Estimation of Multiple Period Expected Shortfall and Median Shortfall for Risk Management

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Abstract

With the regulatory requirements for risk management, Value at Risk (VaR) has become an essential tool in determining capital reserves to protect the risk induced by adverse market movements. The fact that VaR is not coherent has motivated the industry to explore alternative risk measures like expected shortfall. The first objective of this paper is to propose statistical methods for estimating multiple-period expected shortfall under GARCH models. In addition to the expected shortfall, we investigate a new tool called median shortfall to measure risk. The second objective of this paper is to develop backtesting methods for assessing the performance of expected shortfall and median shortfall estimators from statistical and financial perspectives. By applying our expected shortfall estimators and other existing approaches to seven international markets, we demonstrate the superiority of our methods with respect to statistical and practical evaluations. Our expected shortfall estimators likely provide an unbiased reference for setting the minimum capital required for safeguarding against expected loss.

Keywords: conditional kurtosis; expected shortfall; risk management; value at risk; volatility.

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1 Introduction

Managing risk in financial institutions has been the focus of much interest in many countries. In a qualitative sense, risk management amounts to setting up suitable regulations, monitoring systems and using proper disclosure mechanisms. From a quantitative perspective, the use of appropriate risk measures is an important issue. In developing criteria to describe measures of risk, Artzner et al. (1997, 1999) introduced coherent properties that proper risk measures should obey. The properties are related to the 'benefit of diversification' that financial participants would like to see in their portfolios. The subadditivity condition is one of the coherent properties which cannot be satisfied by the common risk measure, Value at Risk (VaR). Therefore, it is of practical need to investigate alternative risk measures that are coherent. Among all coherent measures, expected shortfall (Acerbi et al., 2001; Acerbi and Tasche, 2002; Tasche, 2002) is regarded as a good supplement to VaR, not just because it is optimal in some sense (Inui and Kijima, 2005), but also because it is closely linked to VaR. By definition, expected shortfall is interpreted as the mean loss when the loss in the investment exceeds the VaR level. Therefore, the expected shortfall gives an estimate of the amount of capital depreciated under worst-case scenarios that are quantified by VaR. One important question is how to estimate the multiple-period expected shortfall while capturing the term structure of volatility in the market. This paper develops a new method for estimating the multiple-period expected shortfall that is computationally feasible to use in practice. For a more general discussion of risk measures, we recommend McNeil et al. (2005) and Szego (2005).

Estimation methods for expected shortfall follow one of the two approaches, unconditional and conditional. Common methods using unconditional approach include sample averaging based on order statistics (Yoshiba and Yamai, 2002) and estimators using extreme value theories (Cotter and Dowd, 2006). Acerbi and Tasche (2002) showed that the sample average estimator is consistent. While there are technical difficulties in using a conditional approach, researchers believe that it is necessary to incorporate dynamic changes in the market to reflect the most updated risk level. Guidolin and Timmermann (2006) empirically investigated different models that produce expected shortfall forecasts. They used classical Monte Carlo methods to calculate the forecasts. One major limitation of any Monte Carlo method is that the computational effort can be too heavy to make the method widely applicable. The first objective of this paper is to develop a new multiple-period expected shortfall estimator based on the conditional kurtosis idea in Wong and So (2003) in the GARCH framework. Putting the estimation in the GARCH framework helps us to incorporate recent market volatility changes in our estimation. Generally speaking, we make use of conditional kurtosis to use the 'fat-tailed information' of the multiple-period return distributions to derive an elegant estimator that is simple to use and is easy to calculate. An analytical formula is proposed to replace the use of the Monte Carlo scheme. Advantages of our proposed method are twofold. Firstly, the conditional kurtosis and the analytical formula help us to understand the impact of tail fatness in the return distribution on the quantification of the average loss under worst-case scenarios. Secondly, our estimator, while having similar performance, is computationally much more efficient than the Monte Carlo estimator. As an useful extension, we also investigate a new tool to determine the median shortfall, which is defined as the median loss when the loss exceeds the VaR.

One important factor in developing a new risk estimator is to identify suitable assessment tools. For VaR, we can perform backtesting using both the unconditional and conditional coverage tests in Christoffersen (1998). The second objective of this paper is to propose tests for expected shortfall on the unbiasedness and financial cost functions that are related to the 'real' cost of generating poor expected shortfall estimates. The test of unbiasedness is carried out by a bootstrap method. The cost functions are defined based upon the belief that we want to keep the capital reserve in protecting risk small and adequate. In short, our expected shortfall estimator is examined based on statistical and financial tools that provide information on any deficiency in the estimator from theoretical and financial perspectives. The rest of this paper is organized as follows. Section 2 gives a brief review of common volatility models for risk calculations. Section 3 describes expected and median shortfalls as risk measures. Section 4 presents a new method for computing multiple-period expected shortfall estimation using conditional kurtosis. We outline assessment methods for expected shortfall and present empirical results from seven financial markets in Section 5. Concluding remarks are given in Section 6.

2 Common volatility models for risk calculations

Let r_t be the return at time t and let Ω_t be the publicly available information up to time t. The aggregate return, $R_{t,h}$, at time t for horizon h is given by $R_{t,h} = r_{t+1} + \cdots + r_{t+h}$, which represents the multiple-period return on investment from time t to time $t+h$. In general, if C is the current market value of a portfolio, a loss in the portfolio from time t to time $t+h$ can be calculated from

$$L_{t,h} = -CR_{t,h}.$$

Classical risk calculations usually involve quantifying the variability of $R_{t,h}$ by suitable statistical measures, like the standard deviation, the mean absolute deviation and the interquartile range. The above measures are based on the overall variation of the distribution of $R_{t,h}$. To focus on loss in an investment, it is necessary to use downside risk measures that are primarily related to the left side of the distribution of $R_{t,h}$ rather than the entire return domain. In recent years, it has become a norm for financial market risk managers to consider how extreme the loss of a portfolio would be with a predetermined probability level.

The downside risk measure, VaR or value at risk, is widely used by financial institutions because it was adopted in the capital adequacy framework. It is the maximum loss of a portfolio in a given holding period, h, with a predetermined probability, $1-p$, with p taking a small value. In general, the h-period VaR of that portfolio is given by

$$\mathrm{VaR}_{h,p} = -CV_{h,p}\,, \tag{1}$$

where $V_{h,p}$ is the pth percentile of the aggregate return distribution, that is, $P(L_{t,h} \leq \mathrm{VaR}_{h,p}) = 1-p$ or $P(R_{t,h} \leq V_{h,p}|\Omega_t) = p$ and the negative of it is also the VaR of a \$1 portfolio. Producing a reliable estimator for VaR entails the study of extreme percentiles of the conditional distribution of $R_{t,h}$ given the current information at time t with appropriate statistical models for returns. One popular model used by market practitioners is RiskMetrics pioneered by J.P. Morgan. The RiskMetrics model can be stated as

$$r_t = \sigma_t \varepsilon_t, \qquad \varepsilon_t \sim N(0,1), \tag{2}$$

$$\sigma_t^2 = (1-\lambda)\, r_{t-1}^2 + \lambda \sigma_{t-1}^2, \tag{3}$$

where σ_t is the variance of r_t given Ω_t. This model aims at accounting for the changing conditional variance σ_t by an exponetially weighted moving average. As suggested by J.P. Morgan, the decay factor, λ, is set at 0.94 for daily data and 0.97 for monthly data.

It is well known that the RiskMetrics model is an Integrated GARCH(1,1) model. In this paper, we also consider an asymmetric GARCH model to capture the volatility asymmetry that is observed in financial markets. We will see that capturing volatility asymmetry is a crucial factor in accurately estimating VaR and expected shortfalls. In general GARCH models, the conditional variance of r_t is independent of the sign of the pervious return r_{t-1}. However, there is strong evidence from market data that volatility responds differently to the rise and drop of the market. Therefore, the Quadratic GARCH model (Engle, 1990; Campbell and Hentschel, 1992 and Sentana, 1995) that we use here is more compatible with the stylized facts of asymmetric volatility than are the usual GARCH models. In particular, we model one-period returns using a QGARCH(1,1) model with t error:

$$r_t = \mu + \bar{r}_t, \qquad \bar{r}_t = \sigma_t \varepsilon_t, \qquad \varepsilon_t \sim t_\nu, \tag{4}$$

$$\sigma_t^2 = \alpha_0 + \alpha_1 \left(\bar{r}_{t-1} - b\right)^2 + \beta_1 \sigma_{t-1}^2, \tag{5}$$

where μ is the unconditional mean of the one-period return and t_ν is the standardized t distribution with the standard deviation equal to one. The parameter b is used to capture the volatility asymmetry effect. According to Wong and So (2003), a VaR estimator based on exact conditional variance is given by

$$\hat{V}_{h,p}^{[1]} = h\mu + \sqrt{Var(R_{t,h}|\Omega_t)}\Phi^{-1}(p),$$

where $Var(R_{t,h}|\Omega_t)$ is the exact conditional variance of $R_{t,h}$ under QGARCH, which can be obtained using the method described by Wong and So (2003). The VaR under RiskMetrics is given by

$$\hat{V}_{h,p}^{[2]} = \sqrt{h}\sigma_{t+1}\Phi^{-1}(p),$$

where $\Phi^{-1}(p)$ the pth percentile of the standard normal distribution. The above formula is derived from the usual square root of time rule and by assuming that the conditional distribution of $R_{t,h}$ is normal.

3 Mulitple period expected shortfall and median shortfall

VaR is easy to use as it is a cut-off value that separates future loss events into risky and non-risky scenarios. It is widely accepted in financial institutions for measuring risk and for determining suitable amounts of capital reserves, but it suffers from a number of weaknesses. VaR does not provide any information about the size of the potential loss when it is exceeded. In addition, Artzner et al. (1997, 1999) showed that VaR is not a coherent risk measure because it is not subadditive. This means that if two portfolios are merged to form a combined portfolio, the VaR of the combined portfolio may be greater than the sum of the VaRs of the two separate portfolios. The non-subadditivity feature of VaR violates the consensus regarding the benefit of risk diversification. Using VaR may then discourage the common financial practice of diversifying risk by combining risk positions. In the above regard, VaR should be used with care and alternative risk measures are indispensable.

A new risk measure called expected shortfall, which is closely related to VaR, was proposed by Acerbi et al. (2001). The h-period expected shortfall with probability p, denoted as $\mathrm{ES}_{h,p}$, is defined as the average loss of the worst $p \times 100\%$ scenarios of the portfolio in h periods. Statistically speaking, $\mathrm{ES}_{h,p}$ is related to VaR by (Acerbi and Tasche, 2002):

$$\mathrm{ES}_{h,p} = \frac{1}{p}\int_0^p \mathrm{VaR}_{h,u} du = -C\frac{1}{p}\int_0^p V_{h,u} du. \tag{6}$$

For example, if we believe that the average loss on the worst 1% of the possible outcomes for a portfolio in 10 days is \$1 million, then the 10-day expected shortfall with 1% probability is \$1 million. As in the VaR formula in (1), the h-period expected shortfall, $\mathrm{ES}_{h,p}$, can be written as $-CE_{h,p}$, where

$$E_{h,p} = \frac{1}{p}\int_0^p V_{h,u} du \tag{7}$$

is the negative expected shortfall of a \$1 portfolio and is governed by the conditional distribution of $R_{t,h}$ given Ω_t. It is obvious from the definition of expected shortfall in (6) that it is always greater than or equal to VaR (i.e., $\mathrm{ES}_{h,p} = -CE_{h,p} \geq -CV_{h,p} =$

$\text{VaR}_{h,p}$ implying $E_{h,p} \leq V_{h,p}$) with the same holding period, h, and probability, p. Acerbi and Tasche (2002) showed that expected shortfall, unlike VaR, is coherent. From the definition, expected shortfall measures the expected value of the worst losses and so it gives risk managers an estimate of their portfolios' losses under financial turmoil or distress. The probability, p, provides flexibility to risk professionals for assessment of different levels of financial instability that lead to losses in their portfolios. When the distribution of $R_{t,h}$ is continuous as assumed under RiskMetrics and QGARCH, the expected shortfall is shown to be identical to the tail conditional expectation (Acerbi and Tasche, 2002); that is,

$$\begin{aligned} \text{ES}_{h,p} = E(L_{t,h}|L_{t,h} \geq \text{VaR}_{h,p}) \quad &\Longleftrightarrow -C\text{E}_{h,p} = E(-CR_{t,h}| - CR_{t,h} \geq -C\text{V}_{h,p}), \\ &\Longleftrightarrow \text{E}_{h,p} = E(R_{t,h}|R_{t,h} \leq \text{V}_{h,p}), \end{aligned} \tag{8}$$

which is the expected return when the return is below $V_{h,p}$. Therefore, under continuous distributions, the expected shortfall can also be interpreted as the mean loss when VaR is exceeded. In addition to the expected loss, we define the median shortfall as

$$\text{MS}_{h,p} = Median(L_{t,h}|L_{t,h} \geq \text{VaR}_{h,p}),$$

which is simply the median loss when VaR is exceeded. As a quantile-based measure, it is trivial to see that $\text{MS}_{h,p} = \text{VaR}_{h,p/2}$. As in the VaR formula in (1) and the expected shortfall formula in (8), the h-period median shortfall, $\text{MS}_{h,p}$, can be written as $-CM_{h,p}$, where

$$\text{M}_{h,p} = Median(R_{t,h}|R_{t,h} \leq \text{V}_{h,p}).$$

Both the expected shortfall and the median shortfall provide information about the loss beyond VaR. While the expected shortfall is always coherent, the median shortfall is coherent when the losses or returns of assets follow elliptically contoured distributions, which are true for many financial market assets. It can be shown using the results in McNeil et al. (2005) that, under the QGARCH model in (4) with the normal conditional distribution of $R_{t,h}$,

$$\hat{E}_{h,p}^{[1]} = h\mu - \frac{\sqrt{Var(R_{t,h}|\Omega_t)}}{p}\phi\left(\Phi^{-1}(p)\right), \qquad \hat{M}_{h,p}^{[1]} = h\mu + \sqrt{Var(R_{t,h}|\Omega_t)}\Phi^{-1}(p/2), \tag{9}$$

where $\phi(\cdot)$ is the probability density function of the standard normal. In particular, under the RiskMetrics model in (2),

$$\hat{E}_{h,p}^{[2]} = -\frac{\sqrt{h}}{p}\sigma_{t+1}\phi\left(\Phi^{-1}(p)\right), \qquad \hat{M}_{h,p}^{[2]} = \sqrt{h}\sigma_{t+1}\Phi^{-1}(p/2). \tag{10}$$

In the subsequent discussion, we develop other estimators for $E_{h,p}$ and $M_{h,p}$ with the following notation adopted throughout the paper:

$\mathrm{VaR}_{h,p} = -CV_{h,p}$,	$P(R_{t,h} \leq V_{h,p} \vert \Omega_t) = p$
$\mathrm{ES}_{h,p} = -CE_{h,p}$,	$E_{h,p} = E(R_{t,h} \vert R_{t,h} \leq V_{h,p})$
$\mathrm{MS}_{h,p} = -CM_{h,p}$,	$M_{h,p} = Median(R_{t,h} \vert R_{t,h} \leq V_{h,p})$

4 Mulitple period shortfall estimation using conditional kurtosis

Although the RiskMetrics model is usually regarded as a market benchmark for computing risk measures, the conditional normal assumption on $R_{t,h}$ given Ω_t for deriving (9) and (10) may not be compatible with the properties of real data. In fact, Wong and So (2003) showed that even under the RiskMetrics model where ε_t is normal, the conditional distribution of $R_{t,h}$ should have a fat tail when $h > 1$. By using normal approximation of the conditional distribution, it is likely that the formulas in (9) and (10) lead to underestimation of the expected shortfall, $\mathrm{ES}_{h,p}$, and the median shortfall, $\mathrm{MS}_{h,p}$, especially when p is small. In this paper, we propose a new estimator under the QGARCH model in (4) for the expected shortfall, which is simple to use and incorporates the excess kurtosis property of $R_{t,h}$ in the calculation.

The main idea is to use the exact second to fourth conditional moments of $R_{t,h}$ derived by Wong and So (2003) to work out a suitable distribution that reproduces most of the distributional properties of $R_{t,h}$. Following Wong and So (2003), the h-period return, $R_{t,h}$, distribution is approximated by the skewed t-distribution (introduced by Theodossiou,

1998) with the probability density function

$$f(x) = \begin{cases} C\left(1 + \frac{2}{\nu-2}\left(\frac{x+a}{\theta(1-\tau)}\right)^2\right)^{-\left(\frac{\nu+1}{2}\right)} & \text{if } x < -a, \\ C\left(1 + \frac{2}{\nu-2}\left(\frac{x+a}{\theta(1+\tau)}\right)^2\right)^{-\left(\frac{\nu+1}{2}\right)} & \text{if } x \geq -a, \end{cases} \tag{11}$$

where τ and ν are the parameters of the distribution,

$$C = \frac{B\left(\frac{3}{2}, \frac{\nu-2}{2}\right)^{\frac{1}{2}} S(\tau)}{B\left(\frac{1}{2}, \frac{\nu}{2}\right)^{\frac{3}{2}}}, \quad \theta = \frac{\sqrt{2}}{S(\tau)},$$

$$a = \frac{2\tau B\left(1, \frac{\nu-1}{2}\right)}{S(\tau)\, B\left(\frac{1}{2}, \frac{\nu}{2}\right)^{\frac{1}{2}} B\left(\frac{3}{2}, \frac{\nu-2}{2}\right)^{\frac{1}{2}}}, \qquad S(\tau) = \left[1 + 3\tau^2 - \frac{4\tau^2 B\left(1, \frac{\nu-1}{2}\right)^2}{B\left(\frac{1}{2}, \frac{\nu}{2}\right) B\left(\frac{3}{2}, \frac{\nu-2}{2}\right)}\right]^{\frac{1}{2}},$$

and $B(.)$ is the beta function. The above distribution is suitably scaled to have a mean of zero, a variance of one, and third and fourth moments given by

$$E[x^3] = \frac{4\tau(1+\tau^3)B(2, \frac{\nu-3}{2})B(\frac{1}{2}, \frac{\nu}{2})^{\frac{1}{2}}}{B(\frac{3}{2}, \frac{\nu-2}{2})^{\frac{3}{2}} S(\tau)^3} - 3a - a^3 \qquad \text{and} \tag{12}$$

$$E[x^4] = \frac{3(\nu-2)(1+10\tau^2+5\tau^4)}{(\nu-4)S(\tau)^4} - 4aE[x^3] - 6a^2 - a^4. \tag{13}$$

This skewed t-distribution is more flexible than the normal distribution because it can explain possible asymmetry and excess kurtosis in the aggregate return distribution. Wong and So (2003) developed recursive formulas to compute the exact second to fourth moments of $R_{t,h}|\Omega_t$ when the one-period returns follow a general QGARCH(p,q) model. We adopt their methods to find $E\left[(R_{t,h} - h\mu)^i \mid \Omega_t\right]$ for $i = 2$ to 4 under QGARCH. The parameters τ and ν of the above skewed t-distribution are then obtained by matching the exact second to fourth moments to those of $R_{t,h}|\Omega_t$. In other words, we compute the parameters by solving the two moment equations,

$$\frac{E\left[(R_{t,h} - h\mu)^3 \mid \Omega_t\right]}{var(R_{t,h} \mid \Omega_t)^{\frac{3}{2}}} = E[x^3] \qquad \text{and} \qquad \frac{E\left[(R_{t,h} - h\mu)^4 \mid \Omega_t\right]}{var(R_{t,h} \mid \Omega_t)^2} = E[x^4],$$

where $E[x^3]$ and $E[x^4]$ are from (12) and (13). The numerical solutions of the parameters τ and ν are found by a computer optimization subroutine LMDIF of the MINPACK package (see Moré et al., 1980 for details) that minimizes the squared distance between the above skewness and kurtosis with those of the skewed t-distribution. Based on the skewed-t distribution, explicit formulas for the estimator of the VaR ($V_{h,p}$), the expected shortfall ($E_{h,p}$) and the median shortfall ($M_{h,p}$) can be derived as follows:

$$\hat{V}_{h,p}^{[3]} = h\mu + \sqrt{Var\left(R_{t,h}|\Omega_t\right)}F^{-1}\left(p\right), \tag{14}$$

$$\hat{E}_{h,p}^{[3]} = \begin{cases} h\mu - \sqrt{Var\left(R_{t,h}|\Omega_t\right)}\left[\frac{\theta^2(1-\tau)^2 f\left(F^{-1}(p)\right)}{p}\left(\frac{\nu-2+2\beta_1(p)^2}{2(\nu-1)}\right)+a\right], \\ \qquad \text{if } F^{-1}(p) < -a \\ h\mu + \sqrt{Var\left(R_{t,h}|\Omega_t\right)}\left\{\frac{\theta^2}{p}\left[4\tau C\left(\frac{\nu-2}{2(\nu-1)}\right)-(1+\tau)^2 f(F^{-1}(p))\left(\frac{\nu-2+2\beta_2(p)^2}{2(\nu-1)}\right)\right]-a\right\}, \\ \qquad \text{if } F^{-1}(p) \geq -a \end{cases} \tag{15}$$

$$\hat{M}_{h,p}^{[3]} = h\mu + \sqrt{Var\left(R_{t,h}|\Omega_t\right)}F^{-1}\left(p/2\right), \tag{16}$$

where $f(\cdot)$ is the skewed-t density in (11), $F^{-1}(p)$ is the pth percentile of the skewed-t distribution, $\beta_1\left(p\right) = \frac{F^{-1}(p)+a}{\theta(1-\tau)}$ and $\beta_2\left(p\right) = \frac{F^{-1}(p)+a}{\theta(1+\tau)}$. The proof of (15) is given in the Appendix while the formulas for $V_{h,p}^{[3]}$ and $M_{h,p}^{[3]}$ are obtained from Wong and So (2003). The novelty of using (15) and (16) lies in the fact that they are (i) very easy to use because of the explicit analytical forms; (ii) computationally efficient; and (iii) reliable and effective from both statistical and financial points of view as illustrated in the next section. The formula in (15) is valid for $R_{t,h}$ following the asymmetric t distribution in (11). In particular, under the RiskMetrics model that the conditional third moment of $R_{t,h}$ equals to zero, that is, $E[R_{t,h}^3|\Omega_t] = 0$, we have $\mu = a = \tau = 0$ and $\theta = \sqrt{2}$. The expected shortfall estimator, $\hat{E}_{t,p}^{[3]}$, will be reduced to

$$\hat{E}_{h,p}^{[4]} = -\sqrt{h}\sigma_{t+1}\left[\frac{f\left(F^{-1}\left(p\right)\right)}{p}\left(\frac{\nu-2+\left(F^{-1}(p)\right)^2}{\nu-1}\right)\right], \tag{17}$$

where $f(\cdot)$ and $F^{(-1)}(\cdot)$ now refer to the standardized t distribution with a mean of zero and a variance of one and ν is obtained explicitly from the formulas in Wong and So (2003) as

$$\nu = \frac{6-4K}{3-K},$$

where

$$K = \frac{3}{h}\left[1+\left(\frac{G^h-1}{h(G-1)}-1\right)\left(\frac{6H}{G-1}+1\right)\right],$$

$G = 2(1-\lambda)^2+1$ and $H = 1-\lambda+\frac{\lambda}{3}$. Therefore, our expected shortfall estimator under the RiskMetrics model can be implemented easily in any software that produces standard t percentiles. We can also observe that when ν tends to infinity, $F(\cdot)$ will converge to the standard normal distribution and so $\hat{E}^{[3]}_{h,p}$ and $\hat{E}^{[4]}_{h,p}$ will be reduced to $\hat{E}^{[1]}_{h,p}$ and $\hat{E}^{[2]}_{h,p}$, respectively. In short, the expected shortfall estimator, $\hat{E}^{[3]}_{h,p}$, in (15) developed here encompasses the volatility asymmetry property in financial data and the fat-tailed characteristic of $R_{t,h}|\Omega_t$ through the use of the QGARCH model and the exact conditional kurtosis of $R_{t,h}$. Its special case, $\hat{E}^{[4]}_{h,p}$, also provides a very convenient way under the RiskMetrics framework to assess risk using a coherent measure while accounting for the fat-tailed property of the conditional distribution of $R_{t,h}$.

5 Empirical studies

In this section, we apply our new expected shortfall (15) and median shortfall (16) estimators to real data and compare their performance with the estimators given in (9) and (10), which assume normality of $R_{t,h}|\Omega_t$ and two Monte Carlo estimators. The financial data we use are the AOI (Australia) from 1984 to 2006; the CAC40 (France) from 1988 to 2006; the DAX (Germany) from 1988 to 2006; the FTSE 100 (UK) from 1984 to 2006; the HSI (Hong Kong) from 1984 to 2006; the Nikkei 225 (Japan) from 1984 to 2006 and the S & P 500 (USA) from 1984 to 2006. The time series for these data have at least nineteen years of daily observations. We consider two models in our analysis, namely the QGARCH(1,1) model with t-distributed error, ε_t as given in (4) and (5) and the RiskMetrics model as given in (2) and (3). The parameters of the QGARCH models are obtained by maximum likelihood estimation whereas in the RiskMetrics model, the decay factor is set to $\lambda = 0.94$ as suggested by the RiskMetrics Group.

5.1 Data analysis design

We include in the data analysis three types of estimators. They are the estimators based on exact conditional variance, $\hat{E}^{[1]}_{h,p}$, $\hat{M}^{[1]}_{h,p}$, $\hat{E}^{[2]}_{h,p}$ and $\hat{M}^{[2]}_{h,p}$, the estimators based on exact conditional kurtosis, $\hat{E}^{[3]}_{h,p}$, $\hat{M}^{[3]}_{h,p}$, $\hat{E}^{[4]}_{h,p}$ and $\hat{M}^{[4]}_{h,p}$, and Monte Carlo estimators that are based on independent samples from the conditional distribution

$$P(R_{t,h} \leq x|\Omega_t) = \int\limits_{R_{t,h} \leq x} f\left(R_{t,h}|\Omega_{t+h-1}\right) \prod_{i=1}^{h-1} f\left(r_{t+i}|\Omega_{t+i-1}\right) d\left(r_{t+1}, \cdots, r_{t+h}\right).$$

The equation for $P(R_{t,h} \leq x|\Omega_t)$ is due to the decomposition $f\left(r_{t+1}, \cdots, r_{t+h-1}, R_{t,h}|\Omega_t\right) = f(R_{t,h}|\Omega_{t+h-1}) \prod_{i=1}^{h-1} f\left(r_{t+i}|\Omega_{t+i-1}\right)$. Sampling of $r_{t+1}, ..., r_{t+h}$ from the joint density $f\left(r_{t+1}, \cdots, r_{t+h}|\Omega_t\right)$ under the QGARCH model given in (4) and (5) can be done by the method of decomposition (Tanner, 1993, pp. 30-33) as follows. Given Ω_t, σ_{t+1}^2 is known. For $i = 1$, ..., N where N is the number of replications, we

1. simulate $r_{t+1}^{(i)} \sim \mu + \sigma_{t+1} t_v$ and set $j = 2$,
2. calculate $\sigma_{t+j}^{(i)}$ from (5) using $r_{t+j-1}^{(i)}$,..., $r_{t+1}^{(i)}$ and Ω_t,
3. simulate $r_{t+j}^{(i)} \sim \mu + \sigma_{t+j}^{(i)} t_v$,
4. repeat steps 2 and 3 for $j = 3$, ..., h.

Then, $(r_{t+1}^{(i)}, ..., r_{t+h}^{(i)})$ is a draw from the joint density $f(r_{t+1}, \cdots, r_{t+h} \mid \Omega_t)$ and

$$R_{t,h}^{(i)} = r_{t+1}^{(i)} + \cdots + r_{t+h}^{(i)}, \qquad i = 1, ..., N,$$

forms an independent sample from $f(R_{t,h} \mid \Omega_t)$. In this paper, we generate $N = 200,000$ Monte Carlo observations of $R_{t,h}$ using the RiskMetrics and QGARCH models. Monte Carlo estimators of $V_{h,p}$, denoted by $\hat{V}_{h,p}^{[5]}$ for QGARCH and $\hat{V}_{h,p}^{[6]}$ for RiskMetrics, are formed by the empirical pth percentiles of $R_{t,h}^{(i)}$. The respective expected shortfall estimators will be

$$\hat{E}_{h,p}^{[j]} = \frac{1}{Np} \sum_{i=1}^{N} R_{t,h}^{(i)} I(R_{t,h}^{(i)} \leq \hat{V}_{h,p}^{[j]}),$$

where $I(\cdot)$ is an indicator function and $j = 5$ or 6. Similarly, we can have the median shortfall Monte Carlo estimators given by $\hat{M}_{h,p}^{[j]} = \hat{V}_{h,p/2}^{[j]}$, $j = 5, 6$. This Monte Carlo method does not require any assumption about the distribution, $R_{t,h}|\Omega_t$. It can produce good estimates of $V_{h,p}$, $E_{h,p}$ and $M_{h,p}$ if the number of Monte Carlo replications is large enough such that the distribution $R_{t,h}|\Omega_t$ is well approximated by Monte Carlo samples.

In our data analysis design, we consider six VaR, expected shortfall and median shortfall estimators, $\hat{V}_{h,p}^{[j]}$, $\hat{E}_{h,p}^{[j]}$ and $\hat{M}_{h,p}^{[j]}$, $j = 1, ..., 6$. They are constructed by the two models, RiskMetrics and QGARCH, and three estimation methods, i.e., exact variance, exact kurtosis and Monte Carlo. They include, respectively, 'Exact variance + QGARCH', 'Exact variance + RiskMetrics', 'Exact kurtosis + QGARCH', 'Exact kurtosis + RiskMetrics', 'Monte Carlo + QGARCH' and 'Monte Carlo + RiskMetrics' summarized in the following table.

j	Estimation method	Model	Risk estimators
1	Exact variance	QGARCH	
2	Exact variance	RiskMetrics	
3	Exact kurtosis	QGARCH	$\hat{V}_{h,p}^{[j]}$, $\hat{E}_{h,p}^{[j]}$, $\hat{M}_{h,p}^{[j]}$
4	Exact kurtosis	RiskMetrics	
5	Monte Carlo	QGARCH	
6	Monte Carlo	RiskMetrics	

The analysis is conducted with VaR, expected shortfall and median shortfall estimates of the above six estimators computed for $h = 5$, 10 and 20 and probabilities $p = 1\%$, 2.5% and 5%. We use five years of data ($t = 1$ to m, where m is about 1250) to find the maximum likelihood estimates for QGARCH and subsequently the six estimates for the three risk measures. The actual h-period returns, $R_{t,h}$, for $h = 5$, 10 and 20 are also computed from the daily returns of the market indices. Then, the estimation window is shifted forward by one day and the QGARCH parameters are re-estimated using the daily return, r_t, where $t = 2$ to $m + 1$. The computation of the VaR, expected shortfall (ES) and median shortfall (MS) estimates and the actual multiple-period returns are performed again at the time point $m + 1$. This rolling window analysis is repeated until the whole validation period, 1989 to 2006 (1993 to 2006 for the French CAC40 and German DAX), is covered. In the end, the VaR, ES and MS estimates together with the actual multiple-period returns, $R_{t,h}$, for $h = 5$, 10 and 20 are obtained at $t = m, ..., m + n$ for validating the six estimators, where $n = 4500$ (3500 for the French CAC40 and German DAX).

5.2 Volatility model fitting and VaR estimation

With the rolling sample mechanism in doing the QGARCH parameter estimation, we are able to incorporate possible changes in the dynamic structure of the returns series. To understand how parameter estimates vary over time, we plot μ, α_0, α_1, β_1, b and ν of the QGARCH model in (4) and (5) for S & P 500 computed using five years of observations up to time t in Figure 1, where t is from 1989 to 2006. It is clear that all parameters change slowly with time. For example, μ is predominantly positive except for the period 2002-2006. From the GARCH parameters, we can identify an upward trend in α_1 meaning that the last-day return's impact on the volatility forecasts gradually increases. Similarly, we observe a downward trend in β_1. As expected, the values of b are positive, confirming the volatility asymmetry that the downside movement in the market causes greater effects on the volatility forecasts than does the upside movement. The degrees of freedom is less than six most of the time except in the period when μ is negative. To summarize the time-varying parameter estimates for all seven markets, we calculate the mean and standard deviation of all the estimates in Table 1. We can see that the means of α_1 range from 0.07 (AOI) to 0.18 (Nikkei) whereas the means of β_1 are well above 0.7. All the means of ν are less than 10 with the smallest two appearing in HSI and S & P 500, confirming the typical leptokurtosis in the GARCH error distribution. The standard deviations of the parameter estimates tell us that the time series variation of the parameters cannot be ignored especially when the tail behavior (indicated by ν) is also changing over time. Therefore, we emphasize the use of the rolling-sample mechanism when doing the real data illustration.

We apply the six VaR estimators, $\hat{V}_{h,p}^{[j]}$, j=1, ..., 6, to the seven indices. When we use the QGARCH model to do the estimation, updated parameter estimates are adopted. For each combination of the probability, p, and the horizon, h, we calculate the empirical coverage, $\hat{p}$, the proportion of $R_{t,h}$ that falls below $\hat{V}_{h,p}^{[j]}$ of the seven indices in the eighteen-year validation period from 1989 to 2006 for AOI, FTSE, HSI and S & P 500 and the fourteen-year validation period from 1993 to 2006 for CAC40 and DAX. A good estimator shows that the empirical coverage $\hat{p}$ is close to p or $\hat{p}/p$ is close to one. Table 2 presents the ratio $\hat{p}/p$ for $h = 10$. The closest-to-one ratios among the six VaR estimators are put in boxes to highlight the best performing estimator. From the table, we observe that

$\hat{V}_{h,p}^{[3]}$ (Exact kurtosis + QGARCH) and $\hat{V}_{h,p}^{[5]}$ (Monte Carlo + QGARCH) generate similar empirical coverages. For $p = 1\%$, all the best performed cases are achieved by the 'Exact kurtosis + QGARCH' or 'Monte Carlo + QGARCH' estimators. The results indicate that the two estimators based on the QGARCH model, which account for the fat-tailed properties of $R_{t,h}|\Omega_t$ are superior to the other estimators in estimating 1% VaR. When p increases, there is the tendency that the difference among the six estimators decreases. To summarize the results for $h = 5$ and 20, which can be provided by the authors, as in $h = 10$, $\hat{V}_{h,p}^{[3]}$ and $\hat{V}_{h,p}^{[5]}$ perform similarly well compared with the other four estimators. For example, with $p = 1\%$, the two estimators produce the closest coverage to p, that is, $\hat{p}/p$ is closest to one, in six indices. When $p = 5\%$, there is negligible difference among the six estimators and they all perform equally well.

5.3 Expected shortfall and median shortfall forecasting

We perform the rolling window analysis described in Section 5.1 to produce out-of-sample forecasts of the expected shortfall and the median shortfall. The six estimators, $\hat{E}_{h,p}^{[j]}$ and $\hat{M}_{h,p}^{[j]}$, $j = 1, ..., 6$, are applied to the seven index data sets to obtain the ES and MS with holding periods of $h = 5, 10$ and 20 and $p = 1\%, 2.5\%$ and 5%. Figure 2 shows the time series plots of ES in percentages based on $\hat{E}_{10,1\%}^{[3]}$, the ES estimator under 'Exact kurtosis + QGARCH' and $\hat{E}_{10,1\%}^{[2]}$, the ES estimator under 'Exact variance + RiskMetrics' for S & P 500. Plots of ES in other cases are available upon request. The two plots follow similar time trends in which we observe larger ES in the second half of the validation period. The exact kurtosis estimator, $\hat{E}_{10,1\%}^{[3]}$, generally gives a larger ES than $\hat{E}_{10,1\%}^{[2]}$ does, which is based on the RiskMetrics model. Also, the former forecast is more volatile whereas the RiskMetrics forecast is smoother. To compare the performance of the two ES estimators, we add the ES residual, which is defined as $Loss - ES$ in percentages, that is $100(\hat{E}_{h,p}^{[j]} - R_{t,h})$, to the time series plots whenever there is VaR exceedance of $R_{t,h} \leq \hat{V}_{h,p}^{[j]}$. VaR exceedance occurs when the protection by VaR is insufficient to compensate for the loss incurred. All ES residuals are labeled by circles. We find more circles in the plot of the RiskMetrics model because there are more exceedances produced using RiskMetrics as documented in Section 5.2. It is evident that most ES residuals from the RiskMetrics model are positive, indicating that $\hat{E}_{10,1\%}^{[2]}$ is biased. In other words, there is a tendency that using $\hat{E}^{[2]}_{10,1\%}$ will underestimate the ES. The underestimation is quite severe in the sense that some ES residuals exceed 5%. On the contrary, the ES residuals generated from the exact kurtosis estimator, $\hat{E}^{[3]}_{10,1\%}$, are evenly distributed around zero, suggesting unbiased estimation of the ES. Even though the estimator produces more negative ES residuals than does the RiskMetrics estimator, $\hat{E}^{[2]}_{10,1\%}$, all residuals are within 5% in magnitude. In short, the exact kurtosis estimator gives better and more reliable ES forecasts than does the estimator derived from the RiskMetrics model and it produces smaller ES residuals. Figure 3 presents the MS forecasts in percentages from the 'Exact kurtosis + QGARCH' estimator $\hat{M}^{[3]}_{10,1\%}$ and the MS estimator 'Exact variance + RiskMetrics' $\hat{M}^{[2]}_{10,1\%}$ for S & P 500. Using the same idea as for ES, we also add the MS residual defined as $Loss - MS$ in percentages, that is, $100(\hat{M}^{[j]}_{h,p} - R_{t,h})$, whenever there is VaR exceedance of $R_{t,h} \leq \hat{V}^{[j]}_{h,p}$. Similar to the results in the ES analysis, we observe substantial bias in the MS residuals of the RiskMetrics model in which most MS residuals are positive. The plot shows that $\hat{M}^{[2]}_{10,1\%}$ from the RiskMetrics model underestimates the loss (in %) of $-R_{t,h}$ with the 'loss greater than MS' probability $P(-R_{t,h} \geq -\hat{M}^{[2]}_{10,1\%}) = P(\hat{M}^{[2]}_{10,1\%} - R_{t,h} \geq 0)$ much greater than 0.5. The situation with $\hat{M}^{[3]}_{10,1\%}$ based on the QGARCH model with exact kurtosis is more encouraging. The MS residuals are very balanced on both sides of zero. Therefore, it is likely that $\hat{M}^{[3]}_{10,1\%}$ will give reliable MS forecasts that approximately half of the MS residuals are positive. Although we do not show results of other indices and other combinations of h and p, they are very consistent with those shown in Figures 2 and 3. The above presents some evidence of outperformance of the ES and MS estimators under 'Exact kurtosis + QGARCH' over that under 'Exact variance + RiskMetrics'. Therefore, forecasting ES and MS using exact kurtosis with QGARCH is a promising alternative to using the RiskMetrics model. Further assessment results are provided in the next subsection.

5.4 Assessing the expected shortfall and median shortfall forecasts

To investigate the relative performance of the six ES and MS estimators, we conduct statistical tests and calculate assessment measures that are essential from the regulatory perspective. We define the standardized ES residual as

$$e_{t,h}^{[j]} = \frac{100(\hat{E}_{h,p}^{[j]} - R_{t,h})}{\sqrt{Var(R_{t,h}|\Omega_t)}},$$

which is the difference between the multiple-period loss, $-R_{t,h}$, and the ES forecast, $-\hat{E}_{h,p}^{[j]}$, in percentages, standardized by the multiple-period volatility forecast, $\sqrt{Var(R_{t,h}|\Omega_t)}$. For $\hat{E}_{h,p}^{[j]}$ to be an unbiased estimator of $E_{h,p}$ in (8), the mean of $e_{t,h}^{[j]}$ has to be zero. Therefore, to infer statistically the reliability of the six ES estimators, we perform the bootstrap one-sample t-test described by Efron and Tibshirani (1993) for the null hypothesis, $H_0 : E(e_{t,h}^{[j]}) = 0$, versus the alternative hypothesis, $H_1 : E(e_{t,h}^{[j]}) \neq 0$. We obtain two-sided p-values of the bootstrap test by using bootstrap samples of 10,000 observations. Test results for $h = 10$ are given in Table 3. The cases where H_0 is not rejected at the 5% level of significance are highlighted in bold letters. For $\hat{E}_{h,p}^{[3]}$ and $\hat{E}_{h,p}^{[5]}$, most of the test results are insignificant whereas, for the other estimators, most of the results are statistically significant. The bootstrap test indicates that both $\hat{E}_{h,p}^{[3]}$ based on 'Exact kurtosis + QGARCH' and $\hat{E}_{h,p}^{[5]}$ based on 'Monte Carlo + QGARCH' likely generate unbiased forecasts for the multiple-period expected shortfall. On the other hand, all test statistics for other estimators are positive, indicating that the estimators underestimate the expected shortfall. This agrees with what is observed in Figure 2 that most ES residuals from the RiskMetrics model are positive. We also test the unbiasedness for $h = 5$ and 20. The results are consistent with those for $h = 10$ that most scenarios for $\hat{E}_{h,p}^{[3]}$ and $\hat{E}_{h,p}^{[5]}$ are insignificant and all test statistics of the other four estimators are greater than zero. While the estimators based on 'Exact kurtosis + RiskMetrics' and 'Monte Carlo + RiskMetrics' can incorporate the fourth moment information in the ES calculation, they still lead to substantial biased though the test statistics reveal that the bias may not be as severe as the two common estimators, $\hat{E}_{h,p}^{[1]}$ and $\hat{E}_{h,p}^{[2]}$, based on exact variance. The above findings indicate that it is important to use an appropriate volatility model and to account for the exact kurtosis.

Other than the bootstrap t-test, we also compute the cost functions,

$$C1 = \frac{1}{g} \sum_{t=m}^{m+n} | - R_{t,h} + \hat{E}_{h,p}^{[j]}| \; I(t \text{ is an exceedance}) \tag{18}$$

and

$$C2 = \frac{1}{g} \sum_{t=m}^{m+n} (-R_{t,h} + \hat{E}_{h,p}^{[j]})^2 \; I(t \text{ is an exceedance}), \tag{19}$$

where $g = \sum_{t=m}^{m+n} I(t \text{ is an exceedance})$ is the number of days in the validation period that losses, $-R_{t,h}$, exceed the VaR forecast and $I(\cdot)$ is the indicator function that $I(A) = 1$ if event A occurs and $I(A) = 0$ otherwise. The two cost functions are set up by the objective that when there is exceedance, we want the ES estimate to be as close as possible to the return, $R_{t,h}$, such that the capital reserve determined by the ES estimate is at the 'minimally sufficient' level to protect against financial risk. By doing that, the opportunity cost for reserving capital for risk management can be reduced. On the other hand, when the ES estimate is lower than the actual loss, by having smaller C1 and C2 means that the capital reserved based on the ES estimate does not deviate from the actual loss by much and this reduces the impact on the financial system when VaR is exceeded. The C1 and C2 differ only in how we define the distance between the loss ($-R_{t,h}$) and the ES forecast ($-\hat{E}_{h,p}^{[j]}$). In the cost function, C1, there is a smaller penalty on large discrepancies between the loss and the ES forecasts, and so it is less affected by adverse market events, which usually induce large discrepancies, whereas C2 originates from the usual least square principle in statistical inference. Obviously, smaller values for both C1 and C2 are desirable for good ES estimators. Table 4 gives the cost function values for $h = 10$. The smallest C1 and C2 among the six estimators are put in boxes. Using C1, $\hat{E}_{h,p}^{[5]}$ performs the best for p=1% and p=2.5% while $\hat{E}_{h,p}^{[1]}$ gives the smallest C1 for all indices for p=5%. $\hat{E}_{h,p}^{[3]}$ produces very similar C1 and C2 as $\hat{E}_{h,p}^{[5]}$ does. In terms of C2, the best estimators appear to be $\hat{E}_{h,p}^{[3]}$ and $\hat{E}_{h,p}^{[5]}$. Tables 5 and 6 present the cost function values for $h = 5$ and 20. Using C1 as the criterion, $\hat{E}_{h,p}^{[3]}$ and $\hat{E}_{h,p}^{[5]}$ are more reliable for $p = 1\%$ and the three estimators based on QGARCH yield similar performance for p=2.5% and 5%. According to C2, $\hat{E}_{h,p}^{[3]}$ and $\hat{E}_{h,p}^{[5]}$ outperform the other four estimators. Generally speaking, the two estimators, $\hat{E}_{h,p}^{[3]}$ and $\hat{E}_{h,p}^{[5]}$, which make use of the fat-tailed information of $R_{t,h}$ perform the best. In practice, $\hat{E}_{h,p}^{[3]}$ is preferable to $\hat{E}_{h,p}^{[5]}$ because $\hat{E}_{h,p}^{[3]}$ is calculated using the explicit formula in (15) and is more efficient than the Monte Carlo estimator computationally.

For the assessment of MS forecasts, we use the MS residuals denoted by $me_{t,h}^{[j]} = 100(\hat{M}_{t,h}^{[j]} - R_{t,h})$ again. If the MS estimators are good, we expect to have $P(me_{t,h}^{[j]} >$

0) = 0.5. Therefore, we perform a standard binomial test on the null hypothesis, $H_0 : P(me_{t,h}^{[j]} > 0) = 0.5$, versus the two-sided alternative, $H_1 : P(me_{t,h}^{[j]} > 0) \neq 0.5$. Table 7 gives p-values of the test for $h = 10$. The cases with the p-values greater than 0.05 are highlighted by bold letters. It is evident that H_0 is not rejected in most cases associated with $\hat{M}_{t,h}^{[3]}$ and $\hat{M}_{t,h}^{[5]}$ whereas, with the other estimators, most of the test results are significant. We also conduct the binomial test for $h = 5$ and 20. It turns out that the two estimators, $\hat{M}_{t,h}^{[3]}$ and $\hat{M}_{t,h}^{[5]}$, generally perform better in that the binomial test cannot reject $H_0 : P(me_{t,h}^{[j]} > 0) = 0.5$ in more than 80% of the cases. To enrich the practical relevance of the assessment results, we use the same idea in constructing C1 and C2 to produce their MS version by replacing $\hat{E}_{t,h}^{[j]}$ with $\hat{M}_{t,h}^{[j]}$ in (18) and (19). We describe the summary findings from the C1 and C2 assessment of the MS forecasts here (detailed tables are available upon request). As in the ES forecasts, the smallest C1 and C2 are attained by using either $\hat{M}_{t,h}^{[3]}$ or $\hat{M}_{t,h}^{[5]}$. This is true across different h and p. Indeed, the two estimators produce very close C1 and C2 and so their coherent performance is understandable. Therefore, the exact kurtosis not only helps to generate good multiple-period ES forecasts, it also can produce reliable multiple-period MS forecasts.

To give an overall picture of how the six ES estimators perform, we rank them according to C1 and C2 in each combination of indices, p and h. Estimators with smaller C1 or C2 are given lower ranks. By averaging the ranks of the six estimators based on their cost function values in Tables 4 to 6, we can assess the overall performance of the ES estimators. We repeat this ranking procedure with the six MS estimators. The average ranks are presented in Table 8. For the expected shortfall estimation, $\hat{E}_{t,h}^{[5]}$ under 'Monte Carlo + QGARCH' has the lowest average ranks for all three p we consider, implying that it gives us 'closer-to-loss' ES forecasts when VaR is exceeded. From the financial point of view, $\hat{E}_{t,h}^{[5]}$ works the best in predicting expected shortfall. The other estimator based on exact kurtosis, namely $\hat{E}_{t,h}^{[3]}$, works similarly well for $p = 1\%$ and 2.5%. As far as the median shortfall is concerned. $\hat{M}_{t,h}^{[3]}$ is the most reliable estimator followed by $\hat{M}_{t,h}^{[5]}$. In short, estimators developed under the frameworks of 'Exact kurtosis + QGARCH' and 'Monte Carlo + QGARCH' are more reliable in the sense that their ES and MS estimators generally produce smaller deviations from the actual loss when the VaR is exceeded.

6 Concluding remarks

Calculating coherent risk measures in financial institutions has become indispensable to risk management and protection. Among coherent risk measures in the literature, ES is one of the most commonly used alternatives to VaR, which is not coherent. Aligning with the requirements of regulatory committees like those recognized in Basel II, it is important to estimate ES, the expected shortfall, in a multiple-period setting. One contribution of this paper is to develop a new ES estimator incorporating the tail information of future multiple-period returns. What we call exact conditional kurtosis is the key to summarizing the tail information, and it enables us to derive a new ES estimator with an explicit formula to facilitate easy implementation in real applications. Another contribution is that we propose assessment methods for ES using statistical tests and cost functions, C1 and C2, that are financially relevant. Bootstrap t-tests are used to test for the unbiasedness of various ES estimators. C1 and C2 are defined based on a distance between the ES and the true loss whenever there is VaR exceedance. By construction, small C1 and C2 are desirable such that the ES is either marginal enough to compensate for the loss or the ES does not deviate too much from the loss when ES is less than the loss. The two cost functions are also used to compare different MS estimators.

We observe that in most cases using the exact conditional kurtosis with the QGARCH model, the unbiasedness hypothesis cannot be rejected. Using C1 and C2, the 'Exact kurtosis + QGARCH' estimator performs similarly well as the 'Monte Carlo + QGARCH' estimator especially when p is small. Therefore, the exact kurtosis estimator not only can generate unbiased estimates, but it also produces risk proxies that match well with true losses. While both the exact kurtosis and Monte Carlo estimators are promising based on the bootstrap test and the financial cost functions, the exact kurtosis estimator, which is based on an explicit formula, is computationally more efficient. Therefore, it is preferable to use the exact kurtosis estimator to keep track of the dynamic risk environment in real applications. From Acerbi (2002), it is also suggested that the exact kurtosis estimator can be extended to spectral risk measures to account for the risk aversion properties of investors.

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Appendix: Proof of $\hat{E}^{[3]}_{h,p}$ in equation (15)

Let's consider the skewed t-distribution in (11). The probability density is

$$f(x) = \begin{cases} f_1(x) & \text{if } x < -a, \\ f_2(x) & \text{if } x \geq -a, \end{cases}$$

where

$$f_1(x) = C\left(1 + \frac{2}{\nu-2}\left(\frac{x+a}{\theta(1-\tau)}\right)^2\right)^{-\left(\frac{\nu+1}{2}\right)} \text{ and } f_2(x) = C\left(1 + \frac{2}{\nu-2}\left(\frac{x+a}{\theta(1+\tau)}\right)^2\right)^{-\left(\frac{\nu+1}{2}\right)}.$$

From the definition in (7) and the equation of $\hat{V}^{[3]}_{h,p}$ from (14), we have

$$\hat{E}^{[3]}_{h,p} = \frac{1}{p}\int_0^p \hat{V}^{[3]}_{h,u} du = h\mu + \sqrt{Var(R_{t,h}|\Omega_t)}\left[\frac{1}{p}\int_0^p F^{-1}(u)\, du\right],$$

where $\hat{V}^{[3]}_{h,p} = h\mu + \sqrt{Var(R_{t,h}|\Omega_t)}F^{-1}(p)$ and $F^{-1}(p)$ is the inverse cumulative distribution function of the skewed t-distribution. Let $u = F(x) = \int_{-\infty}^{x} f(y)\, dy$. We have

$$\frac{1}{p}\int_0^p F^{-1}(u)\, du = \frac{1}{p}\int_{-\infty}^{F^{-1}(p)} F^{-1}(F(x))\, f(x) dx = \frac{1}{p}\int_{-\infty}^{F^{-1}(p)} x f(x) dx. \tag{20}$$

Define

$$g(t) = C\left[1 + \frac{2t^2}{\nu-2}\right]^{-\left(\frac{\nu+1}{2}\right)}.$$

We have

$$\begin{aligned}
& d\left[-g(t)\left(\frac{\nu-2+2t^2}{2(\nu-1)}\right)\right] \\
= \ & d\left[-C\left[1+\frac{2t^2}{\nu-2}\right]^{-\left(\frac{\nu+1}{2}\right)}\left(\frac{\nu-2+2t^2}{2(\nu-1)}\right)\right] \\
= \ & C\left\{\left(\frac{\nu+1}{2}\right)\left[1+\frac{2t^2}{\nu-2}\right]^{-\left(\frac{\nu+1}{2}\right)-1}\frac{4t}{\nu-2}\left(\frac{\nu-2+2t^2}{2(\nu-1)}\right) - \left[1+\frac{2t^2}{\nu-2}\right]^{-\left(\frac{\nu+1}{2}\right)}\left(\frac{2t}{\nu-1}\right)\right\} dt \\
= \ & C\left\{\left(\frac{\nu+1}{\nu-1}\right)\left[1+\frac{2t^2}{\nu-2}\right]^{-\left(\frac{\nu+1}{2}\right)-1} t\left(1+\frac{2t^2}{\nu-2}\right) - \left[1+\frac{2t^2}{\nu-2}\right]^{-\left(\frac{\nu+1}{2}\right)}\left(\frac{2t}{\nu-1}\right)\right\} dt \\
= \ & C\left\{t\left[1+\frac{2t^2}{\nu-2}\right]^{-\left(\frac{\nu+1}{2}\right)}\left(\frac{\nu+1-2}{\nu-1}\right)\right\} dt \\
= \ & t g(t) dt.
\end{aligned} \tag{21}$$

Let $x = \sigma t - a$ and assume, without loss of generality, that $\sigma > 0$. Then, the integrals

$$\int_{l_1}^{l_2} x f_1(x) dx \quad \text{and} \quad \int_{l_1}^{l_2} x f_2(x) dx$$

take the form

$$\begin{aligned}
& C \int_{l_1}^{l_2} x \left(1 + \frac{2}{\nu - 2}\left(\frac{x+a}{\sigma}\right)^2\right)^{-\left(\frac{\nu+1}{2}\right)} dx \\
= \ & C \int_{(l_1+a)/\sigma}^{(l_2+a)/\sigma} (\sigma t - a)\left(1 + \frac{2t^2}{\nu - 2}\right)^{-\left(\frac{\nu+1}{2}\right)} \sigma dt \\
= \ & \int_{(l_1+a)/\sigma}^{(l_2+a)/\sigma} \sigma^2 t C \left(1 + \frac{2t^2}{\nu - 2}\right)^{-\left(\frac{\nu+1}{2}\right)} dt - a \int_{(l_1+a)/\sigma}^{(l_2+a)/\sigma} \sigma C \left(1 + \frac{2t^2}{\nu - 2}\right)^{-\left(\frac{\nu+1}{2}\right)} dt \\
= \ & \int_{(l_1+a)/\sigma}^{(l_2+a)/\sigma} \sigma^2 t g(t) dt - a \int_{(l_1+a)/\sigma}^{(l_2+a)/\sigma} \sigma g(t) dt \\
= \ & \int_{(l_1+a)/\sigma}^{(l_2+a)/\sigma} \sigma^2 d\left[-g(t)\left(\frac{\nu - 2 + 2t^2}{2(\nu - 1)}\right)\right] - a \int_{(l_1+a)/\sigma}^{(l_2+a)/\sigma} \sigma g(t) dt \qquad \text{by } (21) \\
= \ & \sigma^2 \left[g\left(\frac{l_1+a}{\sigma}\right)\left(\frac{\nu - 2 + 2\left(\frac{l_1+a}{\sigma}\right)^2}{2(\nu-1)}\right) - g\left(\frac{l_2+a}{\sigma}\right)\left(\frac{\nu - 2 + 2\left(\frac{l_2+a}{\sigma}\right)^2}{2(\nu-1)}\right)\right] \\
& -a \int_{l_1}^{l_2} f_i(x) dx, \qquad (22)
\end{aligned}$$

assuming that the integration is with respect to $f_i(x)$, for $i = 1, 2$.

Case 1: $F^{-1}(p) < -a$

Let $\beta_1(p) = \frac{F^{-1}(p)+a}{\theta(1-\tau)}$. Assume, without loss of generality, that $\theta(1-\tau) > 0$. Substituting $l_1 = -\infty$, $l_2 = F^{-1}(p)$ and $\sigma = \theta(1-\tau)$ in (22), we get

$$\begin{aligned}
& \frac{1}{p} \int_{-\infty}^{F^{-1}(p)} x f(x) dx = \frac{1}{p} \int_{-\infty}^{F^{-1}(p)} x f_1(x) dx \\
= \ & \frac{\theta^2(1-\tau)^2}{p}\left[-g(\beta_1(p))\right]\left(\frac{\nu - 2 + 2\beta_1(p)^2}{2(\nu-1)}\right) - \frac{a}{p} \int_{-\infty}^{F^{-1}(p)} f_1(x) dx \\
= \ & -\frac{\theta^2(1-\tau)^2 f_1\left(F^{-1}(p)\right)}{p}\left(\frac{\nu - 2 + 2\beta_1(p)^2}{2(\nu-1)}\right) - a.
\end{aligned}$$

The last equality follows by noting that $g(\beta_1(p)) = f_1(F^{-1}(p))$ and $f_1(x) = f(x)$ when $x < F^{-1}(p) < -a$.

Case 2: $F^{-1}(p) \geq -a$

Let $\beta_2(p) = \frac{F^{-1}(p)+a}{\theta(1+\tau)}$. Assume, without loss of generality, that both $\theta(1+\tau)$ and $\theta(1-\tau)$ are positive. Using the results in (22) and following Case 1, we get

$$\frac{1}{p}\int\limits_{-\infty}^{F^{-1}(p)} xf(x)dx = \frac{1}{p}\left[\int\limits_{-\infty}^{-a} xf_1(x)dx \int\limits_{-a}^{F^{-1}(p)} xf_2(x)dx\right]$$

$$= \frac{\theta^2(1-\tau)^2}{p}\left[-g\left(0\right)\left(\frac{\nu-2}{2(\nu-1)}\right)\right] - \frac{a}{p}\int_{-\infty}^{-a} f_1(x)dx +$$
$$\frac{\theta^2(1+\tau)^2}{p}\left[g\left(0\right)\left(\frac{\nu-2}{2(\nu-1)}\right) - g\left(\beta_2(p)\right)\left(\frac{\nu-2+2\beta_2(p)^2}{2(\nu-1)}\right)\right] - \frac{a}{p}\int_{-a}^{F^{-1}(p)} f_2(x)dx$$

$$= \frac{\theta^2(1-\tau)^2}{p}\left[-C\left(\frac{\nu-2}{2(\nu-1)}\right)\right] - \frac{a}{p}\int_{-\infty}^{-a} f(x)dx +$$
$$\frac{\theta^2(1+\tau)^2}{p}\left[C\left(\frac{\nu-2}{2(\nu-1)}\right) - g\left(\beta_2(p)\right)\left(\frac{\nu-2+2\beta_2(p)^2}{2(\nu-1)}\right)\right] - \frac{a}{p}\int_{-a}^{F^{-1}(p)} f(x)dx$$

$$= \frac{\theta^2}{p}\left[4\tau C\left(\frac{\nu-2}{2(\nu-1)}\right) - (1+\tau)^2 f_2(F^{-1}(p))\left(\frac{\nu-2+2\beta_2(p)^2}{2(\nu-1)}\right)\right] - a.$$

The last two equalities follow by noting that $g(\beta_2(p)) = f_2(F^{-1}(p))$, $f_1(x) = f(x)$ when $x < -a$ and $f_2(x) = f(x)$ when $x \geq -a$.

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Table 1: The mean and standard deviation (in parentheses) of the parameter estimates in the QGARCH model.

	AOI	CAC	DAX	FTSE	HSI	NIKKEI	SP500
μ	0.04	0.03	0.04	0.04	0.07	0.00	0.04
	(0.03)	(0.04)	(0.04)	(0.03)	(0.06)	(0.06)	(0.04)
α_0	0.04	0.14	0.09	0.05	0.14	0.19	0.03
	(0.05)	(0.28)	(0.12)	(0.11)	(0.13)	(0.45)	(0.04)
α_1	0.07	0.09	0.13	0.09	0.12	0.18	0.08
	(0.03)	(0.09)	(0.09)	(0.09)	(0.05)	(0.16)	(0.05)
β_1	0.86	0.80	0.84	0.85	0.80	0.74	0.87
	(0.07)	(0.25)	(0.10)	(0.17)	(0.12)	(0.23)	(0.08)
b	0.43	0.57	0.49	0.28	0.46	0.46	0.65
	(0.23)	(0.42)	(0.20)	(0.29)	(0.23)	(0.23)	(0.29)
ν	9.06	9.04	7.25	9.32	5.24	6.21	5.84
	(2.35)	(3.57)	(1.69)	(3.26)	(0.69)	(1.35)	(1.14)

Table 2: Ratio of the proportion $\hat{p}$ of 10-day returns less than the VaR estimates to the actual probability. Ratios $\hat{p}/p$ that are closest to one are in boxes.

	AOI	CAC	DAX	FTSE	HSI	NIKKEI	SP500
				$p = 1\%$			
Exact variance + QGARCH	1.50	[1.12]	1.70	1.70	2.39	1.62	1.25
Exact variance + RiskMetrics	1.58	1.47	2.01	1.83	2.36	2.17	1.30
Exact kurtosis + QGARCH	[1.03]	[0.88]	[0.95]	[1.29]	[1.60]	0.80	0.77
Exact kurtosis + RiskMetrics	1.54	1.28	1.90	1.68	2.16	1.99	1.19
Monte Carlo + QGARCH	1.10	[0.88]	1.07	[1.29]	1.77	[1.11]	[0.86]
Monte Carlo + RiskMetrics	1.54	1.25	1.93	1.65	2.18	1.97	1.17
				$p = 2.5\%$			
Exact variance + QGARCH	1.44	0.88	1.19	1.31	1.59	[1.04]	1.07
Exact variance + RiskMetrics	1.29	[1.00]	1.35	1.32	1.45	1.45	[0.95]
Exact kurtosis + QGARCH	[1.01]	0.59	0.85	[1.08]	[1.34]	0.74	0.72
Exact kurtosis + RiskMetrics	1.23	0.99	1.31	1.32	1.43	1.41	0.94
Monte Carlo + QGARCH	1.08	0.62	[0.97]	1.12	1.38	0.86	0.81
Monte Carlo + RiskMetrics	1.23	0.99	1.33	1.32	1.44	1.40	0.94
				$p = 5\%$			
Exact variance + QGARCH	1.22	0.88	1.06	1.08	1.26	0.88	[0.87]
Exact variance + RiskMetrics	[1.14]	0.94	1.06	1.06	[1.14]	1.24	0.79
Exact kurtosis + QGARCH	1.15	0.76	0.93	[1.02]	1.26	0.74	0.80
Exact kurtosis + RiskMetrics	1.16	[0.96]	1.08	1.06	1.17	1.26	0.81
Monte Carlo + QGARCH	1.20	0.82	[1.00]	1.07	1.32	[0.89]	0.82
Monte Carlo + RiskMetrics	1.16	0.95	1.08	1.06	1.17	1.25	0.81

Table 3: Test statistics and p-values (in parentheses) of the Bootstrap t-test for $h = 10$ days.

	AOI	CAC	DAX	FTSE	HSI	NIKKEI	SP500
				$p = 1\%$			
Exact variance + QGARCH	4.53	3.87	3.73	3.49	6.58	5.53	4.73
	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)
Exact variance + RiskMetrics	4.40	4.13	3.73	4.22	6.16	7.80	4.69
	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)
Exact kurtosis + QGARCH	**-0.19**	**0.40**	**-0.10**	**-0.24**	**1.32**	-2.46	**-1.85**
	(0.86)	**(0.69)**	**(0.92)**	**(0.82)**	**(0.20)**	(0.02)	**(0.08)**
Exact kurtosis + RiskMetrics	2.63	3.13	**2.05**	2.22	5.13	6.48	3.23
	(0.02)	(0.01)	**(0.06)**	(0.04)	(0.00)	(0.00)	(0.00)
Monte Carlo + QGARCH	**-0.55**	**0.30**	**-0.65**	**-0.39**	**1.26**	-2.41	**-1.96**
	(0.59)	**(0.76)**	**(0.52)**	**(0.70)**	**(0.22)**	(0.02)	**(0.07)**
Monte Carlo + RiskMetrics	2.60	3.27	**1.96**	2.32	5.07	6.60	3.38
	(0.02)	(0.00)	**(0.06)**	(0.03)	(0.00)	(0.00)	(0.00)
				$p = 2.5\%$			
Exact variance + QGARCH	3.03	2.94	4.28	3.91	6.81	5.73	3.67
	(0.01)	(0.01)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)
Exact variance + RiskMetrics	4.40	4.84	4.80	5.35	6.87	7.41	4.63
	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)
Exact kurtosis + QGARCH	**-0.29**	2.29	**0.70**	**0.89**	**2.06**	**-0.96**	**-1.05**
	(0.77)	(0.03)	**(0.49)**	**(0.38)**	**(0.05)**	**(0.33)**	**(0.29)**
Exact kurtosis + RiskMetrics	3.07	3.54	3.54	3.33	5.83	6.17	3.19
	(0.01)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)
Monte Carlo + QGARCH	**-0.73**	**1.95**	**0.12**	**0.66**	2.27	**0.13**	**-0.89**
	(0.48)	**(0.06)**	**(0.91)**	**(0.53)**	(0.03)	**(0.89)**	**(0.38)**
Monte Carlo + RiskMetrics	2.60	3.53	3.46	3.31	5.79	6.22	3.27
	(0.02)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)
				$p = 5\%$			
Exact variance + QGARCH	3.88	2.19	3.81	4.65	6.89	5.32	4.48
	(0.00)	(0.04)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)
Exact variance + RiskMetrics	4.57	3.75	5.48	5.83	7.01	6.63	4.79
	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)
Exact kurtosis + QGARCH	**-1.60**	**-0.67**	**-0.47**	**1.11**	2.09	**-0.33**	**-1.88**
	(0.12)	**(0.51)**	**(0.65)**	**(0.27)**	0.04	**(0.75)**	**(0.07)**
Exact kurtosis + RiskMetrics	3.15	2.58	4.27	4.53	6.01	5.43	3.59
	(0.00)	(0.01)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)
Monte Carlo + QGARCH	**-1.67**	**-0.92**	**-0.23**	**0.85**	2.14	**-0.14**	**-0.80**
	(0.10)	**(0.35)**	**(0.82)**	**(0.40)**	(0.04)	**(0.89)**	**(0.44)**
Monte Carlo + RiskMetrics	3.13	2.63	4.33	4.50	6.01	5.44	3.57
	(0.00)	(0.01)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)

Table 4: Expected shortfall cost functions, C1 and C2, for $h = 10$ days. The smallest C1 and C2 are in boxes

	AOI	CAC	DAX	FTSE	HSI	NIKKEI	SP500
				C1, $p = 1\%$			
Exact variance + QGARCH	1.27	2.31	2.84	1.22	4.42	2.75	1.89
Exact variance + RiskMetrics	1.42	2.73	3.22	1.47	4.98	2.70	1.97
Exact kurtosis + QGARCH	0.94	1.65	1.82	1.18	3.50	1.33	1.23
Exact kurtosis + RiskMetrics	1.33	2.80	2.99	1.43	4.94	2.43	1.70
Monte Carlo + QGARCH	0.92	1.64	1.76	1.18	3.40	1.26	1.13
Monte Carlo + RiskMetrics	1.33	2.80	2.99	1.43	4.93	2.43	1.70
				C1, $p = 2.5\%$			
Exact variance + QGARCH	1.06	2.46	2.30	1.20	3.30	2.31	1.50
Exact variance + RiskMetrics	1.28	2.54	2.60	1.38	4.10	2.80	1.70
Exact kurtosis + QGARCH	1.08	1.89	1.95	1.14	3.07	2.02	1.34
Exact kurtosis + RiskMetrics	1.29	2.56	2.50	1.39	4.14	2.69	1.67
Monte Carlo + QGARCH	1.07	1.95	1.92	1.14	3.03	1.84	1.25
Monte Carlo + RiskMetrics	1.29	2.55	2.50	1.38	4.14	2.69	1.67
				C1, $p = 5\%$			
Exact variance + QGARCH	0.91	1.75	1.97	1.19	2.82	1.94	1.23
Exact variance + RiskMetrics	1.16	2.13	2.34	1.36	3.67	2.52	1.55
Exact kurtosis + QGARCH	1.03	1.93	2.11	1.22	2.99	2.03	1.47
Exact kurtosis + RiskMetrics	1.19	2.17	2.37	1.36	3.73	2.49	1.58
Monte Carlo + QGARCH	1.01	1.90	2.06	1.21	2.93	1.99	1.37
Monte Carlo + RiskMetrics	1.19	2.17	2.37	1.36	3.74	2.49	1.58
				C2, $p = 1\%$			
Exact variance + QGARCH	3.34	9.52	16.05	3.19	42.13	10.42	5.28
Exact variance + RiskMetrics	4.47	11.72	21.87	3.84	41.79	9.34	6.46
Exact kurtosis + QGARCH	1.79	3.85	5.57	2.27	27.65	2.44	2.23
Exact kurtosis + RiskMetrics	3.82	11.27	18.35	3.42	39.97	7.73	4.81
Monte Carlo + QGARCH	1.73	3.76	5.49	2.30	27.58	2.37	1.90
Monte Carlo + RiskMetrics	3.78	11.24	18.33	3.40	39.89	7.81	4.81
				C2, $p = 2.5\%$			
Exact variance + QGARCH	2.60	11.55	12.82	3.23	28.73	8.95	4.76
Exact variance + RiskMetrics	3.35	11.72	16.01	3.95	32.91	12.13	5.65
Exact kurtosis + QGARCH	2.00	6.46	7.75	2.50	22.28	6.17	2.68
Exact kurtosis + RiskMetrics	3.26	11.19	15.02	3.71	32.80	11.59	5.20
Monte Carlo + QGARCH	1.99	6.86	8.10	2.53	22.40	5.47	2.45
Monte Carlo + RiskMetrics	3.24	11.16	14.98	3.71	32.82	11.57	5.20
				C2, $p = 5\%$			
Exact variance + QGARCH	2.01	7.79	9.85	3.14	21.55	7.58	3.82
Exact variance + RiskMetrics	2.70	9.45	12.50	3.69	27.00	10.83	5.28
Exact kurtosis + QGARCH	1.88	6.43	8.34	2.71	19.83	6.22	3.44
Exact kurtosis + RiskMetrics	2.74	9.51	12.46	3.60	27.36	10.71	5.30
Monte Carlo + QGARCH	1.87	6.64	8.45	2.74	19.62	6.13	3.22
Monte Carlo + RiskMetrics	2.73	9.50	12.45	3.60	27.41	10.68	5.30

Table 5: Expected shortfall cost functions, C1 and C2, for $h = 5$ days. The smallest C1 and C2 are in boxes

	AOI	CAC	DAX	FTSE	HSI	NIKKEI	SP500
				C1, $p = 1\%$			
Exact variance + QGARCH	1.19	1.27	[1.26]	0.88	3.19	2.32	1.09
Exact variance + RiskMetrics	1.28	1.75	1.85	1.01	3.20	2.39	1.26
Exact kurtosis + QGARCH	1.09	1.04	1.49	0.86	2.36	1.31	0.94
Exact kurtosis + RiskMetrics	1.22	1.91	1.83	0.99	3.05	2.19	1.18
Monte Carlo + QGARCH	[1.08]	[1.03]	1.48	[0.85]	[2.35]	[1.29]	[0.79]
Monte Carlo + RiskMetrics	1.22	1.90	1.83	0.99	3.05	2.20	1.18
				C1, $p = 2.5\%$			
Exact variance + QGARCH	[0.84]	[0.95]	[1.10]	[0.81]	2.46	[1.51]	1.02
Exact variance + RiskMetrics	1.03	1.27	1.44	0.94	2.87	1.85	1.15
Exact kurtosis + QGARCH	0.89	1.17	1.26	0.84	[2.28]	1.60	[0.82]
Exact kurtosis + RiskMetrics	1.04	1.34	1.47	0.93	2.83	1.82	1.14
Monte Carlo + QGARCH	0.88	1.13	1.24	0.84	[2.28]	1.54	[0.82]
Monte Carlo + RiskMetrics	1.04	1.34	1.47	0.93	2.83	1.81	1.14
				C1, $p = 5\%$			
Exact variance + QGARCH	[0.78]	[0.94]	[1.13]	[0.83]	[2.10]	[1.24]	[0.92]
Exact variance + RiskMetrics	0.85	1.15	1.43	0.95	2.49	1.56	1.08
Exact kurtosis + QGARCH	0.81	1.04	1.29	0.86	2.11	1.56	0.96
Exact kurtosis + RiskMetrics	0.86	1.18	1.46	0.95	2.50	1.55	1.10
Monte Carlo + QGARCH	0.80	1.01	1.27	0.85	[2.10]	1.42	0.94
Monte Carlo + RiskMetrics	0.86	1.18	1.46	0.95	2.50	1.55	1.10
				C2, $p = 1\%$			
Exact variance + QGARCH	4.29	3.89	5.25	2.31	21.49	8.57	2.84
Exact variance + RiskMetrics	4.67	6.19	7.92	2.55	20.36	8.52	2.93
Exact kurtosis + QGARCH	3.30	2.88	4.24	1.83	[12.01]	[3.13]	1.45
Exact kurtosis + RiskMetrics	4.35	6.93	7.70	2.34	18.65	7.39	2.49
Monte Carlo + QGARCH	[3.29]	[2.84]	[4.22]	[1.80]	12.24	3.17	[1.26]
Monte Carlo + RiskMetrics	4.36	6.90	7.74	2.34	18.61	7.39	2.49
				C2, $p = 2.5\%$			
Exact variance + QGARCH	2.50	2.50	3.48	1.78	16.04	5.11	2.58
Exact variance + RiskMetrics	2.87	3.23	4.98	2.06	17.06	6.34	2.91
Exact kurtosis + QGARCH	[2.19]	2.44	3.38	[1.56]	[11.97]	4.01	[1.48]
Exact kurtosis + RiskMetrics	2.82	3.41	5.02	2.02	16.64	6.10	2.75
Monte Carlo + QGARCH	[2.19]	[2.29]	[3.37]	[1.56]	12.20	[3.93]	1.56
Monte Carlo + RiskMetrics	2.83	3.41	5.03	2.01	16.61	6.08	2.76
				C2, $p = 5\%$			
Exact variance + QGARCH	1.85	2.15	[3.12]	1.68	12.39	[3.59]	2.11
Exact variance + RiskMetrics	2.02	2.63	4.10	1.84	13.53	4.88	2.53
Exact kurtosis + QGARCH	[1.72]	2.11	3.19	[1.57]	[10.85]	4.05	[1.74]
Exact kurtosis + RiskMetrics	2.01	2.69	4.13	1.82	13.46	4.81	2.53
Monte Carlo + QGARCH	[1.72]	[2.10]	3.15	[1.57]	10.93	3.60	1.79
Monte Carlo + RiskMetrics	2.01	2.69	4.14	1.82	13.46	4.80	2.53

Table 6: Expected shortfall cost functions, C1 and C2, for $h = 20$ days. The smallest C1 and C2 are in boxes

	AOI	CAC	DAX	FTSE	HSI	NIKKEI	SP500
				C1, $p = 1\%$			
Exact variance + QGARCH	1.67	3.17	5.01	1.52	8.53	6.81	2.37
Exact variance + RiskMetrics	2.18	4.73	5.21	2.04	7.84	5.43	2.49
Exact kurtosis + QGARCH	1.72	[2.27]	3.05	[1.36]	[5.96]	3.24	2.00
Exact kurtosis + RiskMetrics	2.44	4.93	4.83	2.08	7.08	4.74	2.09
Monte Carlo + QGARCH	[1.63]	[2.27]	[3.01]	[1.36]	6.09	[3.18]	[1.57]
Monte Carlo + RiskMetrics	2.44	4.93	4.83	2.09	7.06	4.74	2.08
				C1, $p = 2.5\%$			
Exact variance + QGARCH	[1.29]	[2.29]	3.19	[1.59]	6.29	5.41	1.55
Exact variance + RiskMetrics	1.80	3.12	4.06	1.93	6.62	4.89	2.15
Exact kurtosis + QGARCH	1.50	2.41	2.95	1.62	5.92	[3.39]	1.70
Exact kurtosis + RiskMetrics	1.98	3.39	4.16	1.92	6.53	4.61	2.14
Monte Carlo + QGARCH	1.46	2.35	[2.80]	[1.59]	[5.76]	3.80	[1.52]
Monte Carlo + RiskMetrics	1.98	3.41	4.16	1.92	6.53	4.61	2.14
				C1, $p = 5\%$			
Exact variance + QGARCH	[1.31]	2.30	2.85	[1.64]	[4.80]	3.73	[1.58]
Exact variance + RiskMetrics	1.62	2.79	3.21	1.98	5.74	3.75	2.31
Exact kurtosis + QGARCH	1.37	2.34	2.98	1.71	5.14	3.77	1.65
Exact kurtosis + RiskMetrics	1.70	2.89	3.29	2.00	5.77	[3.68]	2.37
Monte Carlo + QGARCH	1.36	[2.28]	[2.82]	1.65	4.82	3.79	1.62
Monte Carlo + RiskMetrics	1.70	2.89	3.29	2.00	5.77	[3.68]	2.38
				C2, $p = 1\%$			
Exact variance + QGARCH	6.91	21.50	43.23	4.65	137.80	61.40	7.94
Exact variance + RiskMetrics	8.27	31.82	49.10	7.22	120.60	43.07	10.89
Exact kurtosis + QGARCH	5.05	[10.28]	[16.78]	2.71	[83.36]	[16.77]	5.54
Exact kurtosis + RiskMetrics	8.60	34.10	42.83	7.44	111.20	35.05	7.67
Monte Carlo + QGARCH	[4.74]	10.30	16.94	[2.70]	85.22	19.57	[3.68]
Monte Carlo + RiskMetrics	8.59	34.12	43.05	7.51	110.80	34.95	7.64
				C2, $p = 2.5\%$			
Exact variance + QGARCH	4.32	15.07	26.01	5.10	97.03	48.18	5.16
Exact variance + RiskMetrics	5.64	20.72	32.31	6.53	90.62	39.49	8.44
Exact kurtosis + QGARCH	3.90	10.82	16.54	4.07	[77.56]	[20.52]	4.10
Exact kurtosis + RiskMetrics	6.17	22.05	32.86	6.21	88.40	36.73	8.25
Monte Carlo + QGARCH	[3.75]	[10.61]	[16.28]	[4.02]	77.61	26.80	[3.62]
Monte Carlo + RiskMetrics	6.18	22.18	32.89	6.22	88.22	36.63	8.27
				C2, $p = 5\%$			
Exact variance + QGARCH	3.98	13.30	21.21	5.25	65.56	31.44	5.31
Exact variance + RiskMetrics	4.92	15.27	23.14	6.70	67.78	27.65	9.70
Exact kurtosis + QGARCH	3.53	10.74	17.18	4.83	60.67	[22.90]	4.43
Exact kurtosis + RiskMetrics	5.17	15.69	23.41	6.65	67.64	26.87	10.08
Monte Carlo + QGARCH	[3.52]	[10.72]	[16.90]	[4.59]	[59.28]	26.01	[4.38]
Monte Carlo + RiskMetrics	5.18	15.72	23.44	6.66	67.55	26.89	10.10

Table 7: P-values of the test for $H_0 : P(me_t > 0) = 0.5$, h=10 days.

	AOI	CAC	DAX	FTSE	HSI	NIKKEI	SP500
				$p = 1\%$			
Exact variance + QGARCH	0.00	0.00	0.01	0.03	0.00	0.00	0.00
Exact variance + RiskMetrics	0.00	0.00	0.02	0.01	0.00	0.00	0.00
Exact kurtosis + QGARCH	**0.47**	**1.00**	**0.60**	**0.15**	**0.12**	**0.86**	**1.00**
Exact kurtosis + RiskMetrics	0.06	0.00	**0.27**	**0.17**	0.00	0.00	0.01
Monte Carlo + QGARCH	**0.78**	**1.00**	**0.62**	**0.35**	**0.25**	**0.88**	**0.33**
Monte Carlo + RiskMetrics	**0.06**	0.00	**0.63**	**0.13**	0.01	0.00	0.00
				$p = 2.5\%$			
Exact variance + QGARCH	**0.81**	**0.42**	0.00	0.00	0.00	0.00	**0.27**
Exact variance + RiskMetrics	0.04	0.00	0.00	0.00	0.00	0.00	0.01
Exact kurtosis + QGARCH	**0.93**	0.01	**0.25**	**0.31**	0.01	**0.81**	**1.00**
Exact kurtosis + RiskMetrics	**0.18**	0.04	0.01	**0.05**	0.00	0.00	**0.21**
Monte Carlo + QGARCH	**0.65**	0.02	**0.19**	**0.24**	0.01	**0.10**	**0.75**
Monte Carlo + RiskMetrics	**0.31**	0.01	0.00	0.03	0.00	0.00	**0.17**
				$p = 5\%$			
Exact variance + QGARCH	0.00	**1.00**	**0.10**	0.00	0.00	0.01	0.00
Exact variance + RiskMetrics	0.03	**0.44**	0.00	0.00	0.00	0.01	0.01
Exact kurtosis + QGARCH	**0.05**	0.01	**0.31**	**0.42**	**0.33**	**0.93**	**0.23**
Exact kurtosis + RiskMetrics	**0.29**	**0.70**	0.00	0.00	0.00	0.04	0.02
Monte Carlo + QGARCH	**0.12**	0.00	**0.65**	**0.52**	**0.41**	**0.59**	**0.77**
Monte Carlo + RiskMetrics	**0.29**	**0.64**	0.00	0.00	0.00	**0.05**	0.03

Table 8: Average ranks of the expected shortfall and median shortfall estimators based on C1 and C2.

	Expected shortfall			Median shortfall		
	$p = 1\%$	$p = 2.5\%$	$p = 5\%$	$p = 1\%$	$p = 2.5\%$	$p = 5\%$
Exact variance + QGARCH	4.0	2.8	2.1	4.6	3.4	3.4
Exact variance + RiskMetrics	5.4	5.3	4.8	5.4	5.5	5.3
Exact kurtosis + QGARCH	1.8	1.9	2.5	1.3	1.2	1.3
Exact kurtosis + RiskMetrics	4.4	4.8	4.9	4.1	4.7	4.7
Monte Carlo + QGARCH	1.3	1.5	1.8	1.7	1.8	1.7
Monte Carlo + RiskMetrics	4.2	4.6	5.0	3.9	4.4	4.6

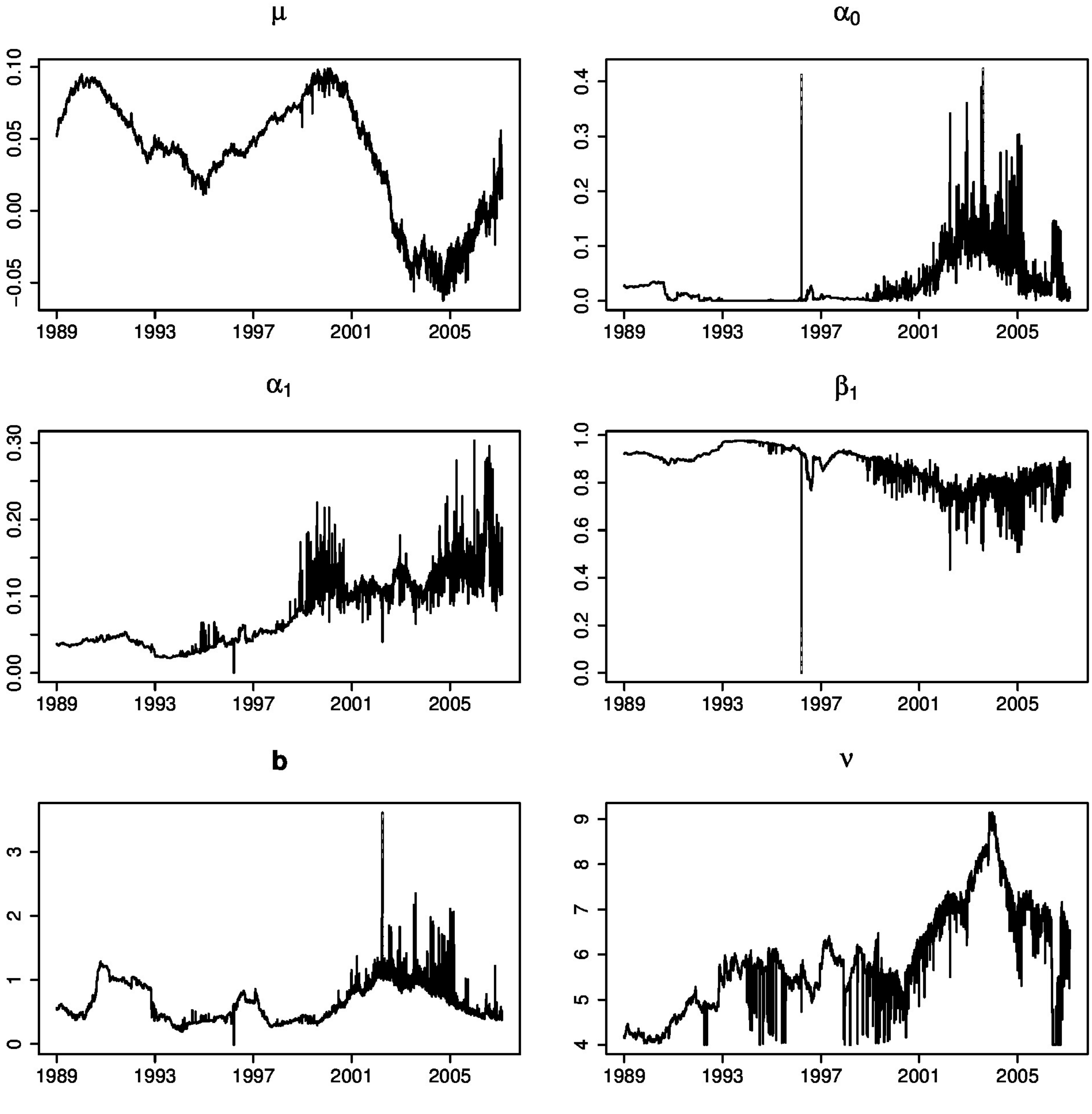

Figure 1: QGARCH parameter estimates of S & P 500 in the validation period.

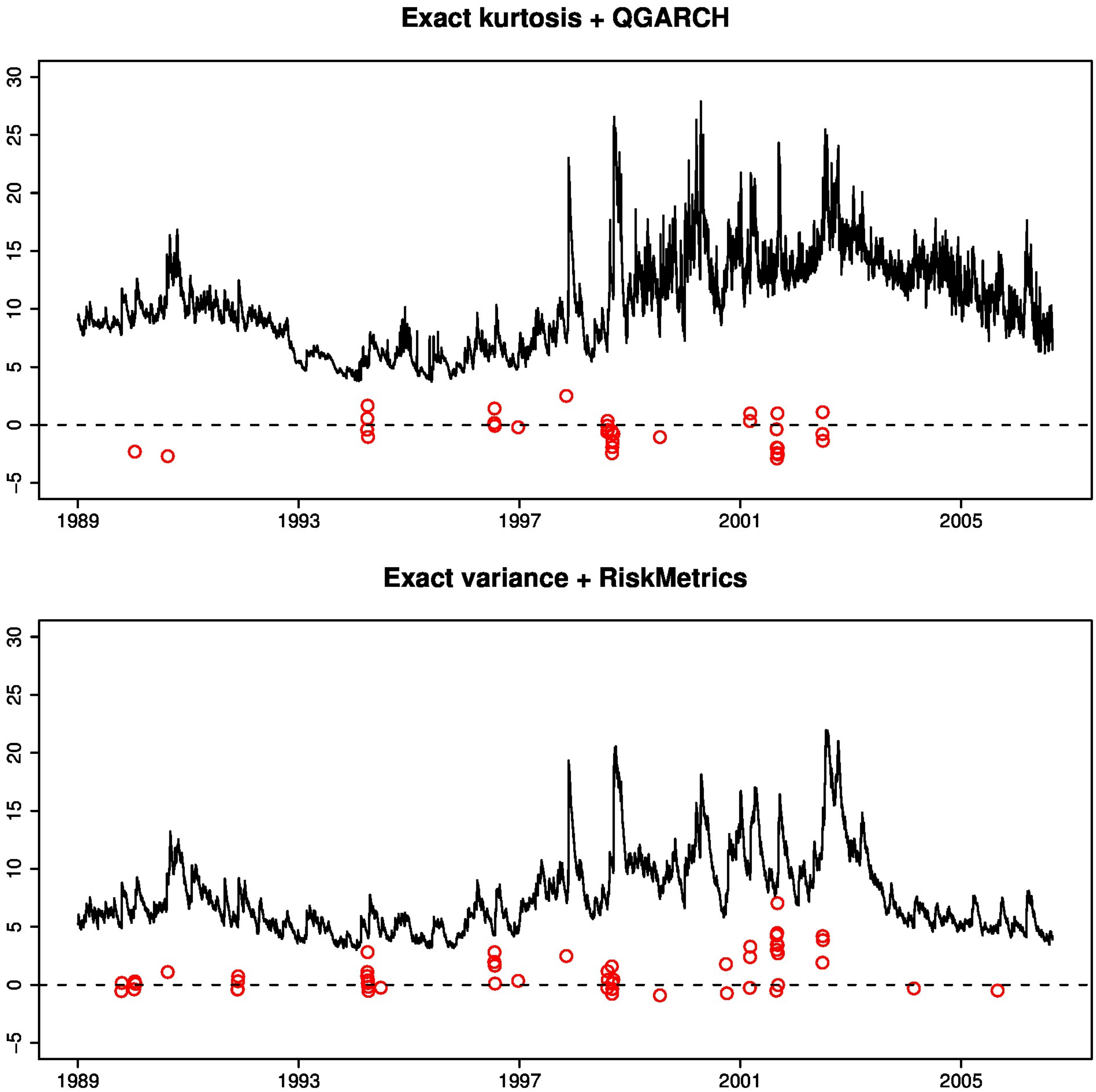

Figure 2: Time series plot of the expected shortfall estimates in percentages using the Exact kurtosis + QGARCH estimator ($\hat{E}^{[3]}_{10,1\%}$) and the Exact variance + RiskMetrics estimator ($\hat{E}^{[2]}_{10,1\%}$) for S & P 500 in the validation period. The ES residuals, $-R_{10,1\%} - \hat{E}^{[j]}_{10,1\%}$, are labeled by circles in the time series plots whenever there is VaR exceedance of $-R_{10,1\%} > \hat{V}^{[j]}_{10,1\%}$ for j=2 and 3.

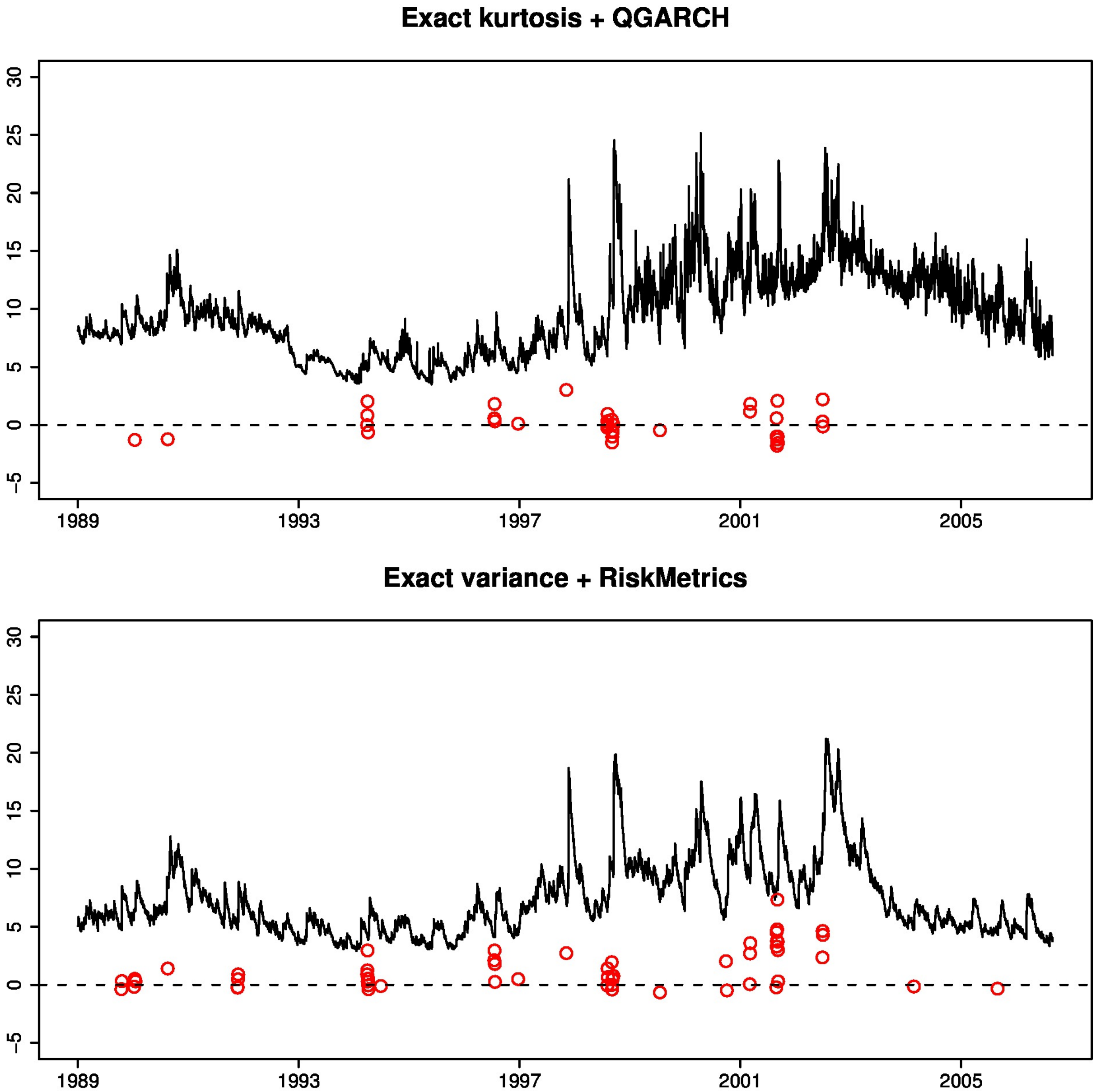

Figure 3: Time series plot of the median shortfall estimates in percentages using the Exact kurtosis + QGARCH estimator ($\hat{M}^{[3]}_{10,1\%}$) and the Exact variance + RiskMetrics estimator ($\hat{M}^{[2]}_{10,1\%}$) for S & P 500 in the validation period. The MS residuals, $-R_{10,1\%} - \hat{M}^{[j]}_{10,1\%}$, are labeled by circles in the time series plots whenever there is VaR exceedance of $-R_{10,1\%} > \hat{V}^{[j]}_{10,1\%}$ for j=2 and 3.