \tag{5.8}$$

has solutions given by (5.3) *and* (5.4). *Suppose that* $\mu : [b,\infty) \to \mathbb{R}$ *is invertible, g has an inverse Laplace transform, and* $\tilde{f} \in L^1[0,\infty)$, *where* $\tilde{f}(y) = \frac{f(\mu^{-1}(y+\mu(b)))}{\rho(\mu^{-1}(y+\mu(b)))}$.

Define the constants $A = \alpha\rho(b) + \beta\rho'(b) + \gamma\rho''(b)$, $B = -\gamma\rho(b)(\mu'(b))^2$, $C = \beta\rho(b)\mu'(b) + \gamma(\rho(b)\mu''(b) + 2\rho'(b)\mu'(b))$, $\chi = \frac{2B}{\pi}\left(c - \int_0^\infty F(\xi)d\xi\right)$, *and* $c = \lim_{z\to b}\frac{f(z)}{\rho(z)}$, *which we assume to exist. Suppose that A and B are nonzero and have opposite signs.*

Let $F(\xi) = \frac{2}{\pi}\int_0^\infty \tilde{f}(\eta)\cos(\xi\eta)d\eta$ *and* $G(\xi) = 2\xi\mathcal{L}^{-1}[g(t);\xi^2]$, *and suppose that* $K(\xi) = G(\xi) - (A - B\xi^2)F(\xi) \in L^2[0,\infty)$. *Then a solution of* (5.8) *satisfying* $u(x,0) = f(x)$ *and* $\alpha u(b,t) + \beta u_x(b,t) + \gamma u_{xx}(b,t) = g(t)$ *is*

$$u(x,t) = \int_0^\infty \varphi(\xi)\rho(x)\cos(\xi(\mu(x) - \mu(b)))e^{-\xi^2 t}d\xi + \int_0^\infty \psi(\xi)\rho(x)\sin(\xi(\mu(x) - \mu(b)))e^{-\xi^2 t}d\xi, \tag{5.9}$$

where

$$\psi(\xi) = -\frac{(A - B\xi^2)((\mathcal{H}_e K)(\xi) - \chi\xi) - C\xi K(\xi)}{(A - B\xi^2)^2 + C^2\xi^2}, \tag{5.10}$$

and

$$\varphi(\xi) = \frac{G(\xi) - C\xi\psi(\xi)}{A - B\xi^2}. \tag{5.11}$$

Proof The derivation of the solutions follows the same lines as in the Black–Scholes case. The initial condition gives $\varphi(\xi) = F(\xi) + (\mathcal{H}_o\psi)(\xi)$, and the boundary condition gives $[A - B\xi^2]\varphi(\xi) + C\xi\psi(\xi) = G(\xi)$. We immediately obtain

$$[A - B\xi^2]\psi(\xi) + C\xi\psi(\xi) = K(\xi).$$

Taking the even Hilbert transform of both sides gives us

$$-(A - B\xi^2)\psi(\xi) + C\xi(\mathcal{H}_o)\psi)(\xi) + \chi\xi = (\mathcal{H}_e K)(\xi). \tag{5.12}$$

We therefore have a pair of simultaneous equations. Solving gives us our expressions for ψ and φ.

Now $w_1(x,t;\xi)$ and $w_2(x,t;\xi)$ are solutions of the PDE, and they have Gaussian decay in ξ. By Riesz's inequality (see [24]) there exists a constant $\mathcal{R}$ such that $\|\mathcal{H}_e K\|_2 \leq \mathcal{R}\|K\|_2$. It immediately follows from Hölder's inequality and our assumptions on F and G that the integral defining the solution is convergent. Similarly, the integrals

$$\int_0^\infty \eta(\xi)\frac{\partial^j}{\partial x^j}w_{1,2}(x,t;\xi)d\xi,$$

$j = 1, 2$, and

$$\int_0^\infty \eta(\xi)\frac{\partial}{\partial t}w_{1,2}(x,t;\xi)d\xi$$

are convergent. Here η is either φ or ψ. Therefore (5.9) is a solution of the PDE, and by construction it satisfies the boundary and initial conditions. □

5.1 Some examples with elementary solutions

Using the previous remarks, we can generate an interesting variety of examples.

Example 5.1 For the heat equation with drift

$$u_t = u_{xx} + au_x,\ x > 0, t > 0, \tag{5.13}$$

we let $u(x,0) = f(x)$ and $\alpha u(0,t) + \beta u_x(0,t) + \gamma u_{xx}(0,t) = e^{-\frac{a^2}{4}t}g(t)$. The elementary solutions are $w_1(x,t;\xi) = e^{-\xi^2 t-\frac{a^2}{4}t}e^{-\frac{a}{2}x}\cos(\xi x)$ and $w_2(x,t;\xi) = e^{-\xi^2 t-\frac{a^2}{4}t}e^{-\frac{a}{2}x}\sin(\xi x)$. Note that the theorem is still valid because the additional factor of $e^{-\frac{a^2}{4}t}$ will cancel when we apply the boundary conditions.

Example 5.2 An important class of stochastic processes are squared Bessel processes; see [27]. An n-dimensional squared Bessel process is the squared distance from the origin of an n-dimensional Brownian motion. For the three-dimensional case, we have

$$u_t = 2xu_{xx} + 3u_x,\ x \geq b > 0, t > 0. \tag{5.14}$$

The elementary solutions are

$$w_1(x,t;\lambda) = \frac{1}{\sqrt{x}}e^{-\lambda^2 t}\cos(\lambda(\sqrt{2x} - \sqrt{2b}))$$

and

$$w_2(x,t;\lambda) = \frac{1}{\sqrt{x}}e^{-\lambda^2 t}\sin(\lambda(\sqrt{2x} - \sqrt{2b})).$$

Choosing $u(x,0) = 0$, $b = 1$, and the boundary condition $u(1,t) = \frac{1}{\sqrt{t+1}}$, we can explicitly solve the corresponding BVP. Indeed, Theorem 5.1 gives

$$u(x,t) = \frac{e^{-\frac{(\sqrt{x}-1)^2}{2(t+1)}}}{\sqrt{t+1}\sqrt{x}} - \frac{e^{-\frac{(\sqrt{x}-1)^2}{2(t+1)}}\operatorname{erf}\left(\frac{\sqrt{x}-1}{\sqrt{2}\sqrt{t(t+1)}}\right)}{\sqrt{t+1}\sqrt{x}}. \tag{5.15}$$

In Fig. 2, we plot solution (5.15) for $x \in (1,4)$ and $t \in (0,5)$.

Example 5.3 We consider the PDE

$$u_t = u_{xx} + 2(\tanh x)u_x,\ x > 0, t > 0. \tag{5.16}$$

The elementary solutions are

$$w_1(x,t;\xi) = e^{-\xi^2 t}\frac{\cos(\xi x)}{\cosh x},\ w_2(x,t;\xi) = e^{-\xi^2 t}\frac{\sin(\xi x)}{\cosh x}.$$

The solution to the BVP will be of the form (5.9).

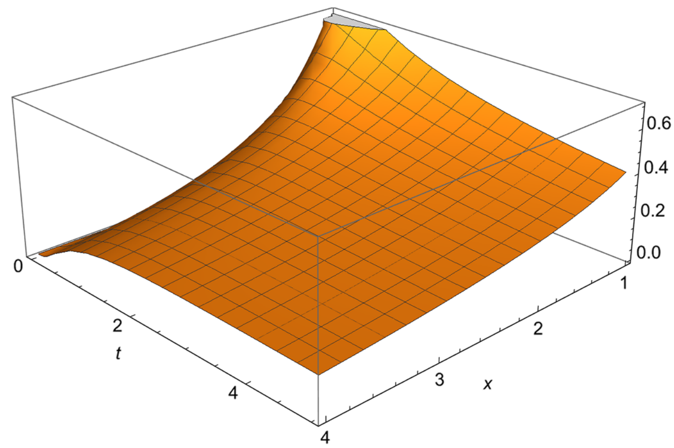

Figure 2 Plot of (5.15) for $x \in (1,4)$ and $t \in (0,5)$, which corresponds to the solution of the PDE (5.14) with $b = 1$, initial condition $u(x,0) = 0$, and boundary condition $u(1,t) = \frac{1}{\sqrt{t+1}}$

We also remark that the PDE $u_t = u_{xx} + 2(\coth x)u_x$ arises in hyperbolic geometry. See the discussion in [28]. The elementary solutions are

$$w_1(x,t;\xi) = e^{-\xi^2 t}\frac{\cos(\xi x)}{\sinh x},\ w_2(x,t;\xi) = e^{-\xi^2 t}\frac{\sin(\xi x)}{\sinh x}.$$

To avoid singularities, we must place the lower boundary at $x = b > 0$.

Example 5.4 Next, we consider a family of PDEs of the form

$$u_t = u_{xx} + (2x^2 + a)u_x + (x^4 + ax^2 + 2x)u,\ x > 0, t > 0.$$

The elementary solutions that we use are

$$w_1(x,t;\xi) = e^{-\frac{1}{4}(\xi^2+a^2)t}e^{-\frac{1}{3}x^3-\frac{1}{2}ax}\cos\left(\frac{x\xi}{2}\right),$$
$$w_2(x,t;\xi) = e^{-\frac{1}{4}(\xi^2+a^2)t}e^{-\frac{1}{3}x^3-\frac{1}{2}ax}\sin\left(\frac{x\xi}{2}\right).$$

Variants of the Robin boundary condition can also be studied, and we present some examples.

Example 5.5 Naturally, we cannot obtain a solution where none exists. Boundary conditions must be suitable to the PDE. We illustrate this with the equation

$$u_t = (x^2 - 1)u_{xx} + xu_x,\ x > 1, t > 0. \tag{5.17}$$

We require $u(1,t) = f(x)$ and $\alpha u(1,t) + \beta\sqrt{x^2-1}u_x(x,t)\big|_{x\to 1} = g(t)$. It is not clear that the standard Robin problem at $x = 1$ even has a solution. This is clearly not covered by Theorem 5.1. Nevertheless, the same construction works as in that result. Our elementary solutions are

$$w_1(x,t;\xi) = e^{-\xi^2 t}\cos\left(\xi\cosh^{-1}x\right),\ w_2(x,t;\xi) = e^{-\xi^2 t}\sin\left(\xi\cosh^{-1}x\right).$$

From the initial data we obtain $\varphi(\xi) = F(\xi) + \mathcal{H}_o\psi)(\xi)$ with $F(\xi) = \frac{2}{\pi}\int_0^\infty f(\cosh y)\cos(\xi y)dy$. The boundary condition gives

$$\alpha\varphi(\xi) + \beta\xi\psi(\xi) = 2\xi\mathcal{L}^{-1}[g(t);\xi^2],$$

and we have solved equations like those before.

5.1.1 Extension to time-dependent coefficients

We can also study problems with time-dependent coefficients. A full discussion would be lengthy, so we only make some brief remarks. Suppose that $r : [0,\infty) \to (0,\infty)$ is a continuous positive function. Suppose that the PDE (5.8) has solutions given by (5.3) and (5.4). Then the PDE

$$\frac{1}{r(t)}u_t = P(x)u_{xx} + Q(x)u_x + R(x)u, \; x > b > 0, \tag{5.18}$$

has solutions given by

$$\bar{v}_1(x,t;\xi) = \rho(x)\cos(\xi(\mu(x) - \mu(b)))e^{-\xi^2\int_0^t r(s)ds}, \tag{5.19}$$

$$\bar{v}_2(x,t;\xi) = \rho(x)\sin(\xi(\mu(x) - \mu(b)))e^{-\xi^2\int_0^t r(s)ds}. \tag{5.20}$$

The extension of Theorem 5.8 to (5.18) subject to the boundary condition (1.1) is entirely straightforward, and we leave it to the interested reader. See however the remark in Sect. 7.

Many classes of equations with time-dependent coefficients can be solved, and we make no attempt to list them all here, but, as an example, we simply present equations of the form

$$u_t = \sigma(t)u_{xx} + (A(t) - B(t)x)u_x, \; x > 0, \sigma(t) > 0. \tag{5.21}$$

Here σ, A, B are continuous. Equations of this type admit solutions of the form

$$u_c(x,t) = \exp(-\xi^2k(t))\cos\left(\xi(xe^{-\int_0^t B(s)ds} + C(t))\right), \tag{5.22}$$

$$u_s(x,t) = \exp(-\xi^2k(t))\sin\left(\xi(xe^{-\int_0^t B(s)ds} + C(t))\right), \tag{5.23}$$

where $k(t) = \int_0^t \sigma(y)e^{-2\int_0^y B(s)ds}dy$ and $C(t) = \int_0^t A(y)e^{-\int_0^y B(s)ds}dy$. A variety of boundary value problems for (5.21) can be solved by our method.

We take our solutions to be of the form

$$\begin{aligned} u(x,t) = &\int_0^\infty \varphi(\xi)\exp(-\xi^2k(t))\cos\left(\xi(xe^{-\int_0^t B(s)ds} + C(t))\right)d\xi \\ &+ \int_0^\infty \psi(\xi)\exp(-\xi^2k(t))\sin\left(\xi(xe^{-\int_0^t B(s)ds} + C(t))\right)d\xi. \end{aligned}$$

Since $k(0) = C(0) = 0$, we have

$$u(x,0) = \rho(x)(\widehat{\varphi}_c(x) + \widehat{\psi}_s(x)) = f(x).$$

This is the same condition that we had before. There are various cases where the equations resulting from the boundary conditions can also be solved using the odd and even Hilbert transforms; however, some cases are challenging. If $A(t) = 0$, then the Neumann and Dirichlet problems are reasonably straightforward. For example, the inhomogeneous Dirichlet problem yields

$$u(0,t) = \int_0^\infty \varphi(\xi) e^{-\xi^2 k(t)} d\xi = g(t). \tag{5.24}$$

Setting $\xi^2 = z$, we have the Laplace transform $\tilde{\Phi}(k(t)) = g(t)$. Here $\tilde{\Phi}$ is the Laplace transform of $\tilde{\varphi}(z) = \frac{\varphi(\sqrt{z})}{2\sqrt{z}}$. Since k is increasing, it is invertible, and we write $\tilde{\Phi}(s) = g\left(k^{-1}(s)\right)$; this can be inverted explicitly for a large number of cases. This gives us a solution of the Dirichlet problem.

5.2 A modified Robin problem for the harmonic oscillator

The equation for the harmonic oscillator is

$$u_t = \sigma u_{xx} - \mu x^2 u, \quad x \in \Omega \subseteq \mathbb{R}, \mu > 0. \tag{5.25}$$

This plays an important role in quantum mechanics, though it would normally be in the form $iu_t = \sigma x_{xx} - \mu x^2 u$. The real form also plays an important role in the theory of heat kernels. See Davies' book [28].

It also arises in stochastic analysis. By the Feynman–Kac formula the functional $u(x,t) = \mathbb{E}\left[f(X_t)e^{-\mu\int_0^t X_s^2 ds} \middle| X_0 = x\right]$, where $X = \{X_t; t \geq 0\}$ is a Brownian motion, is given by the solution of the problem

$$u_t = \frac{1}{2}u_{xx} - \mu x^2 u, x \in \mathbb{R},$$
$$u(x,0) = f(x).$$

If $f(x) = e^{-\lambda x}$, then the solution of the initial value problem gives the Laplace transform of the joint density of $(X_t, \int_0^t X_s^2 ds)$. Our intention is to solve a modified Robin problem of a special form. However, we note here that the Dirichlet and Neumann problems can easily be solved by our method. In fact, there are numerous problems that we can solve. The standard Robin problem reduces to the solution of a different type of integral equation of Laplace transform convolution type. We will present it elsewhere.

We will solve the problem

$$u_t = u_{xx} - x^2 u, \ x \geq 0,$$
$$u(x,0) = f(x),$$
$$\alpha u(0,t) + \beta e^{-2t} u_x(0,t) = g(t).$$

We will suppose that α and β are nonzero. It is worth noting that if we map this to the heat equation, then we obtain a very complicated moving boundary problem, which appears to be intractable.

The boundary condition describes a situation where the lower boundary starts off as partially absorbing and partially reflecting and exponentially decays to a purely absorbing boundary. Such a situation can arise in many settings, such as the design of materials which reflect, say, alpha or beta particles. These can cause serious damage to living tissue.

We can let u be the amount of radioactive material that has been absorbed. As the protective material breaks down, the proportion reflected decreases to zero, and so the boundary will become purely absorbing. For small t, the reflectivity will decay linearly, but after a certain point is reached, the reflectivity will evaporate exponentially fast. These are important considerations in the design of systems to protect against radiation. There is a very vast literature on this subject, and we can only suggest a survey [29]. Our problem may be regarded as a toy problem that could be of value. However, we present it here purely for its mathematical interest. We prove the following theorem.

Theorem 5.2 *Let $f \in L^1([0,\infty)$ and $g(t) = \int_0^\infty G(x)e^{-xt}dx$, where $G \in L^1\left([0,\infty), \frac{dx}{\Gamma\left(\frac{1+x}{4}\right)}\right)$. Then the boundary value problem*

$$\begin{aligned} u_t &= u_{xx} - x^2 u, \; x \geq 0, \\ u(x,0) &= f(x), \\ \alpha u(0,t) + \beta e^{-2t}u_x(0,t) &= g(t), \end{aligned}$$

where α, β are nonzero, has a solution given by

$$u(x,t) = \int_0^\infty \varphi(\lambda)w_1(x,t;\lambda)d\lambda + \int_0^\infty \psi(\lambda)w_2(x,t;\lambda)d\lambda. \tag{5.26}$$

Here

$$\begin{aligned} w_1(x,t;\lambda) &= \exp\left(t + \frac{1}{2}x^2 - \frac{1}{4}\lambda^2(e^{4t}-1)\right)\cos(\lambda x e^{2t}), \\ w_2(x,t;\lambda) &= \exp\left(t + \frac{1}{2}x^2 - \frac{1}{4}\lambda^2(e^{4t}-1)\right)\sin(\lambda x e^{2t}), \end{aligned}$$

$$\psi(\lambda) = \frac{\alpha\left[\alpha(\mathcal{H}_e F)(\lambda) - (\mathcal{H}_e K)(\lambda) + \frac{\beta\lambda}{\alpha}K(\lambda) - \lambda\beta F(\lambda)\right]}{\alpha^2 + \beta^2\lambda^2},$$

$\varphi(\lambda) = \frac{1}{\alpha}K(\lambda) - \frac{\lambda\beta}{\alpha}\psi(\lambda)$, $F(\lambda) = \frac{2}{\pi}\int_0^\infty e^{-\frac{1}{2}y^2}f(y)\cos(\lambda y)dy$, $K(\lambda) = \frac{\lambda}{2}\mathcal{L}^{-1}\left[\bar{g}(s); \frac{\lambda^2}{4}\right]$, and $\bar{g}(s) = \frac{1}{(1+s)^{\frac{1}{4}}}g\left(\frac{1}{4}\ln(1+s)\right)$.

Proof We begin with the following solutions of the PDE:

$$w_1(x,t;\lambda) = \exp\left(t + \frac{1}{2}x^2 - \frac{1}{4}\lambda^2(e^{4t}-1)\right)\cos(\lambda x e^{2t}), \tag{5.27}$$

$$w_2(x,t;\lambda) = \exp\left(t + \frac{1}{2}x^2 - \frac{1}{4}\lambda^2(e^{4t}-1)\right)\sin(\lambda x e^{2t}). \tag{5.28}$$

It is straightforward to check that these satisfy the equation for every λ. (Notice that they are not however of the form (5.1) or (5.2).)

As usual, we construct a new solution of the PDE by setting

$$u(x,t) = \int_0^\infty \varphi(\lambda) w_1(x,t;\lambda)d\lambda + \int_0^\infty \psi(\lambda) w_2(x,t;\lambda)d\lambda. \tag{5.29}$$

The initial condition then implies that

$$e^{\frac{1}{2}x^2}\left[\widehat{\varphi}_c(x) + \widehat{\psi}_s(x)\right] = f(x). \tag{5.30}$$

Reasoning as previously, we deduce that

$$\varphi(\lambda) = \frac{2}{\pi}\int_0^\infty e^{-\frac{1}{2}y^2} f(y)\cos(\lambda y)dy + (\mathcal{H}_o\psi)(\lambda). \tag{5.31}$$

We set $F(\lambda) = \frac{2}{\pi}\int_0^\infty e^{-\frac{1}{2}y^2} f(y)\cos(\lambda y)dy$.

Now we easily see that

$$u(0,t) = \int_0^\infty \varphi(\lambda)\exp\left(t - \frac{1}{4}\lambda^2(e^{4t}-1)\right)d\lambda \tag{5.32}$$

and

$$u_x(0,t) = \int_0^\infty \lambda\psi(\lambda)\exp\left(3t - \frac{1}{4}\lambda^2(e^{4t}-1)\right)d\lambda. \tag{5.33}$$

From the boundary condition we have

$$\alpha\int_0^\infty \varphi(\lambda)e^{-\frac{\lambda^2}{4}(e^{4t}-1)}d\lambda + \beta\int_0^\infty \lambda\psi(\lambda)e^{-\frac{\lambda^2}{4}(e^{4t}-1)}d\lambda = e^{-t}g(t). \tag{5.34}$$

Now we set $s = e^{4t}-1$, which implies $t = \frac{1}{4}\ln(1+s)$. We also put $\lambda^2 = 4z$. This transforms (5.34) into

$$\alpha\int_0^\infty \frac{\varphi(2\sqrt{z})}{\sqrt{z}}\exp(-zs)\,dz + 2\beta\int_0^\infty \psi(2\sqrt{z})\exp(-zs)\,dz = \bar{g}(s), \tag{5.35}$$

where

$$\bar{g}(s) = \frac{1}{(1+s)^{\frac{1}{4}}}g\left(\frac{1}{4}\ln(1+s)\right).$$

Inverting the Laplace transform gives

$$\alpha\frac{\varphi(2\sqrt{z})}{\sqrt{z}} + 2\beta\psi(2\sqrt{z}) = \mathcal{L}^{-1}\left[\bar{g}(s);z\right], \tag{5.36}$$

and so

$$\alpha\varphi(\lambda) + \beta\lambda\psi(\lambda) = \frac{\lambda}{2}\mathcal{L}^{-1}\left[\bar{g}(s);\frac{\lambda^2}{4}\right] = K(\lambda), \tag{5.37}$$

which immediately yields

$$\alpha(F(\lambda) + (\mathcal{H}_o\psi)(\lambda)) = K(\lambda) - \beta\lambda\psi(\lambda). \tag{5.38}$$

Taking the even Hilbert transform of both sides produces the result

$$\alpha(\mathcal{H}_eF)(\lambda) - \alpha\psi(\lambda)) = (\mathcal{H}_eK)(\lambda) - \beta\lambda(\mathcal{H}_o\psi)(\lambda). \tag{5.39}$$

However we also have

$$(\mathcal{H}_o\psi)(\lambda) = \frac{1}{\alpha}\left[K(\lambda) - \beta\lambda\psi(\lambda)\right] - F(\lambda),$$

whence

$$\alpha(\mathcal{H}_eF)(\lambda) - \alpha\psi(\lambda)) = (\mathcal{H}_eK)(\lambda) - \beta\lambda\left[\frac{1}{\alpha}\left[K(\lambda) - \beta\lambda\psi(\lambda)\right] - F(\lambda)\right]. \tag{5.40}$$

Solving for ψ gives

$$\psi(\lambda) = \frac{\alpha\left[\alpha(\mathcal{H}_eF)(\lambda) - (\mathcal{H}_eK)(\lambda) + \frac{\beta\lambda}{\alpha}K(\lambda) - \lambda\beta F(\lambda)\right]}{\alpha^2 + \beta^2\lambda^2}. \tag{5.41}$$

From this we can find φ. Proving the convergence of the integrals for these choices of φ and ψ proceeds much as in our previous example. Since the details are similar, we focus on the function K, which has a different structure than in the previous case. To proceed, we suppose that

$$g(s) = \int_0^\infty G(x)e^{-xs}dx. \tag{5.42}$$

Then we immediately see that

$$\frac{1}{(1+s)^{\frac{1}{4}}}g\left(\frac{1}{4}\ln(1+s)\right) = \int_0^\infty \frac{G(x)}{(1+s)^{\frac{1+x}{4}}}dx. \tag{5.43}$$

Taking the inverse Laplace transform, we obtain

$$\mathcal{L}^{-1}\left[\frac{1}{(1+s)^{\frac{1}{4}}}g\left(\frac{1}{4}\ln(1+s)\right); z\right] = \int_0^\infty \frac{G(x)}{\Gamma\left(\frac{1+x}{4}\right)}e^{-\frac{(x+1)z}{4}}z^{\frac{x}{4}-\frac{3}{4}}dx. \tag{5.44}$$

See [19] for the inverse Laplace transform.

Thus we obtain the expression

$$K(\lambda) = \left(\frac{\lambda}{2}\right)^{-\frac{1}{2}}e^{-\frac{\lambda^2}{4}}\int_0^\infty \frac{G(x)}{\Gamma\left(\frac{1+x}{4}\right)}e^{-\frac{x\lambda^2}{16}}\left(\frac{\lambda}{2}\right)^{\frac{x}{2}}dx. \tag{5.45}$$

If we suppose that $G \in L^1\left([0,\infty), \frac{dx}{\Gamma\left(\frac{1+x}{4}\right)}\right)$, then an application of the dominated convergence theorem gives

$$\lim_{\lambda\to\infty} \int_0^\infty \frac{1}{\Gamma\left(\frac{1+x}{4}\right)} G(x) e^{-\frac{x\lambda^2}{16}} \left(\frac{\lambda}{2}\right)^{\frac{x}{2}} dx = 0. \tag{5.46}$$

We also have

$$\int_0^\infty \frac{1}{\Gamma\left(\frac{1+x}{4}\right)} G(x) e^{-\frac{x\lambda^2}{16}} \left(\frac{\lambda}{2}\right)^{\frac{x}{2}} dx\Big|_{\lambda=0} = 0. \tag{5.47}$$

Now the Lebesgue integral of a measurable function returns a uniformly continuous function; see [30]. Hence the integral in (5.45) is bounded, and we have the inequality

$$|K(\lambda)| \le \frac{C}{\sqrt{\lambda}} e^{-\frac{\lambda^2}{4}} \tag{5.48}$$

for some positive constant C depending on G. Thus the even Hilbert transform of K exists. We can prove that $(\mathcal{H}_e K)(\lambda) w_k(x,t;\lambda)$ is integrable for $k = 1,2$ by similar means to the Black–Scholes equation case. If we suppose that f is integrable, then similar statements can be made for the even Hilbert transform of F. The remainder of the proof is similar to our previous examples. □

6 The Robin problem for a five-dimensional squared Bessel process

The method is effective for elementary solutions of greater complexity. For example, solutions of the form

$$w_1(x,t) = (\rho_1(x,t;\xi)\cos(\xi A_1(x,t)) + \rho_2(x,t;\xi)\sin(\xi A_2(x,t)))e^{-\xi^2 t}, \tag{6.1}$$

$$w_2(x,t) = (\rho_3(x,t;\xi)\cos(\xi A_1(x,t)) + \rho_4(x,t;\xi)\sin(\xi A_2(x,t)))e^{-\xi^2 t} \tag{6.2}$$

can be used. It is possible to formulate an analogue of Theorem 5.1 for equations with elementary solutions of this form. However, we omit this for brevity. Instead, we content ourselves by working through the details of a particular example of interest. We solve the problem

$$u_t = 2xu_{xx} + 5u_x,\ x > b, t > 0, \tag{6.3}$$

$$u(x,0) = f(x), x > b, \tag{6.4}$$

$$\alpha u(b,t) + \beta u_x(b,t) = g(t). \tag{6.5}$$

We assume that f is continuously differentiable. This is associated with a five-dimensional squared Bessel process and is not covered by Theorem 5.1. However, our method can be extended to cover it. We look for a solution that can be written as

$$u(x,t) = \int_0^\infty \varphi(\xi) w_1(x,t;\xi) d\xi + \int_0^\infty \psi(\xi) w_2(x,t;\xi) d\xi, \tag{6.6}$$

where

$$w_1(x,t;\xi) = e^{-\xi^2 t}x^{-3/2}\left(2\xi\sqrt{x}\sin(\xi m(x)) + \sqrt{2}\cos(\xi m(x))\right), \tag{6.7}$$

$$w_2(x,t;\xi) = e^{-\xi^2 t}x^{-3/2}\left(\sqrt{2}\sin(\xi m(x)) - 2\xi\sqrt{x}\cos(\xi m(x))\right) \tag{6.8}$$

with $m(x) = \sqrt{2x} - \sqrt{2b}$.

By the change of variables $x = \frac{1}{2}(y+\sqrt{2b})^2$ the initial condition reduces to

$$\begin{aligned}&\int_0^\infty \varphi(\xi)(\cos(\xi y) + (y+\sqrt{2b})\xi\sin(\xi y))d\xi \\ &\qquad + \int_0^\infty \psi(\xi)(\sin(\xi y) - (y+\sqrt{2b})\xi\cos(\xi y))d\xi \\ &= \frac{1}{\sqrt{2}}\left(\frac{1}{2}(y+\sqrt{2b})^2\right)^{3/2} f\left(\frac{1}{2}(y+\sqrt{2b})^2\right) = \tilde{f}(y).\end{aligned}$$

This seems to be quite different to the previous cases that we have encountered. However, we can rewrite the above as

$$\left(1 - (y+\sqrt{2b})\frac{d}{dy}\right)(\widehat{\varphi}_c(y) + \widehat{\psi}_s(y)) = \tilde{f}(y) \tag{6.9}$$

or, equivalently,

$$\left(\frac{d}{dy} - \frac{1}{y+\sqrt{2b}}\right)(\widehat{\varphi}_c(y) + \widehat{\psi}_s(y)) = -\frac{1}{y+\sqrt{2b}}\tilde{f}(y). \tag{6.10}$$

Introducing the integrating factor $\dfrac{1}{y+\sqrt{2b}}$, we have

$$\frac{d}{dy}\left(\frac{1}{y+\sqrt{2b}}\left(\widehat{\varphi}_c(y) + \widehat{\psi}_s(y)\right)\right) = -\frac{1}{(y+\sqrt{2b})^2}\tilde{f}(y). \tag{6.11}$$

Hence

$$\widehat{\varphi}_c(y) + \widehat{\psi}_s(y) = (y+\sqrt{2b})\left(\int_0^y \frac{-1}{(\eta+\sqrt{2b})^2}\tilde{f}(\eta)d\eta + I\right) = F(y), \tag{6.12}$$

where I is a constant of integration. To determine the value of I, we need an initial condition, and we notice that $\widehat{\psi}_s(0) = 0$ and $\widehat{\varphi}_c(0) = \int_0^\infty \varphi(\xi)d\xi$. Thus we need the value of the integral $\int_0^\infty \varphi(\xi)d\xi$. This same constant also arises from the boundary conditions. We will see that it can be obtained by solving a pair of simultaneous equations.

From (6.12) we obtain the familiar condition

$$\varphi(\xi) = \widehat{F}(\xi) + (\mathcal{H}_o\psi)(\xi) \tag{6.13}$$

with $\widehat{F}(\xi) = \frac{2}{\pi}\int_0^\infty F(\eta)\cos(\eta\xi)d\eta$.

We remark here that for the squared Bessel processes of odd order higher than five, condition (6.13) is obtained by solving equations of Euler type for $\widehat{\varphi}_c(y) + \widehat{\psi}_s(y)$ and the initial conditions are found by solving a system of linear equations. However, the details are quite involved and so will be presented elsewhere.

The Robin boundary condition gives us

$$\frac{((2b\alpha - 3\beta) + 2b\beta\xi^2)\varphi(\xi)}{2b^{5/2}} + \frac{3\beta - 2b\alpha}{b^2}\xi\psi(\xi) = 2\xi\mathcal{L}^{-1}[g(t);\xi^2] = G(\xi). \tag{6.14}$$

This is equivalent to

$$(A + B\xi^2)(\mathcal{H}_o\psi)(\xi) + C\xi\psi = G(\xi) - (A + B\xi^2)\widehat{F}(\xi) \tag{6.15}$$

with $A = \frac{(2b\alpha-3\beta)}{2b^{5/2}}$ and $C = \frac{3\beta-2b\alpha}{b^2}$. We assume that $A + B\xi^2$ has no real roots. We take the even Hilbert transform of both sides, just as before. This gives

$$-A\psi + B\left(-\frac{2\xi}{\pi}P\int_0^\infty \mathcal{H}_o\psi(\eta)d\eta - \xi^2\psi(\xi)\right) + C\xi\,(\mathcal{H}_o\psi)(\xi) = J(\xi), \tag{6.16}$$

where $J(\xi) = (\mathcal{H}_e K)(\xi)$ and $K(\xi) = G(\xi) - (A + B\xi^2)\widehat{F}(\xi)$. Now as before, we require

$$P\int_0^\infty (\mathcal{H}_o\psi)(\eta)d\eta = P\int_0^\infty (\varphi(\eta) - \widehat{F}(\eta))d\eta. \tag{6.17}$$

We will drop the principal value by assuming suitable integrability. The procedure for determining the value of the integral $\int_0^\infty \varphi(\xi)d\xi$ will be addressed below.

From (6.15) we see that

$$(\mathcal{H}_o\psi)(\xi) = \frac{K(\xi) - C\xi\psi(\xi)}{A + B\xi^2}. \tag{6.18}$$

Together with (6.16), we have a pair of simultaneous equations for ψ and $\mathcal{H}_o\psi$, and we obtain

$$-\frac{(A + B\xi^2)^2 + C^2\xi^2}{A + B\xi^2}\psi(\xi) = (\mathcal{H}_e K)(\xi) + B\frac{2\xi}{\pi}\int_0^\infty \mathcal{H}_o\psi(\eta)d\eta - k(\xi), \tag{6.19}$$

where $k(\xi) = \frac{C\xi K(\xi)}{A+B\xi^2}$. This gives

$$\psi(\xi) = -(A + B\xi^2)\frac{(\mathcal{H}_e K)(\xi) + B\frac{2\xi}{\pi}\left(\int_0^\infty(\varphi(\eta) - \widehat{F}(\eta))d\eta\right) - \frac{C\xi K(\xi)}{A+B\xi^2}}{(A + B\xi^2)^2 + C^2\xi^2}. \tag{6.20}$$

From this we obtain φ. To complete the calculation, we need to determine $\int_0^\infty \varphi(\xi)d\xi$. First, from the construction of the solution we have

$$u(b,0) = f(b) = b^{-3/2}\sqrt{2}\int_0^\infty \varphi(\xi)d\xi - \frac{2}{b}\int_0^\infty \xi\psi(\xi)d\xi. \tag{6.21}$$

Integrating both sides of (6.14) gives

$$A\int_0^\infty \varphi(\xi)d\xi + B\int_0^\infty \xi^2\varphi(\xi)d\xi + C\int_0^\infty \xi\psi(\xi)d\xi = \int_0^\infty G(\xi)d\xi. \tag{6.22}$$

We next calculate $u_x(b,0)$ and after rearranging obtain

$$\int_0^\infty \xi\psi(\xi)d\xi = -\frac{\sqrt{2a}}{3}\int_0^\infty \xi^2\varphi(\xi)d\xi + \frac{1}{\sqrt{2b}}\int_0^\infty \varphi(\xi)d\xi + f'(b). \tag{6.23}$$

Basic algebra leads to the pair of simultaneous equations

$$\begin{aligned} H_1\int_0^\infty \varphi(\xi)d\xi + H_2\int_0^\infty \xi^2\varphi(\xi)d\xi &= \frac{b^{3/2}}{\sqrt{2}}f(b) - \frac{\sqrt{2b}}{C}\int_0^\infty G(\xi)d\xi, \\ H_3\int_0^\infty \varphi(\xi)d\xi + H_4\int_0^\infty \xi^2\varphi(\xi)d\xi &= \frac{b^2}{3}f'(b) - \int_0^\infty \frac{G(\xi)}{C}d\xi, \end{aligned}$$

where $H_1 = \left(1 + \frac{\sqrt{2b}A}{C}\right)$, $H_2 = \frac{\sqrt{2b}B}{C}$, $H_3 = \left(\sqrt{\frac{b}{2}} + \frac{A}{C}\right)$, $H_4 = \left(\frac{B}{C} - \frac{\sqrt{2b}}{3}\right)$. This gives $\int_0^\infty \varphi(\xi)d\xi = \gamma$, where

$$\gamma = \frac{H_4\left(\frac{b^{3/2}}{\sqrt{2}}f(b) - \frac{\sqrt{2b}}{C}\int_0^\infty G(\xi)d\xi\right) - H_2\left(\frac{b^2}{3}f'(b) - \int_0^\infty \frac{G(\xi)}{C}d\xi\right)}{H_1H_4 - H_2H_3}, \tag{6.24}$$

provided that $H_1H_4 - H_2H_3 \neq 0$. If $g(t) = 0$, then this simplifies to

$$\int_0^\infty \varphi(\xi)d\xi = \frac{\frac{b^{3/2}}{\sqrt{2}}H_4 f(b) - \frac{b^2}{3}H_2 f'(b)}{H_1H_4 - H_2H_3}. \tag{6.25}$$

We immediately obtain

$$F(y) = (y + \sqrt{2b})\left(\int_0^y \frac{-1}{(\eta + \sqrt{2b})^2}\tilde{f}(\eta)d\eta + \frac{\gamma}{\sqrt{2b}}\right).$$

The question then arises as to how we are to interpret the Fourier cosine transform of $y + \sqrt{2b}$? This is in terms of distributions. Clearly, $\int_0^\infty \delta(\xi)\cos(\xi y)d\xi = 1$. Hence $\frac{2}{\pi}\int_0^\infty \cos(\xi y)dy = \delta(\xi)$. Now let ϕ be a suitable test function of Schwartz class. We define the distribution $\Lambda(\phi) = \int_0^\infty y\widehat{\phi_c}(y)dy$ and as a distribution $\left(\int_0^\infty y\cos(\xi)dy, \phi\right) = \Lambda(\phi)$. Note that solutions (6.7) and (6.8) are in the Schwartz class, so that $\Lambda(w_1)$ and $\Lambda(w_2)$ are well defined and can be computed easily.

It is possible to give conditions that guarantee the convergence and to prove that we do indeed have a solution of the Robin problem. This proceeds along the lines of the treatment given in the cases of the Black–Scholes equation and the harmonic oscillator. To avoid repeating the same arguments as before, we will omit this analysis. However, we are able to construct a solution to the problem by this method, which again does not require us to know the fundamental solution.

7 Future directions

In this paper, we have introduced a new method, which yields new representations for the solution of important boundary value problems. The idea of constructing solutions to boundary value problems from elementary solutions of the PDE without use of a fundamental solution is potentially very important. Naturally, there are many open problems.

First, we would like a fuller characterization of the types of equations for which the method is effective. We have one result along those lines, and others are possible. We can give a characterization of PDEs that possess solutions of the forms (6.1) and (6.2). We proceed as in the discussion at the start of Sect. 5. We substitute the solutions into an arbitrary linear PDE, which leads to conditions that guarantee that we have a solution.

There are also particular cases that we have not gone into detail to keep the current paper to a manageable length. For most of these cases, the analysis is basically the same as that presented for our main problems. In many cases, they are easier because certain terms disappear from the equations. For example, in the derivation of Theorem 3.3, assuming that $\left(\alpha + \frac{\mu\beta}{b}\right) = 0$ significantly reduces the difficulty in determining φ and ψ. We simply invert a single Laplace transform to obtain ψ, and φ can be determined from this.

We do not yet know the full range of equations with time-dependent coefficients that can be studied by this technique. However, we made some preliminary remarks on this earlier. If we consider equation (5.18), then the extension of Theorem 5.1 to equations of this form is not at all difficult. We simply construct a solution

$$\begin{aligned} u(x,t) &= \int_0^\infty \varphi(\xi)\rho(x)\cos(\xi(\mu(x)-\mu(b)))e^{-\xi^2\int_0^t r(s)ds}d\xi \\ &\quad + \int_0^\infty \psi(\xi)\rho(x)\sin(\xi(\mu(x)-\mu(b)))e^{-\xi^2\int_0^t r(s)ds}d\xi. \end{aligned}$$

Note that $k(t) = \int_0^t r(s)ds$ is increasing and so is invertible. We have

$$u(x,0) = \rho(x)\left(\widehat{\varphi}_c(\mu(x)-\mu(b)) + \widehat{\psi}_s(\mu(x)-\mu(b))\right) = f(x),$$

and the equations arising from the boundary conditions can be solved by the use of the odd and even Hilbert transforms. The analysis is similar to the case where $r(s) = 1$, and the solution is a modification of that given in Theorem 5.1. The solution is in terms of the inverse Laplace transform of $g(k^{-1}(s))$, as with our example for the equation (5.21).

For (5.21), it is clear that there are numerous possible cases that can be considered. We have seen that the Dirichlet problem is straightforward if we set $A(t) = 0$, which corresponds to a time-dependent Ornstein–Uhlenbeck process. The Neumann problem can also be solved in the same manner. For different choices of the coefficients, a variety of boundary value problems can be solved. There are other time-dependent equations that can be studied, but we will not discuss them here.

Another application of these techniques is to equations with nonstandard boundary conditions. We mentioned previously that certain moving boundary problems can be solved. We will give one example. Consider the PDE $u_t = u_{xx} - xu$, $x \geq t^2$, subject to $u(t^2,t) = g(t)$, $u(x,0) = f(x)$. The PDE has solutions

$$u_1(x,t;\xi) = e^{\frac{1}{3}t^3 - xt - \xi^2 t}\cos(\xi(x-t^2)), \tag{7.1}$$

$$u_2(x,t;\xi) = e^{\frac{1}{3}t^3 - xt - \xi^2 t}\sin(\xi(x-t^2)). \tag{7.2}$$

We can solve a number of different problems with these elementary solutions; see [22]. To solve the moving boundary problem, we define

$$u(x,t) = \int_0^\infty \varphi(\xi)u_1(x,t;\xi)d\xi + \int_0^\infty \varphi(\xi)u_2(x,t;\xi)d\xi. \tag{7.3}$$

Then $u(x,0) = \widehat{\varphi}_c(x) + \widehat{\psi}_s(x) = f(x)$. The moving boundary condition leads to

$$\int_0^\infty \varphi(\xi)e^{-\xi^2 t}d\xi = e^{2t^3/3}g(t). \tag{7.4}$$

This reduces to a Laplace transform, and inverting the transform gives us φ, and from this we obtain ψ. There are other moving boundary problems that we can study for different PDEs. This is ongoing work.

We also remark that different families of elementary solutions that do not involve sines and cosines can also be used in our basic construction. There are also PDEs that possess elementary solutions for which the integral equations arising from (1.2) can be solved by means other than the Hilbert transform. Much work remains to be done on these problems; however, see preprint [31] for work in this direction.

Appendix

The classical approach to solving the Robin problem when g is nonzero is presented here. For a more technical treatment of the solution of boundary problems for parabolic operators, we recommend [1]. We wish to solve

$$u_t(S,t) = Lu(S,t), \tag{A.1}$$

$$u(S,0) = f(S), \tag{A.2}$$

$$\alpha u(b,t) + \beta u_S(b,t) + \gamma u_{SS}(b,t) = g(t), \tag{A.3}$$

with $S \geq b$, $t > 0$, $b > 0$, and L a positive second-order operator in S. In our case, L is the second-order Black–Scholes operator:

$$Lu(S,t) = \frac{1}{2}\sigma^2 S^2 u_{SS}(S,t) + rSu_S(S,t).$$

To solve problem (A.1)–(A.3), we set $u(S,t) = v(S,t) + h(S,t)$, where

$$\alpha v(b,t) + \beta v_S(b,t) + \gamma v_{SS}(b,t) = 0,$$

and

$$\alpha h(b,t) + \beta h_S(b,t) + \gamma h_{SS}(b,t) = g(t). \tag{A.4}$$

This gives

$$\begin{cases} v_t(S,t) = Lv(S,t) + K(S,t), \\ v(S,0) = f(S) - h(S,0), \\ \alpha v(b,t) + \beta v_S(b,t) + \gamma v_{SS}(b,t) = g(t), \end{cases} \tag{A.5}$$

where $K(S,t) = Lh - h_t(S,t)$ and $S > 0$, $t > 0$. We assume that $g(t)$ is differentiable and integrable. Then we have the following result. We omit the proof, which is available on request.

Proposition A.1 *Let $h(S,t)$ be a function that satisfies (A.4), and let $q(S,y,t)$ be a fundamental solution of $q_t = Lq$ such that $\alpha q(b,y,t) + \beta q_S(b,y,t) + \gamma q_{SS}(b,y,t) = 0$. Then the solution of the boundary value problem (A.1)–(A.3) can be written as*

$$\begin{aligned} u(S,t) =& h(S,t) + \int_b^\infty (f(y) - h(y,0))q(S,y,t)dy \\ &+ \int_0^t \int_b^\infty \left(Lh(y,\tau) - \frac{\partial h}{\partial t}(y,\tau) \right) q(S,y,t-\tau)dyd\tau. \end{aligned}$$

In general, the function $h(S,t)$ is not unique, and different choices for it may lead to different representations of the solution to the BVP. The fundamental solution in this case is given by (2.21) in which

$$\begin{aligned} \tilde{p}(S,y,t) =& \frac{S^\mu b^{-\frac{\ln\left(\frac{y}{b}\right)+2\ln(S)}{\sigma^2 t} - \mu} M(S,t)N(S,t)}{4\sqrt{2\pi\,\Delta} y \tilde{\beta} \left(\sigma^2 t\right)^{3/2} \left(\tilde{\alpha} + \mu\left(\tilde{\beta} + \gamma\mu\right)\right)} e^{-\frac{y\left(\tilde{\beta}+\sqrt{\Delta}\right)}{2\gamma}} \\ &\times \exp\left(-\frac{4\left(-4\ln(b)\ln(S) + \ln^2(b) + \ln^2(S)\right) + 4\ln^2\left(\frac{y}{b}\right) + \left(\sigma^2 - 2r\right)^2 t^2}{8\sigma^2 t} \right), \end{aligned}$$

where

$$\begin{aligned} M(S,t) =& \left(-\sqrt{\Delta}\tilde{\beta} + R\left(2\gamma f(b)\left(\tilde{\alpha} + \mu\left(\tilde{\beta} + \gamma\mu\right)\right) - \tilde{\beta}\left(\tilde{\beta} + 2\gamma\mu\right)\right)\right. \\ &\left. \tilde{\beta}\left(\tilde{\beta} + 2\gamma\mu\right) + 2\sqrt{\Delta}\tilde{\beta} e^{\frac{y\left(\tilde{\beta}+2\gamma\mu+\sqrt{\Delta}\right)}{2\gamma}} + \sqrt{\Delta}\tilde{\beta}\left(-e^{\frac{\sqrt{\Delta}y}{\gamma}}\right)\right), \end{aligned}$$

$$R = \left(e^{\frac{\sqrt{\Delta}y}{\gamma}} - 1\right),$$

$$\begin{aligned} N(s,t) =& \left(\sqrt{2\pi}\sigma^3 t^{3/2} \tilde{\alpha} b^{\frac{\log\left(\frac{y}{b}\right)}{\sigma^2 t}} e^{\frac{\ln^2\left(\frac{y}{b}\right)+\ln^2(b)+\ln^2(S)}{2\sigma^2 t}} Z(S,y) + 2b^{\frac{\ln(S)}{\sigma^2 t}} S^{-\frac{\ln\left(\frac{y}{b}\right)}{\sigma^2 t}} \right. \\ &\times \left(b^{\frac{2\ln\left(\frac{y}{b}\right)}{\sigma^2 t}} \left(\sigma^2 t\tilde{\beta} + \gamma\ln\left(\frac{y}{b}\right)\right) + S^{\frac{2\ln\left(\frac{y}{b}\right)}{\sigma^2 t}} \left(\sigma^2 t\tilde{\beta} - \gamma\ln\left(\frac{y}{b}\right)\right)\right) \\ &\left. - 2\gamma\ln\left(\frac{S}{b}\right) S^{-\frac{\ln\left(\frac{y}{b}\right)}{\sigma^2 t}} b^{\frac{2\ln\left(\frac{y}{b}\right)+\ln(S)}{\sigma^2 t}} - 2\gamma\ln\left(\frac{S}{b}\right) b^{\frac{\ln(S)}{\sigma^2 t}} S^{\frac{\ln\left(\frac{y}{b}\right)}{\sigma^2 t}} \right), \end{aligned}$$

and $Z(S,y) = \operatorname{erf}\left(\frac{\ln\left(\frac{S}{b}\right)+\ln\left(\frac{y}{b}\right)}{\sqrt{2\sigma^2 t}}\right) - \operatorname{erf}\left(\frac{\ln\left(\frac{S}{y}\right)}{\sqrt{2\sigma^2 t}}\right)$.

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