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Rhaly terraced sequences their generalizations, properties and applications

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Abstract

This paper links terraced matrices with other well-known integer sequences, such as the Hankel matrices and related Fibonacci and Lucas matrices. These, in turn, are connected with related results of Macmahon and Sloane as well as we introduce the r-Terraced matrix as a generalization of the Terraced matrix, along with its symmetric counterpart, the symmetric r-Terraced matrix. We derive key properties of these matrices, including their spectral and Euclidean norms, upper bounds for their spreads, and characteristic polynomials. To validate and exemplify the theoretical findings, we apply them to Fibonacci numbers, providing illustrative examples that strengthen the theory and confirm its accuracy. In addition to the theoretical results, we investigated how the choice of the parameter r and the matrix dimension affect the upper bounds of the spread. Our findings reveal that selecting values of $r < 1$ and using lower-dimensional matrices lead to tighter upper bounds while reducing computational complexity. These results highlight the practical benefits of our approach, particularly in optimization-related applications where efficiency is crucial.

Keywords Catalecticant matrix · Characteristic polynomial · Euclidean norm · Fibonacci numbers · Frobenius norm · Spectral norm · Terraced matrix · Lucas sequence · Sequence bisection · Spreads

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1 Introduction

Terraced matrices are a specialized class of structured matrices characterized by a distinct pattern in their entries. They are particularly significant in the study of sequence spaces and summability theory. A terraced matrix is defined by its action on sequences, often transforming a given sequence into another by applying specific coefficients arranged in a terraced pattern.

Absolutely summing behavior: Terraced matrices have been studied for their absolutely summing properties, which are crucial in understanding their behavior in various sequence spaces. Research has shown that certain conditions are necessary and sufficient for a terraced matrix to be absolutely summing.

Applications in summability theory: These matrices play a role in summability theory, where they are used to analyze and transform series and sequences. Their structured form allows for the examination of convergence properties and the summation of divergent series.

In Almasri (2016), Almasri provides a comprehensive study on the conditions under which terraced matrices are absolutely summing, offering valuable insights into their mathematical properties and applications. This paper provides a solid foundation for understanding the significance and applications of terraced matrices in mathematical analysis.

Rhaly terraced matrix sequences (Rhaly 1989) offer opportunities to link them with other well-known integer sequences, such as the Fibonacci sequence. The Rhaly terraced matrices have the form

$$R_a = \begin{bmatrix} t_1 & 0 & 0\ 0 & \dots \\ t_2 & t_2 & 0\ 0 & \dots \\ t_3 & t_3 & t_3\ 0 & \dots \end{bmatrix}, \tag{1.1}$$

So, we plan to examine extensions such as the Rhaly terraced Fibonacci matrix

$$R_F = \begin{bmatrix} 1 & 0 & 0\ 0 & \dots \\ 1 & 1 & 0\ 0 & \dots \\ 2 & 2 & 2\ 0 & \dots \\ 3 & 3 & 3\ 3 & \dots \end{bmatrix}, \tag{1.2}$$

in which the sums of the rows result in the sequence $\{nF_n\}$, and the sums of the leading diagonals result in the sequence $\{A_n\} = \{1, 1, 3, 5, 10, 16, 29, 45, 75, 115, \dots\}$. These two sequences are respectively A045925 and A000990 of Sloane's *OIES* (2024). MacMahon (1924) found the non-zero general terms of $\{A_n\}$ as

$$t_n = \sum_{k=\left\lfloor \frac{n+1}{2} \right\rfloor}^{n} F_{k+1} \tag{1.3}$$

For example,

$$t_5 = F_3 + F_4 + F_5 = 10.$$

Other valuable studies on matrices are the studies carried out by adapting number sequences to matrices. In these studies, circulant matrices were generally considered and many important properties of these matrices, such as their norm properties, characteristic equations, determinants, eigenvalues, and principal minors, were examined (Kuloğlu et al. 2023a, b; Kumari et al. 2023; Kızılateş and Tuğlu 2016, 2018; Prasad et al. 2024). Also, Akbıyık et al. (2021) A symmetric matrix family constructed with harmonic Pell numbers as its elements, along with its Hadamard exponential matrix. Several linear algebraic properties and inequalities investigated using matrix norms. Additionally, summation identities for harmonic Pell numbers are derived. Shi and Kızılateş (2024) new generalization of the Frank matrix and its some properties studied.

Durna and Yıldırım (2016) defined the generalized terraced matrix by using the generalized Cesàro matrix, and gave some properties of this matrix.

Definition 1.1 (Horn 1990) Let $X = [x_{ij}]$, $i, j = 1,2, 3, \ldots, n$ and $Y = [y_{ij}]$, $i, j = 1,2, 3, \ldots, n$ be $n \times n$ matrices, the Hadamard products of this matrix are defined as follows:

$$X \circ Y = [x_{ij} y_{ij}],\ i, j = 1,2, 3, \ldots, n \tag{1.4}$$

Definition 1.2 (Horn 1990) For any given matrix X, the spectral norm, l_p norm, Euclidean norm are given by the following equations, respectively:

$$\|X\|_2 = \sqrt{max\lambda_k(X^H X)},\ 1 \le k \le n \tag{1.5}$$

Here, $\lambda_k(X^H X)$ represents an eigenvalue of $X^H X$, where X^H denotes the conjugate transpose of the matrix X.

$$\|X\|_p = \sqrt[p]{\sum_{i,j=1}^{n} |x_{ij}|^p},\ p \ge 2 \tag{1.6}$$

$$\|X\|_E = \sqrt{\sum_{i,j=1}^{n} |x_{ij}|^2} \tag{1.7}$$

Definition 1.3 Let $X = [x_{ij}]$, $i, j = 1,2, 3, \ldots, n$, $n \times n$ matrices. The spread of a matrix,

$$S(X) = \underbrace{max}_{i,j}|\lambda_i - \lambda_j|,$$

$$S(X) \le \sqrt{2\|X\|_E^2 - \frac{2}{n}|tr(X)|^2} \tag{1.8}$$

here, λ_i represents an eigenvalue of X matrix.

Lemma 1.4 *(Horn and Johnson 1991) Let* $X = [x_{ij}]$, $i, j = 1,2, 3, \ldots, n$ *and* $Y = [y_{ij}]$, $i, j = 1,2, 3, \ldots, n$ be $m \times n$ *matrices, following equality holds:*

$$\|X \circ Y\|_2 \le r(X)c(Y) \tag{1.9}$$

Here, $r(X) = max\sqrt{\sum_{j=1}^{n} |x_{ij}|^2}$, $1 \le i \le m$ *and* $c(Y) = max\sqrt{\sum_{i=1}^{m} |y_{ij}|^2}$, $1 \le j \le n$.

Matrix analysis is fundamental in various fields, with the spread—defined as the difference between the largest and smallest eigenvalues—being a key spectral property. Optimizing the spread is crucial for applications such as coding theory and signal processing. Prior studies (Horn and Johnson 2013; Golub and Van Loan 2013) show that smaller matrix dimensions and appropriate scaling parameters improve computational efficiency and spectral quality.

This work focuses on matrices constructed with Fibonacci sequences and scaling parameters. Increasing the matrix size leads to higher upper bounds for the spread, confirming trends noted by Parlett (1998).

Our primary goal in this study is to define generalized Terraced matrices, obtain different norms and extensions of these matrices, and obtain matrices that will find a place in many applied branches of science other than mathematics through applications. In this context, all theorems and propositions given at the end of Sect. 3 and Sect. 4 are proven to be correct with Fibonacci numbers by application with the examples, and then it is aimed to reduce the upper bounds to 1 with different n and r values.

The study reveals that tuning the parameter r provides a powerful mechanism to control the spectral behaviour of matrices constructed from Fibonacci numbers. Lower r values foster spectral stability with minimal spread, beneficial in control systems and computational methods requiring fats convergence. Conversely, higher r values produce matrices with broad spectral dispersion, useful in fields like encryption, randomness modelling and systems requiring wide frequency responses.

Importantly, the inherent connection to the golden ratio through the Fibonacci diagonal entries ensures that even small adjustments to r can produce significant spectral shifts. As the matrix size decreases and structures become more complex, this sensitivity becomes even more pronounced, offering rich avenues for future exploration in mathematical modelling, engineering applications and data driven technologies.

The analysis demonstrates that varying r provides a powerful mechanism to control both the spread and the Euclidean norm of the Fibonacci-based matrix. By carefully selecting r, practitioners can tailor the matrix's stability and spectral properties to specific applications. Future research could explore higher-order recurrence relations or the use of other integer sequences to broaden the spectrum of potential applications.

Also, larger upper bound values were obtained at the same r and n values with generalized symmetric Rhaly r-Terraced Matrices and generalized r-Terraced matrices, and generalizations of operations were tried to be made with generalized r-terraced matrices.

2 Sums and products of the terraced Fibonacci and Lucas matrices

We next check the Rhaly terraced Lucas matrix

$$R_L = \begin{bmatrix} 1 & 0 & 0 & 0 & \dots \\ 3 & 3 & 0 & 0 & \dots \\ 4 & 4 & 4 & 0 & \dots \\ 7 & 7 & 7 & 7 & \dots \end{bmatrix}, \tag{2.1}$$

The leading diagonals seem to yield an *OEIS* sequence A028831,

$$\{B_n\} \equiv \{1{,}3,\ 7{,}11{,}22{,}43{,}83{,}159{,}307, \dots\}$$

the elements of which satisfy a 4th order linear homogeneous recurrence relation

$$B_n = B_{n-1} + B_{n-2} + B_{n-3} + B_{n-4},\ n > 4,$$

with initial conditions $B_1 = 1,\ B_2 = 3,\ B_3 = 7,\ B_4 = 11$. We note that after $n = 3$, $\{B_n\}$ seems to be related to a convolution of the Lucas sequence, and the row sums result in the sequence $\{nL_n\}$[OEIS, A145005], somewhat similar to R_F and $\{nF_n\}$. In turn the, for the

4×4 case, we obtain

$$R_L R_F = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 8 & 0 \\ 0 & 0 & 0 & 21 \end{bmatrix} = R_F R_L \tag{2.2}$$

which is a bisection of the Fibonacci sequence (Sloane 2024), (A00196), It is also well known that $F_n L_n = F_{2n}$ (Hoggatt 1969). If, however, we refine our notation for the Rhaly terraced Fibonacci matrix to

$$R_{F(n)} = \begin{bmatrix} 1 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & \dots \\ 2 & 2 & 2 & 0 & \dots \\ 3 & 3 & 3 & 3 & \dots \end{bmatrix} \tag{2.3}$$

then, for the 4×4 case,

$$R_{L(n)} R_{F(n)} = R_{F(2n)} \tag{2.4}$$

and

$$R_{F(n-1)} + R_{F(n+1)} = R_L, \tag{2.5}$$

A Hankel or catalecticant matrix (Masano 1985) has the symmetric form

$$H_a = \begin{bmatrix} a_1 & a_2 & a_3 & \\ a_2 & a_3 & a_4 & \dots \\ a_3 & a_4 & a_5 & \end{bmatrix} \tag{2.6}$$

and we can define a Hankel symmetric Fibonacci matrix as

$$H_F = \begin{bmatrix} 1 & 1 & 2 & \\ 1 & 2 & 3 & \dots \\ 2 & 3 & 5 & \\ & \dots & & \end{bmatrix}, \tag{2.7}$$

so for the 4×4 case it can be calculated that, for example,

$$R_F H_F = \begin{bmatrix} 1 & 1 & 2 & 3 \\ 2 & 3 & 5 & 8 \\ 8 & 12 & 21 & 33 \\ 20 & 32 & 54 & 87 \end{bmatrix} \neq H_F R_F = \begin{bmatrix} 15 & 14 & 13 & 9 \\ 24 & 23 & 21 & 15 \\ 39 & 37 & 34 & 24 \\ 63 & 60 & 55 & 38 \end{bmatrix}, \tag{2.8}$$

though they have their own independent internal and diagonal patterns; for example, the leading diagonal sums of H_F form the same sequences as the row sums of R_F, namely $\{nF_n\}$. Further explorations are fairly obvious.

3 Extensions of the Rhaly terraced matrices

In this section, r-Terraced $R_{r,n}$ matrices, which are a more general form of Terraced matrices, will be created and the general properties of these matrices will be introduced.

Let $r \in \mathbb{C}$ and $t_1 < t_2 < \ldots t_n$. r-Terraced $R_{r,n}$ is represent as follows:

$$R_{r,n} = \begin{bmatrix} t_1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ rt_2 & t_2 & 0 & 0 & 0 & \cdots & 0 & 0 \\ rt_3 & rt_3 & t_3 & 0 & 0 & \cdots & 0 & 0 \\ rt_4 & rt_4 & rt_4 & t_4 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ rt_n & rt_n & rt_n & rt_n & rt_n & \cdots & rt_n & t_n \end{bmatrix} \tag{3.1}$$

and their elements satisfy:

$$R_{i,j} = \begin{cases} 0, & if\ j > j \\ rt_k, & if\ j < i \\ t_k, & i = j \end{cases}$$

Theorem 3.1 *Let $R_{r,n}(\lambda)$ be the characteristic polynomial of matrix $R_{r,n}$. Then $R_{r,n}(\lambda)$ satisfies recurrence relation as in follows:*

$$R_{r,n}(\lambda) = (t_n - \lambda)R_{r,n-1}(\lambda) \tag{3.2}$$

where $R_{r,1}(\lambda) = (t_1 - \lambda)$.

Proof Let the characteristic polynomials of $R_{r,n}$, $R_{r,n}(\lambda)$.

$$R_{r,n}(\lambda) = \det\big(R_{r,n} - \lambda I\big) = \begin{vmatrix} t_1-\lambda & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ rt_2 & t_2-\lambda & 0 & 0 & 0 & \cdots & 0 & 0 \\ rt_3 & rt_3 & t_3-\lambda & 0 & 0 & \cdots & 0 & 0 \\ rt_4 & rt_4 & rt_4 & t_4-\lambda & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ rt_n & rt_n & rt_n & rt_n & rt_n & \cdots & rt_n & t_n-\lambda \end{vmatrix}$$

$$= (t_1 - \lambda)(t_2 - \lambda)\ldots(t_n - \lambda) = \prod_{i=1}^{n}(t_i - \lambda).$$

When we expand this determinant by the last column, we obtain the recurrence relation for the characteristic polynomial of the r-Terraced matrix as follows:

$$R_{r,n}(\lambda) = (t_n - \lambda)R_{r,n-1}(\lambda)$$

Theorem 3.2 *Assume that $R_{r,n}$ is a matrix in (3.1), the given matrix can be written as:*

$$R_{r,n} = \begin{bmatrix} t_1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ rt_2 & t_2 & 0 & 0 & 0 & \cdots & 0 & 0 \\ rt_3 & rt_3 & t_3 & 0 & 0 & \cdots & 0 & 0 \\ rt_4 & rt_4 & rt_4 & t_4 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ rt_n & rt_n & rt_n & rt_n & rt_n & \cdots & rt_n & t_n \end{bmatrix} = \begin{bmatrix} R_{r,n-1} & 0 \\ rY & t_n \end{bmatrix}$$

where $Y = (t_n, t_n, t_n, \ldots, t_n)$. If $R_{r,n}$ is non-singular matrix, then the inverse of $R_{r,n}$ is

$$R_{r,n}^{-1} = \begin{bmatrix} R_{r,n-1}^{-1} & 0 \\ -rkYR_{r,n-1}^{-1} & k \end{bmatrix}$$

in which $k = \frac{1}{t_n}$.

Proof We can prove this theorem by mathematical induction on n. For $n = 2$ we obtain

$$R_{r,2} = \begin{bmatrix} t_1 & 0 \\ rt_2 & t_2 \end{bmatrix}$$

$$R_{r,2}^{-1} = \frac{1}{\det(R_{r,2})} \begin{bmatrix} t_2 & 0 \\ -rt_2 & t_1 \end{bmatrix} = \begin{bmatrix} \frac{1}{t_1} & 0 \\ \frac{-r}{t_1} & \frac{1}{t_2} \end{bmatrix}$$

On the other hand, for $n = 2$, we obtain

$$R_{r,2}^{-1} = \begin{bmatrix} \frac{1}{t_1} & 0 \\ \frac{-r}{t_2} t_2 \frac{1}{t_1} & \frac{1}{t_2} \end{bmatrix} = \begin{bmatrix} \frac{1}{t_1} & 0 \\ \frac{-r}{t_1} & \frac{1}{t_2} \end{bmatrix}$$

Our conclusion is true for $n = 2$. Assume that our claim is true for $n - 1$. Then identity $R_{r,n-1} R_{r,n-1}^{-1} = I_{n-1}$. We show that the result is true for all n. By multiplying $R_{r,n}$ and $R_{r,n}^{-1}$ together, we find that

$$\begin{bmatrix} R_{r,n-1} & 0 \\ rY & t_n \end{bmatrix} \begin{bmatrix} R_{r,n-1}^{-1} & 0 \\ -rkY R_{r,n-1}^{-1} & k \end{bmatrix} = \begin{bmatrix} R_{r,n-1} R_{r,n-1}^{-1} & 0 \\ rY R_{r,n-1}^{-1} - t_n rkY R_{r,n-1}^{-1} & kt_n \end{bmatrix} = \begin{bmatrix} I_{n-1} & 0 \\ 0 & 1 \end{bmatrix}.$$

Theorem 3.3 *Let the $r-$ Terraced matrix $R_{r,n}$ be as in (3.1)* and $t_n > t_{n-1} > \ldots > t_3 > t_2 > t_1$. *Then we have the upper bound for the spectral norm as follows:*

$$\|R_{r,n}\|_2 \leq \sqrt{\left((n-1)|r|^2 + 1\right) \sum_{i=1}^{n} |t_i|^2} \tag{3.3}$$

In addition, the l_p norm and Euclidean norm of the $r-$ Terraced matrix $R_{r,n}$ provide the following equalities:

$$\|R_{r,n}\|_p^p = \sum_{i=1}^{n} |t_i|^p + |rt_n|^p + \sum_{i=2}^{n} |rt_i|^p + \sum_{i=3}^{n} |rt_i|^p + \cdots + \sum_{i=n-1}^{n} |rt_i|^p \tag{3.4}$$

$$\|R_{r,n}\|_E^2 = \sum_{i=1}^{n} |t_i|^2 + |rt_n|^2 + \sum_{i=2}^{n} |rt_i|^2 + \sum_{i=3}^{n} |rt_i|^2 + \cdots + \sum_{i=n-1}^{n} |rt_i|^2 \tag{3.5}$$

Proof Let $AandB$ form the following matrices:

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ r & 1 & 0 & 0 & \cdots & 0 & 0 \\ r & r & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \cdots & \vdots & \vdots \\ r & r & r & \ddots & \cdots & 1 & 0 \\ r & r & r & r & \cdots & r & 1 \end{bmatrix}$$

and

$$B = \begin{bmatrix} t_1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ t_2 & t_2 & 0 & 0 & \cdots & 0 & 0 & 0 \\ t_3 & t_3 & t_3 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ t_{n-1} & t_{n-1} & t_{n-1} & t_{n-1} & \cdots & t_{n-1} & t_{n-1} & 0 \\ t_n & t_n & t_n & t_n & \cdots & t_n & t_n & t_n \end{bmatrix}$$

Therefore, we obtain

$$\|R_{r,n}\|_2 \leq r_1(A)c_1(B)$$

where

$$r_1(A) = \sqrt{(n-1)|r|^2 + 1}$$

and

$$c_1(B) = \sqrt{\sum_{i=1}^{n} |t_i|^2}$$

Since $R_{r,n} = A \circ B$ and from Lemma 1.4, we obtain

$$\|R_{r,n}\|_2 \leq \sqrt{\left((n-1)|r|^2 + 1\right)\sum_{i=1}^{n} |t_i|^2}$$

From the definition of the l_p norm, we obtain (3.4). Taking $p = 2$ in (3.4) we get the Euclidean norm as in (3.5).

Theorem 3.4 *The coefficients of the characteristic polynomial*

$$R_{r,n}(\lambda) = \beta_n^{(n)}\lambda^n + \beta_{n-1}^{(n)}\lambda^{n-1} + \cdots + \beta_1^{(n)}\lambda + \beta_0^{(n)} \tag{3.6}$$

of the $r-$ Terraced matrix $R_{r,n}$ satisfy

$$\beta_{n-1}^{(n-1)} = 0,\ \beta_{n-2}^{(n-1)} = t_n,\ \beta_{n-i}^{(n-1)} = t_n\beta_{n-i+1}^{(n-1)},\ \beta_{n-1}^{(n)} = \beta_{n-2}^{(n-1)} + \beta_{n-1}^{(n)} - t_n.$$

Proof From (3.6) we have

$$R_{r,n-1}(\lambda) = \lambda^{n-1} + \beta_{n-2}^{(n-1)}\lambda^{n-2} + \cdots + \beta_1^{(n-1)}\lambda + \beta_0^{(n-1)} \tag{3.7}$$

Substituting (3.6), (3.7) in (3.2) and some after calculations we get

$$R_{r,n}(\lambda) = \beta_n^{(n)}\lambda^n + \beta_{n-1}^{(n)}\lambda^{n-1} + \cdots + \beta_1^{(n)}\lambda + \beta_0^{(n)}$$

$$= (t_n - \lambda)\beta_{n-1}^{(n-1)}\lambda^{n-1} + \beta_{n-2}^{(n-1)}\lambda^{n-2} + \beta_{n-3}^{(n-1)}\lambda^{n-3} + \beta_{n-4}^{(n-1)}\lambda^{n-4} + \cdots + \beta_1^{(n-1)}\lambda + \beta_0^{(n-1)}$$

$$\lambda^n\left(\beta_n^{(n)} + \beta_{n-1}^{(n-1)}\right) + \lambda^{n-1}\left(\beta_{n-1}^{(n)} - t_n + \beta_{n-2}^{(n-1)}\right) + \lambda^{n-2}\left(\beta_{n-2}^{(n)} - t_n\beta_{n-2}^{(n-1)} + \beta_{n-3}^{(n-1)}\right)$$
$$+ \lambda^{n-3}\left(\beta_{n-3}^{(n)} - t_n\beta_{n-3}^{(n-1)} + \beta_{n-4}^{(n-1)}\right) + \cdots + \lambda\left(\beta_1^{(n)} - t_n\beta_1^{(n-1)} + \beta_0^{(n-1)}\right) + t_n\beta_0^{(n-1)}$$

Thus, we can express it in the general form:

$$\beta_{n-2}^{(n-1)} = t_n$$

$$\beta_{n-3}^{(n-1)} = t_n\beta_{n-2}^{(n-1)},\ \beta_{n-4}^{(n-1)} = t_n\beta_{n-3}^{(n-1)},\ \beta_{n-i}^{(n-1)} = t_n\beta_{n-i+1}^{(n-1)}$$

Lemma 3.5 *For the $r-$ Terraced matrix $R_{r,n}$, we have*

$$tr\left(R_{r,n}\right) = t_1 + t_2 + t_3 + \cdots + t_n = \sum_{i=1}^{n} t_i$$

Proof This is clear from the definition of the trace of a matrix which is defined as the sum of the elements on its main diagonal.

Theorem 3.6 *The upper bound for the spread of the $r-$ terraced matrix $R_{r,n}$ is given by*

$$S\left(R_{r,n}\right) \leq \sqrt{2\left(\sum_{i=1}^{n}|t_i|^2+|rt_n|^2+\sum_{i=2}^{n}|rt_i|^2+\sum_{i=3}^{n}|rt_i|^2+\cdots+\sum_{i=n-1}^{n}|rt_i|^2\right)-\frac{2}{n}\left(\sum_{i=1}^{n}t_i\right)^2}$$

Proof Using Frobenius norm $\|R_{r,n}\|_E^2$ from Theorem 3.3 and trace formula from Lemma 3.5, we have following equation from Definition 1.3.

$$S\left(R_{r,n}\right) \leq \sqrt{2\|R_{r,n}\|_E^2-\frac{2}{n}\left|tr\left(R_{r,n}\right)\right|^2}$$

$$=\sqrt{2\left(\sum_{i=1}^{n}|t_i|^2+|rt_n|^2+\sum_{i=2}^{n}|rt_i|^2+\sum_{i=3}^{n}|rt_i|^2+\cdots+\sum_{i=n-1}^{n}|rt_i|^2\right)-\frac{2}{n}\left(\sum_{i=1}^{n}t_i\right)^2}$$

Theorem 3.7 *The rank of the $r-$ terraced matrix $R_{r,n}$ is n.*

Proof The rank of a matrix is defined as the maximum number of linearly independent rows or columns in the matrix. Also, the sum eigenvalues of $r-$ terraced matrix $R_{r,n}$ are equal $tr\left(R_{r,n}\right)$.

3.1 Example

In this sub-section, we provide a numerical example to validate our theoretical findings, utilizing Wolfram Alpha for the computations. The example focuses on matrices derived from the Fibonacci numbers F_n. The matrix F_{ij} was constructed with Fibonacci numbers on the main diagonal and sub diagonal entries scaled by the parameter r. The choice of Fibonacci numbers ensures a structured growth pattern influenced by the golden ratio, known for its inherent stability and occurrence in natural growth models. By varying r, we examined how the subdiagonal scaling affects both the spread and the Euclidean norm. In the examples given below, different norm values and therefore different upper bounds for the spread are obtained by increasing the size of the matrix and increasing r in different ways.

To begin, we define the Fibonacci numbers. These numbers are generated by the recurrence formula

$$F_{n+2} = F_{n+1} + F_n,$$

with the initial values $F_0 = 0$ and $F_1 = 1$.

In the example, in order to comply with the Theorem, we shall begin the Fibonacci numbers from 1 and continue as 1, 2, 3, 5, 8, 13, …

Let

$$R_{ij} = F_{ij} = \begin{cases} 0, & if\ j > i \\ r\,F_n, & if\ j < i \\ F_n, & if\ i = j \end{cases}$$

be a matrix as in matrix form in (3.1) for $n = 5$, $r = \frac{1}{2}$. Then we get

$$R_{\frac{1}{2},5} = F_{\frac{1}{2},5} = F_{ij} = \begin{bmatrix} F_1 & 0 & 0 & 0 & 0 \\ \frac{1}{2}F_2 & F_2 & 0 & 0 & 0 \\ \frac{1}{2}F_3 & \frac{1}{2}F_3 & F_3 & 0 & 0 \\ \frac{1}{2}F_4 & \frac{1}{2}F_4 & \frac{1}{2}F_4 & F_4 & 0 \\ \frac{1}{2}F_5 & \frac{1}{2}F_5 & \frac{1}{2}F_5 & \frac{1}{2}F_5 & F_5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 \\ \frac{3}{2} & \frac{3}{2} & 3 & 0 & 0 \\ \frac{5}{2} & \frac{5}{2} & \frac{5}{2} & 5 & 0 \\ 4 & 4 & 4 & 4 & 8 \end{bmatrix}$$

Using Theorems 3.1 and 3.4, the characteristic polynomial of the $r-$ Terraced matrix $F_{\frac{1}{2},5}$ satisfies the recurrence relation:

$$F_{\frac{1}{2},5}(\lambda) = (F_5 - \lambda)F_{\frac{1}{2},4}(\lambda)$$

The coefficients of the characteristic polynomial

$$\beta_5^{(5)}\lambda^5 + \beta_4^{(5)}\lambda^4 + \beta_3^{(5)}\lambda^3 + \beta_2^{(5)}\lambda^2 + \beta_1^{(5)}\lambda + \beta_0^{(5)}$$

of the $r-$ Terraced matrix $R_{\frac{1}{2},5}$ satisfy

$$F_{\frac{1}{2},3}(\lambda) = (F_3 - \lambda)R_{\frac{1}{2},2}(\lambda) = -\lambda^3 + 6\lambda^2 - 11\lambda + 6$$

$$F_{\frac{1}{2},4}(\lambda) = (F_4 - \lambda)R_{\frac{1}{2},3}(\lambda) = \lambda^4 - 11\lambda^3 + 41\lambda^2 - 61\lambda + 30$$

$$F_{\frac{1}{2},5}(\lambda) = (F_5 - \lambda)R_{\frac{1}{2},4}(\lambda) = -\lambda^5 + 19\lambda^4 - 129\lambda^3 + 389\lambda^2 - 518\lambda + 240$$

On the other hand, for the characteristic polynomial $F_{\frac{1}{2},5}(\lambda)$ we obtain, the sum of the roots is 19 and this is equal to the sum of the eigenvalues, with $tr\left(F_{\frac{1}{2},5}(\lambda)\right) = 19$, as can be clearly seen from Theorem 4. By virtue of Theorem 2, $F_{\frac{1}{2},5}$ matrix is written as follows:

$$F_{\frac{1}{2},5} = \begin{bmatrix} F_{\frac{1}{2},4} & 0 \\ \frac{1}{2}Y & F_5 \end{bmatrix},$$

where $Y = (F_5, F_5, \ldots, F_5)$

$$F_{\frac{1}{2},5} = \begin{bmatrix} F_1 & 0 & 0 & 0 & 0 \\ \frac{1}{2}F_2 & F_2 & 0 & 0 & 0 \\ \frac{1}{2}F_3 & \frac{1}{2}F_3 & F_3 & 0 & 0 \\ \frac{1}{2}F_4 & \frac{1}{2}F_4 & \frac{1}{2}F_4 & F_4 & 0 \\ \frac{1}{2}F_5 & \frac{1}{2}F_5 & \frac{1}{2}F_5 & \frac{1}{2}F_5 & F_5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 \\ \frac{3}{2} & \frac{3}{2} & 3 & 0 & 0 \\ \frac{5}{2} & \frac{5}{2} & \frac{5}{2} & 5 & 0 \\ 4 & 4 & 4 & 4 & 8 \end{bmatrix}$$

The inverse matrix of $F_{\frac{1}{2},5}$ is as follows:

$$\left(F_{\frac{1}{2},5}\right)^{-1} = \begin{bmatrix} \left(F_{\frac{1}{2},4}\right)^{-1} & 0 \\ -\frac{1}{2}\frac{1}{F_5}Y\left(F_{\frac{1}{2},4}\right)^{-1} & \frac{1}{F_5} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ -\frac{1}{4} & -\frac{1}{4} & \frac{1}{3} & 0 & 0 \\ -\frac{1}{8} & -\frac{1}{8} & -\frac{1}{6} & \frac{1}{5} & 0 \\ -\frac{1}{16} & -\frac{1}{16} & -\frac{1}{12} & -\frac{1}{10} & \frac{1}{8} \end{bmatrix}$$

The spectral norm of matrix $F_{\frac{1}{2},5}$ is calculated as

$$\left(F_{\frac{1}{2},5}\right)^T F_{\frac{1}{2},5} = \begin{bmatrix} 26{,}5 & 26{,}5 & 26{,}75 & 28{,}5 & 32 \\ 26{,}5 & 28{,}5 & 26{,}75 & 28{,}5 & 32 \\ 26{,}75 & 26{,}75 & 31{,}25 & 28{,}5 & 32 \\ 28{,}5 & 28{,}5 & 28{,}5 & 41 & 32 \\ 32 & 32 & 32 & 32 & 64 \end{bmatrix}$$

$$\det\left(\left(F_{\frac{1}{2},5}\right)^T F_{\frac{1}{2},5} - \lambda I\right) = 0 = -\lambda^5 + 191{,}25\lambda^4 - 5488{,}125\lambda^3 + 44237{,}625\lambda^2 - 108030\lambda + 57600$$

$$\lambda_1 = 0{,}7345,\ \lambda_2 = 3{,}0432,\ \lambda_3 = 7{,}5286,\ \lambda_4 = 21{,}62,\ \lambda_5 = 158{,}3237$$

Spectral norm is $\|F_{\frac{1}{2},5}\|_2 = \sqrt{\lambda_{max}} = \sqrt{158{,}3237} \cong 12{,}5827$

We know that from Theorem 3.3,

$$\|F_{\frac{1}{2},5}\|_2 \leq \sqrt{\left((n-1)|r|^2+1\right)\sum_{i=1}^{n}|F_i|^2}$$
$$= \sqrt{\left(4\frac{1}{4}+1\right)\sum_{i=1}^{5}|F_i|^2} = \sqrt{2\left(F_1^2+F_2^2+F_3^2+F_4^2+F_5^2\right)}$$
$$= \sqrt{2(1+4+9+25+64)} \cong 14{,}36$$

$$\|F_{\frac{1}{2},5}\|_2 = 12{,}5827 \leq 14{,}36$$

as desired.

The Euclidean norm of $F_{\frac{1}{2},5}$ as follows:

$$\|F_{\frac{1}{2},5}\|_E^2 = \sum_{i=1}^{5}|F_i|^2 + \left|\frac{1}{2}F_5\right|^2 + \sum_{i=2}^{5}\left|\frac{1}{2}F_i\right|^2 + \sum_{i=3}^{5}\left|\frac{1}{2}F_i\right|^2 + \sum_{i=4}^{5}\left|\frac{1}{2}F_i\right|^2$$

$$= F_1^2+F_2^2+F_3^2+F_4^2+F_5^2+\left(\frac{1}{2}F_5\right)^2+\left(\frac{1}{2}F_2\right)^2+\left(\frac{1}{2}F_3\right)^2+\left(\frac{1}{2}F_4\right)^2$$
$$+\left(\frac{1}{2}F_5\right)^2+\left(\frac{1}{2}F_3\right)^2+\left(\frac{1}{2}F_4\right)^2+\left(\frac{1}{2}F_5\right)^2+\left(\frac{1}{2}F_4\right)^2+\left(\frac{1}{2}F_5\right)^2 = 191{,}25$$

Finally, let's find an upper bound for the spread value of the $r-$ Terraced matrix $F_{\frac{1}{2},5}$:

$$S\left(F_{\frac{1}{2},5}\right) \leq \sqrt{2\|F_{\frac{1}{2},5}\|_E^2 - \frac{2}{5}\left|tr\left(F_{\frac{1}{2},5}\right)\right|^2}$$
$$= \sqrt{2\left(\sum_{i=1}^{5}|F_i|^2 + \left|\frac{1}{2}F_5\right|^2 + \sum_{i=2}^{5}\left|\frac{1}{2}F_i\right|^2 + \sum_{i=3}^{5}\left|\frac{1}{2}F_i\right|^2 + \sum_{i=4}^{5}\left|\frac{1}{2}F_i\right|^2\right) - \frac{2}{5}\left(\sum_{i=1}^{5}F_i\right)^2}$$
$$= \sqrt{2\left(\begin{array}{c} F_1^2+F_2^2+F_3^2+F_4^2+F_5^2+\left(\frac{1}{2}F_5\right)^2+\left(\frac{1}{2}F_2\right)^2+\left(\frac{1}{2}F_3\right)^2+\left(\frac{1}{2}F_4\right)^2 \\ +\left(\frac{1}{2}F_5\right)^2+\left(\frac{1}{2}F_3\right)^2+\left(\frac{1}{2}F_4\right)^2+\left(\frac{1}{2}F_5\right)^2+\left(\frac{1}{2}F_4\right)^2+\left(\frac{1}{2}F_5\right)^2 \end{array}\right)}$$

$$-\frac{2}{5}(F_1+F_2+F_3+F_4+F_5)^2$$

$$=\sqrt{223{,}6}\cong 14{,}9$$

$$S\left(F_{\frac{1}{2},5}\right)\le 14{,}9.$$

Now let's consider the Fibonacci values of the same size for $r=\frac{1}{4}$:

$$S\left(F_{\frac{1}{4},5}\right)\le\sqrt{2\|F_{\frac{1}{4},5}\|_E^2-\frac{2}{5}\left|tr\left(F_{\frac{1}{4},5}\right)\right|^2}$$

$$=\sqrt{2\begin{pmatrix}F_1^2+F_2^2+F_3^2+F_4^2+F_5^2+\left(\frac{1}{4}F_5\right)^2+\left(\frac{1}{4}F_2\right)^2+\left(\frac{1}{4}F_3\right)^2+\left(\frac{1}{4}F_4\right)^2\\+\left(\frac{1}{4}F_5\right)^2+\left(\frac{1}{4}F_3\right)^2+\left(\frac{1}{4}F_4\right)^2+\left(\frac{1}{4}F_5\right)^2+\left(\frac{1}{4}F_4\right)^2+\left(\frac{1}{4}F_5\right)^2\end{pmatrix}-\frac{2}{5}(F_1+F_2+F_3+F_4+F_5)^2}$$

$$=\sqrt{75{,}475}\cong 8{,}68$$

$$S\left(F_{\frac{1}{4},5}\right)\le 8{,}68$$

Finally, let's consider the Fibonacci values of the same size for $r=2$:

$$S\left(F_{2,5}\right)\le\sqrt{2\|F_{2,5}\|_E^2-\frac{2}{5}\left|tr\left(F_{2,5}\right)\right|^2}$$

$$=\sqrt{2\begin{pmatrix}F_1^2+F_2^2+F_3^2+F_4^2+F_5^2+(2F_5)^2+(2F_2)^2+(2F_3)^2+(2F_4)^2\\+(2F_5)^2+(2F_3)^2+(2F_4)^2+(2F_5)^2+(2F_4)^2+(2F_5)^2\end{pmatrix}-\frac{2}{5}(F_1+F_2+F_3+F_4+F_5)^2}$$

$$=\sqrt{2885{,}6}\cong 56{,}34$$

$$S\left(F_{2,5}\right)\le 56{,}34$$

Now let's find a new upper bound by increasing the matrix size, i.e. for the values $n=6$, $r=\frac{1}{4}$:

$$S\left(F_{\frac{1}{4},6}\right)\le\sqrt{2\|F_{\frac{1}{4},6}\|_E^2-\frac{2}{5}\left|tr\left(F_{\frac{1}{4},6}\right)\right|^2}$$

$$=\sqrt{2\begin{pmatrix}F_1^2+F_2^2+F_3^2+F_4^2+F_5^2+F_6^2+\left(\frac{1}{4}F_6\right)^2+\left(\frac{1}{4}F_2\right)^2+\left(\frac{1}{4}F_3\right)^2+\left(\frac{1}{4}F_4\right)^2\\+\left(\frac{1}{4}F_5\right)^2+\left(\frac{1}{4}F_6\right)^2\\\left(\frac{1}{4}F_3\right)^2+\left(\frac{1}{4}F_4\right)^2+\left(\frac{1}{4}F_5\right)^2+\left(\frac{1}{4}F_6\right)^2+\left(\frac{1}{4}F_4\right)^2+\left(\frac{1}{4}F_5\right)^2+\left(\frac{1}{4}F_6\right)^2\end{pmatrix}-\frac{2}{6}(F_1+F_2+F_3+F_4+F_5+F_6)^2}$$

$$=\sqrt{320{,}45}\cong 17{,}90$$

$$S\left(F_{\frac{1}{4},6}\right)\le 17{,}90$$

The Fibonacci numbers on the diagonal impart a structured spectral pattern influenced by the golden ratio. As r increases, this structured growth is disrupted, highlighting the delicate balance between the diagonal's stability-driven sequence and the sub diagonal's scaling factor.

Smaller r values preserve the inherent stability provided by the Fibonacci sequence, while larger r values stretch the eigenvalues, revealing the underlying sensitivity to perturbations. This interplay is crucial for applications requiring customizable stability profiles.

Here's an improved and more insightful explanation with stronger academic language and reasoning:

The constructed matrix was analyzed by choosing as consecutive Fibonacci numbers and varying the parameter r over the set $\left\{\frac{1}{2}, \frac{1}{4}, 2\right\}$. The analysis revealed that the smallest upper bound for the spread was achieved when $r = \frac{1}{4}$. Upon increasing the matrix dimension to include $F_6 = 13$, the upper bound increased, despite using the same optimal r value.

Now let's make a detailed explanation according to r r values:

1. For $r = \frac{1}{4}$ (Upper Bound $= 8.68$):

   At this smaller r value, the sub diagonal elements are relatively weak compared to the dominant Fibonacci diagonal entries. This leads to a matrix that is strongly diagonally dominant, resulting in eigenvalues that are closely clustered. Particularly useful in signal and coding theory, where closely packed eigenvalues yield more efficient filtering and encoding.
2. For $r = \frac{1}{2}$ (Upper Bound $= 14.9$):

   Doubling parameter r increases the weight of the sub diagonal elements causing a moderate dispersion of the eigenvalues. Here the matrix exhibits less diagonal dominance and the spread grows accordingly. Engineering systems where controlled eigenvalue separation is needed may leverage this configuration.
3. For $r = 2$ (Upper Bound $= 56.34$):

   With a significantly larger r, the sub diagonal elements become comparable to or larger than the main diagonal entries, greatly reducing diagonal dominance. As a result, the spread increases sharply, reflecting a wide dispersion of eigenvalues. Large spreads may be beneficial in cryptographic algorithms requiring eigenvalue diversity to enhance security measures.

This finding highlights two important insights. First, selecting r values less than one generally improves the spread's upper bound, indicating a stabilizing effect on the spectral distribution. Second, increasing the matrix dimension tends to raise the upper bound, implying that larger matrices may not always be advantageous for minimizing spread-related criteria. From both a theoretical and practical perspective, these results suggest that working with lower-dimensional matrices and appropriately chosen parameters particularly with $r < 1$ can yield more favorable spectral properties while reducing computational complexity and time. This balance between matrix size and parameter choice is crucial for efficient and effective matrix-based computations.

In Fig. 1, variation of the upper bound of the spread with respect to the parameter r is given for increasing and decreasing values of r for the same size matrix:

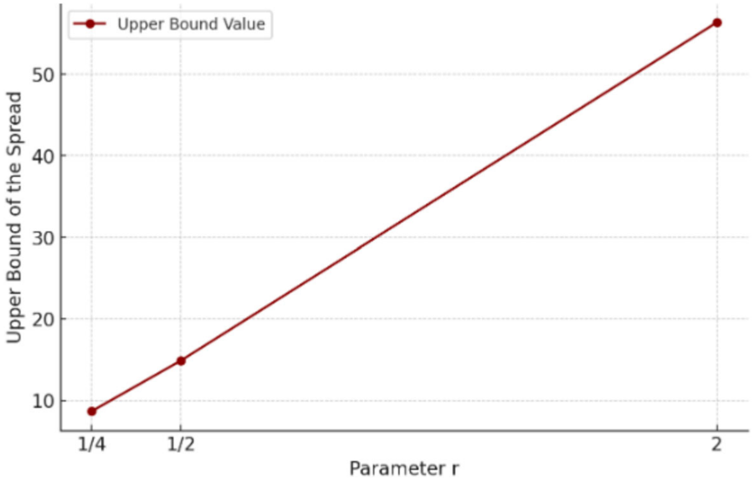

Fig. 1 Variation of the upper bound of the spread with respect to the parameter r

4 Generalized symmetric Rhaly r-terraced matrices

In this section, generalized symmetric $r-$ Terraced $SR_{r,n}$ matrices, which are a more general form of Terraced matrices with symmetric entries, will be created and the general properties of these matrices will be introduced.

Let $r \in \mathbb{C}$ and $t_1 < t_2 < \ldots t_n$. Generalized symmetric $r-$ Terraced $SR_{r,n}$ is represent as follows:

$$SR_{r,n} = \begin{bmatrix} t_1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ t_2 & t_2 & 0 & 0 & 0 & \cdots & 0 & 0r \\ t_3 & t_3 & t_3 & 0 & 0 & \cdots & 0r & 0r \\ t_4 & t_4 & t_4 & t_4 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ t_{n-1} & t_{n-1} & rt_{n-1} & rt_{n-1} & rt_{n-1} & \cdots & rt_{n-1} & 0r \\ t_n & rt_n & rt_n & t_n & t_n & \cdots & rt_n & rt_n \end{bmatrix} \tag{4.1}$$

Theorem 4.1 Let $SR_{r,n}(\lambda)$ be the characteristic polynomial of matrix $SR_{r,n}$.
Then $SR_{r,n}(\lambda)$ satisfies the recurrence relation is as follows:

$$SR_{r,n}(\lambda) = (rt_n - \lambda)SR_{r,n-1}(\lambda) \tag{4.2}$$

where $SR_{r,1}(\lambda) = (t_1 - \lambda)$.

Proof Let the characteristic polynomials of $SR_{r,n}$, $SR_{r,n}(\lambda)$.

$$SR_{r,n}(\lambda) = \det\left(SR_{r,n} - \lambda I\right)$$

$$= \begin{vmatrix} t_1-\lambda & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ rt_2 & t_2-\lambda & 0 & 0 & 0 & \cdots & 0 & 0 \\ rt_3 & rt_3 & t_3-\lambda & 0 & 0 & \cdots & 0 & 0 \\ rt_4 & rt_4 & rt_4 & t_4-\lambda & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ t_{n-1} & t_{n-1} & rt_{n-1} & rt_{n-1} & rt_{n-1} & \cdots & rt_{n-1}-\lambda & 0 \\ t_n & rt_n & rt_n & rt_n & rt_n & \cdots & rt_n & rt_n-\lambda \end{vmatrix}$$

If n is odd;

$$\det\left(SR_{r,n}-\lambda I\right)=(t_1-\lambda)(t_2-\lambda)\ldots\left(t_{\frac{n+1}{2}}-\lambda\right).\left(\boldsymbol{r}t_{\frac{n+1}{2}+1}-\lambda\right)\left(\boldsymbol{r}t_{\frac{n+1}{2}+2}-\lambda\right)\ldots(rt_n-\lambda).$$

If n is even;

$$\det\left(SR_{r,n}-\lambda I\right) = (t_1-\lambda)(t_2-\lambda)\ldots\left(t_{\frac{n}{2}}-\lambda\right).\left(\boldsymbol{r}t_{\frac{n}{2}+1}-\lambda\right)\left(\boldsymbol{r}t_{\frac{n}{2}+2}-\lambda\right)\ldots(rt_n-\lambda).$$

When we expand this determinant by the last column, we obtain the recurrence relation for the characteristic polynomial of the generalized symmetric r-Terraced matrix as follows:

$$SR_{r,n}(\lambda) = (rt_n-\lambda)SR_{r,n-1}(\lambda)$$

Theorem 4.2 Assume that $SR_{r,n}$ is a matrix in *(4.1)*, the given matrix can be written as:

$$SR_{r,n} = \begin{bmatrix} t_1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ t_2 & t_2 & 0 & 0 & 0 & \cdots & 0 & 0r \\ t_3 & t_3 & t_3 & 0 & 0 & \cdots & 0r & 0r \\ t_4 & t_4 & t_4 & t_4 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ t_{n-1} & t_{n-1} & rt_{n-1} & rt_{n-1} & rt_{n-1} & \cdots & rt_{n-1} & 0r \\ t_n & rt_n & rt_n & t_n & t_n & \cdots & rt_n & rt_n \end{bmatrix} = \begin{bmatrix} SR_{r,n-1} & 0 \\ K & rt_n \end{bmatrix}$$

where $K = (t_n, rt_n, rt_n, \ldots, rt_n)$. If $SR_{r,n}$ is non-singular matrix, then the inverse of $SR_{r,n}$ is

$$SR_{r,n}^{-1} = \begin{bmatrix} SR_{r,n-1}^{-1} & 0 \\ -sKSR_{r,n-1}^{-1} & s \end{bmatrix}$$

in which $s = \frac{1}{rt_n}$.

Proof We can prove this theorem by mathematical induction on n. For $n = 2$ we obtain

$$SR_{r,2} = \begin{bmatrix} t_1 & 0 \\ t_2 & rt_2 \end{bmatrix}$$

$$SR_{r,2}^{-1} = \frac{1}{\det(R_{r,2})}\begin{bmatrix} rt_2 & 0 \\ -t_2 & t_1 \end{bmatrix} = \begin{bmatrix} \frac{1}{t_1} & 0 \\ \frac{-1}{rt_1} & \frac{1}{rt_2} \end{bmatrix}$$

On the other hand, for $n=2$, we obtain

$$SR_{r,2}^{-1}=\begin{bmatrix}\frac{1}{t_1} & 0\\ \frac{-1}{rt_2}t_2\frac{1}{t_1} & \frac{1}{rt_2}\end{bmatrix}=\begin{bmatrix}\frac{1}{t_1} & 0\\ \frac{-1}{rt_1} & \frac{1}{rt_2}\end{bmatrix}$$

Our conclusion is true for $n=2$. Assume that our claim is true for $n-1$. Then identity $SR_{r,n-1}SR_{r,n-1}^{-1}=I_{n-1}$. We show that the result is true for all n. By multiplying $SR_{r,n}$ and $SR_{r,n}^{-1}$ together, we find that

$$\begin{bmatrix}SR_{r,n-1} & 0\\ K & rt_n\end{bmatrix}\begin{bmatrix}SR_{r,n-1}^{-1} & 0\\ -sKSR_{r,n-1}^{-1} & s\end{bmatrix}=\begin{bmatrix}SR_{r,n-1}SR_{r,n-1}^{-1} & 0\\ KSR_{r,n-1}^{-1}-t_nrsKSR_{r,n-1}^{-1} & srt_n\end{bmatrix}$$
$$=\begin{bmatrix}1 & 0\\ I_{n-1} & 1\end{bmatrix}.$$

Theorem 4.3 *Let the symmetric $r-$ Terraced matrix $SR_{r,n}$ be as in (4.1) and $t_n>t_{n-1}>\ldots>t_3>t_2>t_1$. Then we have the upper bound for the spectral norm as follows:*

If $r>1$

$$\|SR_{r,n}\|_2\leq\sqrt{\left((n-1)|r|^2+1\right)\sum_{i=1}^{n}|t_i|^2} \tag{4.3}$$

If $r<1$

$$\|SR_{r,n}\|_2\leq\begin{cases}\sqrt{\left(\frac{n}{2}|r|^2+\frac{n}{2}\right)\sum_{i=1}^{n}|t_i|^2}, \text{ If } n \text{ is even}\\ \sqrt{\left(\frac{n+1}{2}+\left(\frac{n+1}{2}-1\right)|r|^2\right)\sum_{i=1}^{n}|t_i|^2}, \text{ If } n \text{ is odd}\end{cases} \tag{4.4}$$

In addition, the l_p norm and Euclidean norm of the symmetric $r-$ Terraced matrix $SR_{r,n}$ provide the following equalities:

If n is even;

$$\|SR_{r,n}\|_p^p=\sum_{i=1}^{\frac{n}{2}}i|t_i|^p+\sum_{i=\frac{n}{2}}^{n}(n-i)|t_{i+1}|^p+\sum_{i=\frac{n}{2}+1}^{n}(2i-n-1)|rt_i|^p \tag{4.5}$$

$$\|SR_{r,n}\|_E^2=\sum_{i=1}^{\frac{n}{2}}i|t_i|^2+\sum_{i=\frac{n}{2}}^{n}(n-i)|t_{i+1}|^2+\sum_{i=\frac{n}{2}+1}^{n}(2i-n-1)|rt_i|^2 \tag{4.6}$$

If n is odd;

$$\|SR_{r,n}\|_p^p=\sum_{i=1}^{\frac{n+1}{2}}i|t_i|^p+\sum_{i=\frac{n+1}{2}+1}^{n}(n-i+1)|t_i|^p+\sum_{i=\frac{n+1}{2}+1}^{n}(i-2)|rt_i|^p \tag{4.7}$$

$$\|SR_{r,n}\|_E^2=\sum_{i=1}^{\frac{n+1}{2}}i|t_i|^2+\sum_{i=\frac{n+1}{2}+1}^{n}(n-i+1)|t_i|^2+\sum_{i=\frac{n+1}{2}+1}^{n}(i-2)|rt_i|^2 \tag{4.8}$$

Proof Let $A' and B$ form the following matrices:

$$A' = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 0 & 0 & \cdots & 0 & r \\ 1 & 1 & 1 & 0 & \cdots & r & r \\ \vdots & \vdots & \vdots & \ddots & \cdots & \vdots & \vdots \\ 1 & 1 & r & \ddots & \cdots & r & r \\ 1 & r & r & r & \cdots & r & r \end{bmatrix}$$

and

$$B = \begin{bmatrix} t_1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ t_2 & t_2 & 0 & 0 & \cdots & 0 & 0 & 0 \\ t_3 & t_3 & t_3 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ t_{n-1} & t_{n-1} & t_{n-1} & t_{n-1} & \cdots & t_{n-1} & t_{n-1} & 0 \\ t_n & t_n & t_n & t_n & \cdots & t_n & t_n & t_n \end{bmatrix}$$

Therefore, we obtain

$$\|SR_{r,n}\|_2 \leq r_1(A')c_1(B)$$

where

$$r_1\left(A'\right) = \sqrt{(n-1)|r|^2 + 1}(\text{If } r > 1)$$

$$r_1\left(A'\right) = \sqrt{\left(\frac{n}{2}|r|^2 + \frac{n}{2}\right)}(\text{If } r < 1 \text{ and } n \text{ is even})$$

$$r_1\left(A'\right) = \sqrt{\left(\frac{n+1}{2} + \left(\frac{n+1}{2} - 1\right)|r|^2\right)}(\text{If } r < 1 \text{ and } n \text{ is odd})$$

and

$$c_1(B) = \sqrt{\sum_{i=1}^{n} |t_i|^2}$$

Since $SR_{r,n} = A' \circ B$ and from Lemma 1.4, we obtain
If $r > 1$

$$\|SR_{r,n}\|_2 \leq \sqrt{\left((n-1)|r|^2 + 1\right) \sum_{i=1}^{n} |t_i|^2}$$

If $r < 1$

$$\|SR_{r,n}\|_2 \leq \begin{cases} \sqrt{\left(\frac{n}{2}|r|^2 + \frac{n}{2}\right) \sum_{i=1}^{n} |t_i|^2}, \; If\ n\ is\ even \\ \sqrt{\left(\frac{n+1}{2} + \left(\frac{n+1}{2} - 1\right)|r|^2\right) \sum_{i=1}^{n} |t_i|^2}, If\ n\ is\ odd \end{cases}$$

From the definition of the l_p norm and Euclidean norm, we get (4.5)–(4.8).

Theorem 4.4 *The upper bound for the spread of the symmetric* $r-$ *Terraced matrix* $SR_{r,n}$ *is given by*

If n *is even;*

$$S\left(SR_{r,n}\right) \leq \sqrt{2\left(\sum_{i=1}^{\frac{n}{2}} i\,|t_i|^2 + \sum_{i=\frac{n}{2}}^{n} (n-i)\,|t_{i+1}|^2 + \sum_{i=\frac{n}{2}+1}^{n} (2i-n-1)\,|rt_i|^2\right) - \frac{2}{n}\left(\sum_{i=1}^{\frac{n}{2}} t_i + \sum_{i=\frac{n}{2}+1}^{n} rt_i\right)^2}$$

If n *is odd;*

$$S\left(SR_{r,n}\right) \leq \sqrt{2\left(\sum_{i=1}^{\frac{n+1}{2}} i\,|t_i|^2 + \sum_{i=\frac{n+1}{2}+1}^{n} (n-i+1)\,|t_i|^2 + \sum_{i=\frac{n+1}{2}+1}^{n} (i-2)\,|rt_i|^2\right) - \frac{2}{n}\left(\sum_{i=1}^{\frac{n+1}{2}} t_i + \sum_{i=\frac{n+1}{2}+1}^{n} rt_i\right)^2}$$

Proof Using Frobenius norm $\|SR_{r,n}\|_E^2$ from Theorem 4.2, Trace formula and Definition 1.3. We achieve the desired results.

4.1 Example

In this sub-section, we provide a numerical example to validate our theoretical findings, utilizing Wolfram Alpha for the computations. The example focuses on matrices derived from the Fibonacci numbers F_n.

Let be a matrix as in matrix form in (4.1) for $n = 5$, $r = \frac{1}{2}$. Then we get

$$SR_{\frac{1}{2},5} = SF_{\frac{1}{2},5} = SF_{ij} = \begin{bmatrix} F_1 & 0 & 0 & 0 & 0 \\ F_2 & F_2 & 0 & 0 & 0 \\ F_3 & F_3 & F_3 & 0 & 0 \\ F_4 & F_4 & \frac{1}{2}F_4 & \frac{1}{2}F_4 & 0 \\ F_5 & \frac{1}{2}F_5 & \frac{1}{2}F_5 & \frac{1}{2}F_5 & \frac{1}{2}F_5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 \\ 3 & 3 & 3 & 0 & 0 \\ 5 & 5 & \frac{5}{2} & \frac{5}{2} & 0 \\ 8 & 4 & 4 & 4 & 4 \end{bmatrix}$$

By virtue of Theorem 4.2, $SF_{\frac{1}{2},5}$ matrix is written as follows:

$$SF_{\frac{1}{2},5} = \begin{bmatrix} SR_{\frac{1}{2},4} & 0 \\ K & rF_5 \end{bmatrix}$$

The inverse matrix of $SF_{\frac{1}{2},5}$,

$$SF^{-1}_{\frac{1}{2},5} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & \frac{1}{3} & 0 & 0 \\ 0 & -\frac{1}{2} & -\frac{1}{3} & \frac{2}{5} & 0 \\ -1 & \frac{1}{2} & 0 & -\frac{2}{5} & \frac{1}{4} \end{bmatrix} = \begin{bmatrix} SF^{-1}_{\frac{1}{2},4} & 0 \\ -K\frac{2}{F_5}SF^{-1}_{\frac{1}{2},4} & \frac{2}{F_5} \end{bmatrix}$$

The spectral norm of matrix $SF_{\frac{1}{2},5}$ is calculated as

$$\left(SF_{\frac{1}{2},5}\right)^T SF_{\frac{1}{2},5} = \begin{bmatrix} 103 & 70 & 53{,}5 & 44{,}5 & 32 \\ 70 & 54 & 37{,}5 & 28{,}5 & 16 \\ 53{,}5 & 37{,}5 & 31{,}25 & 22{,}25 & 16 \\ 44{,}5 & 28{,}5 & 22{,}25 & 22{,}25 & 16 \\ 32 & 16 & 16 & 16 & 16 \end{bmatrix}$$

$$\det\left(\left(SF_{\frac{1}{2},5}\right)^T SF_{\frac{1}{2},5} - \lambda I\right) = \lambda^5 - 226{,}5\lambda^4 + 3776{,}75\lambda^3 - 14752{,}75\lambda^2 + 16577\lambda - 3600$$

$$\lambda_1 = 208{,}744,\ \lambda_2 = 12{,}707,\ \lambda_3 = 3{,}327,\ \lambda_4 = 0{,}284,\ \lambda_5 = 1{,}438$$

Spectral norm is $\|SF_{\frac{1}{2},5}\|_2 = \sqrt{\lambda_{max}} = \sqrt{208{,}744} \cong 14{,}448$

We know that from Theorem 4.3,

$$\|SF_{\frac{1}{2},5}\|_2 \le \sqrt{\left(\frac{n+1}{2} + \left(\frac{n+1}{2} - 1\right)|r|^2\right)\sum_{i=1}^{n} |F_i|^2} = \sqrt{(3{,}5)\,(1+4+9+25+64)} \cong 18{,}98$$

$$\|SF_{\frac{1}{2},5}\|_2 = 14{,}448 \le 18{,}98$$

This shows that inequality is achieved.

The Euclidean norm of $SF_{\frac{1}{2},5}$ as follows:

$$\|SF_{\frac{1}{2},5}\|_E^2 = \sum_{k=1}^{3} k\,|F_k|^2 + \sum_{k=4}^{5} (6-k)\,|F_k|^2$$
$$+ \sum_{k=4}^{5} (k-2)\left|\frac{1}{2}F_k\right|^2 = 1+8+27+50+64+\frac{25}{2}+48 = 210{,}5$$

The upper bound for the spread of the symmetric r-Terraced matrix as follows:

n is odd;

$$S\left(SF_{\frac{1}{2},5}\right)$$
$$\le \sqrt{2\left(\sum_{k=1}^{3} k\,|F_k|^2 + \sum_{k=4}^{5}(6-k)\,|F_k|^2 + \sum_{k=4}^{5}(k-2)\left|\frac{1}{2}F_k\right|^2\right) - \frac{2}{5}\left(\sum_{k=1}^{3} F_k + \sum_{k=4}^{5}\frac{1}{2}F_k\right)^2}$$
$$= \sqrt{421 - \frac{2}{5}\left(1+2+3+\frac{5}{2}+4\right)} = \sqrt{416}$$

So, we get

$$S\left(SF_{\frac{1}{2},5}\right) \le \sqrt{416} = 20.39.$$

5 Conclusion

In this study, we introduced the r-Terraced matrix as a generalization of the Terraced matrix and developed its symmetric counterpart, the symmetric r-Terraced matrix. Key properties of these matrices were derived, including their spectral and Euclidean norms, upper bounds for their spreads, and characteristic polynomials. By applying these theoretical results to Fibonacci numbers, we provided illustrative examples that not only validated the accuracy of our findings but also reinforced the broader applicability of the derived results.

The work presented here contributes to the ongoing study of structured matrices by extending the scope of Terraced matrices and exploring their theoretical and practical implications. The symmetric r-Terraced matrix, in particular, opens up new avenues for analysing symmetric structures in linear algebra and their potential applications.

Also, in this study based on the obtained results, it is observed that selecting values of r less than 1, leads to a smaller upper bound for the spread. This finding suggests that working with lower-dimensional matrices and choosing smaller r values not only yields higher-quality matrices in terms of spread minimization but also reduces computational effort. Therefore, it is both efficient and effective to focus on smaller matrices with carefully selected r values for achieving optimal results.

Suggestions for Future Research:

1. Application to other number sequences: Future studies could explore the application of r-Terraced matrices to other special sequences, such as Lucas numbers or Pell numbers, to investigate whether similar properties and patterns emerge.
2. Numerical analysis and optimization: The derived properties, such as norms and spreads, can be utilized in optimization problems or in the analysis of numerical methods that rely on structured matrices.
3. Generalization to higher dimensions: Extending the r-Terraced matrix framework to higher-dimensional tensors or multi-index matrices could provide deeper insights and broader applications in fields like data analysis and machine learning.
4. Connections with graph theory: The symmetric r-Terraced matrix could be studied in the context of graph theory, particularly for analysing adjacency or Laplacian matrices of structured graphs.

By building on the foundations laid in this work, further research can uncover additional properties and applications of r-Terraced matrices, cementing their role in both theoretical studies and applied mathematics.

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Declarations

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