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Research Article

Development and Applications of Penalty-Based Aggregation Operators in Multicriteria Decision Making

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This article develops a new penalty-based aggregation operator known as the penalty-based induced ordered weighted averaging (P-IOWA) operator which is an extension of penalty-based ordered weighted averaging (P-OWA) operator. Our goal is to figure out how the induced variable realigns penalties when gathering information. We extend the P-OWA and P-IOWA operators with the different means such as generalized mean and quasi-arithmetic mean. This article also includes different families of P-OWA and P-IOWA operators. The value of these new operators is demonstrated through a case study centered on investment matters. This study evaluates the economic and governance performance of seven South Asian nations utilizing nine indicators from 2021 data. The research examines 5 economic indicators including GDP growth, exports and imports (% of GDP), inflation, and labor force metrics, alongside 4 governance indicators focusing on corruption control, government effectiveness, and political stability. We use min–max normalization to standardize the varied values, which originally ranged from 0.5% to 77.7% across various metrics. Following this, the normalized inverse penalty method is used to derive optimal weights for these indicators, tackling the task of combining multidimensional data. Subsequently, we implement and evaluate various penalty-based aggregation methodologies on the normalized data, each offering a distinct approach to penalizing outliers and balancing indicator weights. The study compares the results obtained from these operators to assess their impact on country rankings and overall performance evaluation. This approach allows for a comprehensive comparison of countries' performances, integrating both economic and governance dimensions into a single, quantifiable framework.

Keywords: economic governance relationship; min–max normalization; multicriteria decision making (MCDM); penalty-based aggregation; penalty-based IOWA; penalty-based OWA

1. Introduction

The procedure of bringing together many numerical values into a one value is referred to as aggregation. The numerical function that can perform this process is referred to as the aggregation function [1]. The primary objective of this function is to merge arguments that are usually seen as degrees of preference in fuzzy sets.

In 1965, Zadeh suggested the idea of fuzzy logic [2] which is a suitable technique for dealing with the imprecision and fuzziness of the information available in many real-world situations for decision making [3, 4]. Moreover, in the last few years, the integration of fuzzy set theory with aggregation operator theory has appeared as a powerful tool which is handling decision-making issues with vague or incomplete data [5, 6]. In aggregation operator theory, there are four main aggregation operators such as averaging, conjunctive, disjunctive, and mixed. Among them, the averaging aggregation operator is widely used in the domain of decision making [7]. Several scholars have employed diverse aggregation operators incorporating various concepts in the realm of decision making [8–10].

One of the operators that fall under the category of averaging is referred to as the ordered weighted average (OWA). Yager came up with an idea of an OWA operator [11] for the first time. This OWA operator has widespread application across a variety of domains, including decision making, information fusion, image processing, artificial intelligence, and many others [3, 12, 13]. Several authors attempted to create diverse expansions of OWA operators in various directions, which involve ordered weighted geometric (OWG) averaging operator [14], generalized ordered weighted averaging (GOWA) operator [15], and quasi-arithmetic OWA [16]. Also, it is significant to point out that there is another essential expansion of the OWA operator, which is the induced OWA (IOWA) [17]. IOWA uses inducing variable to weight arguments instead of reordering step which is used in OWA operator. An important advantage of using this approach is that it can make the arrangement of arguments more efficient by taking into account an additional interrelated factor, which is commonly referred to as the order inducing variable. This is in contrast to the traditional method of sorting the arguments according to the values they have been assigned. Inspired by the research conducted on the IOWA operator, numerous scholars have undertaken the task of broadening its conceptual framework across multiple domains [18–20].

In 1993, Yager introduced a remarkable theory of aggregation that is founded on the principle of penalty function [21]. This theory provides a comprehensive approach to investigating the complex issue of aggregating relevant knowledge. A penalty can be interpreted as the disagreement between output and each input. Further, he extended the work and defined different forms of penalty function for different fusion methods [22]. Motivated by this work, many authors extended the theory of penalty-based aggregation in their literature [23–25].

The OWA and IOWA operators give the aggregated value, but in this aggregated value, we are not able to find the connectivity between arguments and its relevant output. Penalty-based ordered weighted averaging (P-OWA) operators explain how each argument is relevant to the required output with the help of OWA-based penalties. In order to provide a more general approach to the P-OWA operator, let us develop a framework by using induced aggregation operators. For doing so, we introduce the penalty-based induced ordered weighted averaging (P-IOWA) operator. It is a penalty aggregation operator with similar characteristics as the P-OWA operator. The main advantage of the P-IOWA operator is that it helps to find inter-relationship between each criterion by using the penalties in the set of arguments considering the complex attitudinal character of the decision maker. For instance, probabilistic OWA combines probability and the OWA operator, assigning varying levels of importance to individual concepts based on their relevance to the specific problem at hand. In contrast, P-IOWA integrates a penalty function with the IOWA operator, ensuring equal significance is placed on all concepts. Additionally, the study also introduces more general approaches by using geometric and generalized means. By doing so, the penalty operators become more robust being able to consider a wide range of particular cases. The article analyzes many of these specific cases and also considers different families of weighting vectors to generate the P-IOWA operator.

In this article, we explain penalty operators and their generalizations with the help of case study. We consider a scenario in which a multinational company (MNC) wants to invest in a South Asian country. However, the situation becomes more convoluted because the criteria for each country are varied. So, we are considering essential investment criteria such as government effectiveness, political stability, GDP growth, and other common criteria of each country. Then aggregating them by penalty-based operators and using ranking method, we choose the required country for investment. The objectives of this research are as follows:

1. Theoretical advancement:
   - Introduced P-IOWA operator
   - Explored novel extensions and generalizations of penalty-based operators
   - Created new families of operators that can handle specific types of data or decision scenarios
   - To analyze the different properties of new P-IOWA operator
2. Develop a comprehensive framework for country selection in investment decisions using P-OWA operators
3. Methodology comparison: to compare different aggregation methods (arithmetic, geometric, harmonic, quadratic, and cubic P-OWA) and assess their effectiveness in this context
4. Comparative analysis of South Asian countries: to evaluate and compare the performance of seven South Asian countries (Bangladesh, India, Maldives, Nepal, Pakistan, Sri Lanka, and Bhutan) across multiple economic and governance criteria
5. Identification of key factors: to identify and analyze factors that impact the chosen economic and governance criteria in these countries
6. Methodological contribution: to potentially contribute to the field of multicriteria decision analysis by applying and comparing various techniques in the context of South Asian countries' economic and governance performance

The composition of this manuscript is arranged in the ensuing manner. The works and foundational aspects pertaining to P-OWA are elucidated in Section 2. A thorough review of penalty function, both OWA and IOWA operators and OWA-based penalties, has been carried out in Section 3. In Section 4, we discuss advantages of penalty-based aggregation operators. In Section 5, we introduce the P-IOWA operator and show the applicability of it in mutual fund selection problem. Also, we analyze its characteristics, as well as generate families of P-IOWA operators. In Section 6, we extend P-IOWA and generate it with generalized mean and quasi-arithmetic mean. Also, we examine the particular cases of these operators. Section 7 constructs theoretical framework for the case study based on decision-making investment problem. In Section 8, we considered the case study using a real-life scenario and discussed in detail. Also, we provide a brief review of the paper's key findings, and lastly concluding remarks of this paper are provided in Section 9.

2. Related Works and Background

The field of penalty-based aggregation functions, which includes P-OWA operators, has seen significant development over the past decade. This approach offers a flexible framework for constructing and analyzing aggregation functions, with applications spanning various domains. Early work in this area focused on establishing the theoretical foundations of penalty-based aggregation. Researchers introduced novel methods for constructing aggregation functions using penalty-based approaches and explored their mathematical properties. This laid the groundwork for future developments and applications. Subsequent research expanded on this foundation, exploring the theoretical properties and practical applications of penalty-based aggregation. Notable contributions include the investigation of relationships between penalty-based functions and consensus measures [23] and the refinement of penalty function definitions and properties [26]. A key development has been the enhancement of OWA operators to improve their robustness, particularly in handling outliers and noisy data. This advancement has broadened the applicability of P-OWA in real-world scenarios where data quality can be inconsistent. Recent years have seen the introduction of weighted penalty-based aggregation methods, further refining the theoretical framework and opening up new avenues for application. The advancement of penalty-based aggregation operators is illustrated in Table 1.

Future work could explore novel extensions of P-OWA operators. This direction builds upon the foundational work of [25] on penalty-based aggregation functions and extends it in line with recent advancements in weighted penalty-based methods [30].

The body of work represented by these papers demonstrates the evolution of penalty-based aggregation functions from their initial conceptualization to their application in various fields and the ongoing refinement of their theoretical foundations. As research continues, we can expect further developments in both the theory and practice of penalty-based aggregation, potentially leading to new insights and applications across multiple disciplines.

3. Preliminaries

In order to enhance the efficiency of the aggregation process, Yager investigated a penalty function–based information aggregation theory. In this section, we will briefly review the fundamental concept of penalty function, OWA operators, and P-OWA operators.

3.1. Penalty Function. Yager suggested the implementation of penalty function which helps in aggregating data. The primary motivation behind our work is Yager's concept of "penalty cost" related to input disagreement with an output. This output value can serve as a fused representation, which can be obtained as the mean, median, or mode of an input vector, and in some cases, it may also be regarded as an ideal value. The greater the number of inputs that disagree with the output, the larger the penalty for the disagreement. We look for an aggregated value which minimizes the penalty; in some sense we look for a consensus value which minimizes the disagreement. In accordance with this definition, the penalty function is defined as follows.

Definition 1 (see [21]). The function $P: R^{n+1} \longrightarrow [0,\infty)$ is a penalty function if it satisfies the following conditions:

1. $P(x_i, y) \geq 0$ for $i = 1, \ldots, n$;
2. $P(x_i, y) = 0$ for all $x_i = y$;
3. For all fixed x, penalty value will be singular element or a range;

   where $x = (x_1, x_2, \ldots, x_n)$ is an input vector while y is an output value or a fused value.

Note: As per the primary two requirements, negative values for penalties are not accepted, and if a complete consensus is achieved, then zero penalty is obtained.

Later on, Calvo, Mesiar, and Yager [31] gave details on weighted aggregation based on penalty function. Many authors have discussed penalty function in their literature [26, 32, 33].

3.2. OWA and IOWA Operators. The OWA operator, developed by Yager, is a remarkable and simple nonlinear aggregation method that has demonstrated its importance in various domains.

Definition 2 (see [34]). An OWA operator defined by OWA: $R^n \longrightarrow R$ is an n-dimensional mapping with weighting vector $W = [w_1, w_2, \ldots, w_n]^T$ and $\sum_{j=1}^{n} w_j = 1$, is defined as follows:

$$\mathrm{OWA}(a_1, a_2, \ldots, a_n) = \sum_{j=1}^{n} w_j b_j, \tag{1}$$

where b_j is the jth largest element in the collection $a_1, a_2, \ldots, a_n$.

Yager modifies the process of reordering and introduces a variable that is induced in the IOWA operator. This variable is associated with each argument and corresponds to the decision maker's comprehension or the important characteristics of the problem.

Definition 3 (see [17]). An IOWA operator of dimension n is a mapping IOWA: $R^n \times R^n \longrightarrow R$ such that it has an associated weighting vector W, such that

$$\mathrm{IOWA}(\langle u_1, a_1\rangle, \langle u_2, a_2\rangle, \ldots, \langle u_n, a_n\rangle) = \sum_{j=1}^{n} w_j b_j, \tag{2}$$

Table 1: Summary of recent P-OWA operator research.

References	Author	Title	Year	Contribution
[25]	Tomasa Calvo, Gleb Beliakov	Aggregation functions based on penalties	2010	Introduces penalty-based aggregation functions with the theoretical framework
[23]	Gleb Beliakov, Tomasa Calvo and Simon James	On penalty-based aggregation functions and consensus	2011	Introduces the concept of OWA-based penalties as a new extension, allowing penalties to be weighted based on their relative sizes rather than just their sources
[26]	Humberto Bustince, Gleb Beliakov, Gracaliz Pereira Dimuro, Benjamin Bedregal, Radko Mesiar	On the definition of penalty functions in data aggregation	2017	Defines and analyzes penalty functions for data aggregation and explores new types of penalty functions and their properties
[27]	Gleb Beliakov	Penalty-based and other representations of economic inequality	2016	Utilizes a penalty-based OWA framework that incorporates generalized OWA operators, establishes conditions for validity, and connects to classical inequality measures
[28]	Gleb Beliakov, Simon James, Tim Wilkin and Tomasa Calvo	Robustifying OWA operators for aggregating data with outliers	2017	The paper effectively employs penalty-based OWA to create a robust aggregation method that can handle outliers while preserving the integrity of the data and the characteristics of the OWA operator
[29]	Pablo Olaso, Karina Rojas, Daniel Gómez, Javier Montero	A generalization of stability for families of aggregation operators	2020	Discusses penalty functions and their implications for aggregation operators, which can be applied to OWA in terms of stability, consistency, and flexibility in handling various aggregation scenarios
[30]	Andrea Stupňanová	Weighted penalty-based aggregation	2021	Introduces normed penalty function for weighted penalty-based aggregation functions and allows consideration of groups of input coordinates with different weights

where b_j is the a_i $(i = 1, 2, \ldots n)$ value of the IOWA pair $\langle u_i, a_i \rangle$ having the jth largest u_i and u_i is the order-inducing variable connected to the a_i.

3.3. P-OWA Operator. In decision-making theory, while compiling information, it is essential to compile all of the relevant data. During the aggregation process, we need to locate all of those irrelevant pieces of information that are not close to the result so that we can remove them. The P-OWA operator is used here to aggregate all penalties that differ between input and output. It provides the problem's lowest penalty value, assisting decision makers in determining which decision best fits the outcome.

Definition 4 (see [23]). A P-OWA operator of dimension $(n+1)$ is a function $P-OWA\colon R^{n+1} \longrightarrow [0,\infty)$ of weighting vector $W = [w_1, w_2, \ldots, w_n]^T$, where $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$ with penalty function $p(x,y)$ having argument vector x and y as fused value such that

$$P-\mathrm{OWA}_W(x_i, y) = \sum_{i,j=1}^{n} w_j p(x_i, y), \tag{3}$$

where the weights are allocated to the penalties based on their respective penalty size using the OWA function.

4. Advantages of Penalty-Based Aggregation Operators

The penalty-based ordered averaging aggregation operator is an extension of the traditional OWA aggregation operator. It introduces a penalty mechanism that provides more nuanced and adaptive decision-making capabilities compared to the standard OWA. The key idea behind the penalty-based operators is to incorporate a penalty function that dynamically adjusts the weights assigned to the ordered input values during the aggregation process. This penalty function takes into account the characteristics of the input values, such as their relative magnitude or position within the ordered sequence. Penalty-based operators enhance sensitivity to extreme values. In contrast, traditional OWA operators treat all ordered values in a uniform manner.

Consider a scenario in which a supply chain organization is aiming to enhance the efficacy of its logistics network; it is imperative to analyze data concerning various metrics pertinent to transportation, warehousing, and distribution operations, encompassing factors such as delivery durations, transportation expenditures, inventory quantities, and customer satisfaction indices. Using the traditional OWA approach, it might assign fixed weights to each of these metrics based on their perceived importance or the company's strategic priorities. For example, it might give a higher weight to delivery times and transportation costs, as these are critical to maintaining an efficient and cost-effective supply chain. However, this approach may not adequately account for the potential impact of extreme or outlier values within your dataset. For instance, there might have been a few instances of exceptionally long delivery times or unusually high transportation costs that could disproportionately skew the aggregated performance metric, even if they are not representative of the overall supply chain's performance.

In this scenario, the P-OWA can provide a more nuanced and adaptive approach. By incorporating a penalty function, we can dynamically adjust the weights assigned to each metric based on their relative deviation from the norm or their potential to introduce undue risk or uncertainty into the aggregation process. Suppose we notice a few delivery times that are significantly longer than the rest; the P-OWA can apply a higher penalty to these outlier values, reducing their influence on the final aggregated performance metric. Conversely, if you have consistently high customer satisfaction scores across the board, the P-OWA can assign them higher weights, ensuring that this positive aspect of your supply chain operations is properly reflected in the overall performance evaluation.

By mitigating the impact of extreme or anomalous values, the P-OWA can provide a more accurate and representative assessment of your supply chain's overall performance. The ability to adjust weights based on the risk or variability associated with each metric can help you better identify and address potential vulnerabilities in your logistics network. The customizable nature of the penalty function allows decision makers to tailor the aggregation process to their specific operational priorities and performance targets. The more nuanced and adaptive performance evaluation enabled by the P-OWA can lead to more informed and effective supply chain optimization decisions.

In summary, the P-OWA extends the traditional OWA by introducing a penalty function that provides a more nuanced, adaptive, and risk-sensitive aggregation mechanism. This enhanced approach can lead to improved decision making in various applications, particularly when dealing with complex or uncertain data.

5. Proposed Work

This section provides the introduction of the P-IOWA operator and discusses its properties. Moreover, it also explores the development of the P-OWA and P-IOWA operator families within the same section.

5.1. P-IOWA Operator. The P-OWA operator has been further elaborated to establish the P-IOWA operator. Unlike the P-OWA operator, the P-IOWA operator's ordering is dependent on the induced variable's value, instead of a step for rearrangement. With the aim of reducing any penalties, we may assign greater weight to the penalty with the smallest value and assign lesser weight to the penalty with the largest value.

Definition 5. A mapping $\mathrm{P-IOWA}\colon R^n \times R^n \times R \longrightarrow R^+$ is called a P-IOWA operator such that it has an associated weighting vector $W = [w_1, w_2, \ldots, w_n]^T$, where $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$ with penalty function $p(x,y)$ having $x =$

$(x_1, \ldots\ldots, x_n)$ as an argument vector and y as fused value such that

$$P - \text{IOWA}(u, x, y) = \sum_{i,j=1}^{n} w_j p(u_i, x_i, y), \quad (4)$$

where the penalties of x are arranged as largest to lowest according to the inducing variable u.

Example 1. Consider the following induced P-IOWA pairs $\langle u_i, x_i \rangle$ as given by $\langle 0.15, 2 \rangle, \langle 0.10, 2 \rangle, \langle 0.20, 3 \rangle, \langle 0.05, 3 \rangle, \langle 0.12, 5 \rangle$ and penalty function $p(x, y) = |x - y|$ with weighting vector $W = \{0.4, 0.3, 0.1, 0.1, 0.1\}$.

Considering y as mean (fused value), then $y = 3$, and calculating penalties using $p(x_i, y) = |x_i - y|$, we get a penalty vector as follows:

$$p(x_i, y) = (1, 1, 0, 0, 2), \quad i = 1 \text{ to } 5. \quad (5)$$

Reordering penalty vector according to induced variable, it becomes $p(0, 1, 2, 1, 0)$. So, using equation (4), we get

$$\begin{aligned} \text{P} - \text{IOWA} &= (0.4)(0) + (0.3)(1) + (0.1)(2) + (0.1)(1) \\ &+ (0.1)(0) = 0.6. \end{aligned} \quad (6)$$

A real-world example was taken to demonstrate the applicability of the IOWA operator in the selection of mutual funds. In the context of the upcoming Example 2, we have selected and extracted data from three different mutual funds, specifically from the financial records of the month of September in the year 2023 [35]. These data have been carefully compiled for analysis and presentation in the subsequent section.

Example 2. In the present context, we will employ the instance of investing in equity mutual funds to elucidate the P-OWA and P-IOWA operators. Equity mutual funds are long-term financial instruments for growth in wealth and come with market risk but tax benefits. These operators are utilized for the purpose of establishing the aggregation of disagreement between input and output values. One may view this disagreement as a value representing a penalty or risk when investing in mutual funds. It is very crucial to find which mutual fund is good for an investment because several parameters are there which are affecting the price of mutual funds. Also, it becomes more complicated when their price fluctuation is minor.

We will now examine three investment possibilities, each with a unique name as indicated in Table 2. In order to analyze these options, we will be observing their daily prices over a span of 22 days.

In Table 3, we have calculated the penalties of each mutual funds by using the penalty function $p(x_i, y) = |x_i - y|$. Experts have assigned a value to the inducing variable on a scale of seven based on seven distinct factors. These factors include the risk factor, liquidity factor, uniformity factor, quality of returns factor, research factor, demand and supply factor, and the companies' earnings and future projects. These factors collectively contribute to the fluctuations in mutual fund prices.

Using the P-OWA and P-IOWA operators, we aggregated the penalties using Table 2 values, which are calculated and listed in Table 4.

It is apparent from Table 4 that the reduction of penalties through the implementation of the P-IOWA operator is noteworthy. Thus, after aggregating penalties using P-OWA and P-IOWA operators, Axis Mid Cap, a mutual fund, is considered as the most preferable option for an investment compared to the other two mutual funds as it offers the lowest penalty value. Additionally, considering penalties as an investment risk value, we can say that P-IOWA reduces the investment risk value, which aids investors in decision making. The utilization of penalty operators enables the comparison of one or more mutual funds with their investment risk value, facilitating investors in making well-informed decisions regarding the most suitable option.

Furthermore, it is possible to establish distinctive operators, such as the ascending and descending P-IOWA operators. Taking an alternative stance on the rearrangement process, we can differentiate between ascending (AP-IOWA) and descending (DP-IOWA) orders. The weights of these operators are assigned according to the equation $w_j = w^*_{n-j+1}$, where w_j is the jth weight of the DP-IOWA operator and w^*_{n-j+1} is the jth weight of the AP-IOWA operator.

According to the definition of penalty function, if the fused value and argument value are the same, then $P - \text{IOWA}(\langle u_1, x_1, y \rangle, \langle u_2, x_2, y \rangle, \ldots, \langle u_n, x_n, y \rangle) = 0$, for all i and also penalties are never negative so $P - \text{IOWA}(\langle u_1, x_1, y \rangle, \langle u_2, x_2, y \rangle, \ldots, \langle u_n, x_n, y \rangle) \geq 0$. The P-IOWA operator is an averaging aggregation operator. Therefore, it fulfills various properties such as commutativity, monotonicity, boundedness, idempotent, non-negativity, and reflexivity.

Property 1 (monotonicity). Let f be a $P - \text{IOWA}$ operator. If $x = \{x_1, x_2, \ldots, x_n\}$ and $z = \{z_1, z_2, \ldots, z_n\}$ are the two arguments for $P - \text{IOWA}$ operator and if $p(u_i, x_i, y) \geq p(u_i, z_i, y)$ for all i, then

$$\begin{aligned} &f(\langle u_1, x_1, y \rangle, \langle u_2, x_2, y \rangle, \ldots, \langle u_n, x_n, y \rangle) \\ &\quad \geq f(\langle u_1, z_1, y \rangle, \langle u_2, z_2, y \rangle, \ldots, \langle u_n, z_n, y \rangle). \end{aligned} \quad (7)$$

Proof 1. Assume we have penalty function $p(x_i, y) = |x_i - y|$ and $p(z_i, y) = |z_i - y|$; if $|x_i - y| \geq |z_i - y|$, according to this induced variable u_i, then we get $P - \text{IOWA}_{(u_i, x_i, y)} \geq P - \text{IOWA}_{(u_i, z_i, y)}$. Hence, we can say that

$$f(\langle u_i, x_i, y \rangle) \geq f(\langle u_i, z_i, y \rangle), \quad (8)$$

where y have the same output related to given arguments which can be any fused value. It is important to acknowledge that not all penalty functions exhibit the property of being monotonic [36]. Therefore, in order to aggregate

Table 2: Data of three equity mutual funds.

Days	Mutual fund price day-wise (in Rs.)		
	Axis mid cap (M_1)	Kotak emerging equity (M_2)	SBI flexi cap (M_3)
01-Sep-2023	89.66	102.98	93.80
04-Sep-2023	89.79	103.56	94.27
05-Sep-2023	90.60	104.57	94.66
06-Sep-2023	90.77	104.81	94.84
07-Sep-2023	91.21	105.13	95.14
08-Sep-2023	91.77	105.71	95.89
11-Sep-2023	92.19	106.46	96.50
12-Sep-2023	90.56	104.10	95.77
13-Sep-2023	90.18	103.84	95.96
14-Sep-2023	90.78	104.35	96.39
15-Sep-2023	90.90	104.22	96.49
18-Sep-2023	90.41	104.08	96.32
20-Sep-2023	90.11	103.72	95.49
21-Sep-2023	89.33	102.86	94.63
22-Sep-2023	89.15	102.74	94.30
y = fused value (μ = average)	$\mu = 90.494$	$\mu = 104.209$	$\mu = 95.363$

Note: Source: [35].

Table 3: Penalties of each mutual fund.

Sr. No.	Weights (w_i)	Inducing variable value (u_i)	Mutual fund risk value (penalties)		
			(M_1)	(M_2)	(M_3)
1	0.1	5.5	0.834	1.229	1.563
2	0.02	5.7	0.704	0.649	1.093
3	0.1	2.7	0.106	0.361	0.703
4	0.02	2.7	0.276	0.601	0.523
5	0.02	5.9	0.716	0.921	0.223
6	0.04	6.4	1.276	1.501	0.527
7	0.02	6.6	1.696	2.251	1.137
8	0.12	2.5	0.066	0.109	0.407
9	0.04	3.0	0.314	0.369	0.597
10	0.05	2.8	0.286	0.141	1.027
11	0.09	2.0	0.406	0.011	1.127
12	0.15	2.2	0.084	0.129	0.957
13	0.09	3.2	0.384	0.489	0.127
14	0.12	6.6	1.164	1.349	0.733
15	0.02	6.7	1.344	1.469	1.063

Table 4: Aggregated value of penalties.

Operators	Aggregation of mutual fund's penalties		
	(M_1)	(M_2)	(M_3)
P-OWA	0.54380115	0.692	0.733
P-IOWA	0.51742	0.654	0.699
Overall reduction	0.02638115 (2.6%)	0.038 (3.8%)	0.034 (3.4%)

information, it becomes important to utilize penalty-based aggregation functions [25]. □

Property 2 (commutativity). If f is a $P-$IOWA operator, then for all P-IOWA pairs, f is commutative if

$$f\left(\langle u_1, x_1, y\rangle, \langle u_2, x_2, y\rangle, \ldots, \langle u_n, x_n, y\rangle\right) = f\left(\langle u_1, y, x_1\rangle, \langle u_2, y, x_2\rangle, \ldots, \langle u_n, y, x_n\rangle\right). \quad (9)$$

Proof 2. Assume we have penalty function

$$\begin{aligned} p(x_i, y) &= |x_i - y|, \quad \text{for all } i = 1, 2, \ldots, n \\ &= |y - x_i|, \\ P - IOWA(u, x, y) &= \sum_{i,j=1}^{n} w_j p(u_i, x_i, y) \\ &= \sum_{i,j=1}^{n} w_j |x_i - y| \\ &= \sum_{i,j=1}^{n} w_j |y - x_i| \\ &= P - IOWA(u, y, x). \end{aligned} \tag{10}$$

Hence, commutativity is proved. □

Property 3 (boundedness). If f is a $P - IOWA$ operator with the argument vector $x = \{x_1, x_2, \ldots, x_n\}$, for all i, then it is bounded if

$$\begin{aligned} \min p\{x_i, y\} &\le f(\langle u_1, x_1, y\rangle, \langle u_2, x_2, y\rangle, \ldots, \langle u_n, x_n, y\rangle) \\ &\le \max p\{x_i, y\}. \end{aligned} \tag{11}$$

Proof 3. Consider $p(x_i, y) = |x_i - y|$ for which upper bound is $\max\{|x_i - y|\} = c$ and lower bound is $\min\{|x_i - y|\} = d$; according to the corresponding induced variable u_i, we can write

$$\begin{aligned} P - IOWA(u, x, y) &= \sum_{i,j=1}^{n} w_j p(u_i, x_i, y) \\ \sum_{i,j=1}^{n} w_j p(u_i, x_i, y) &\ge \sum_{i,j=1}^{n} w_j d \text{ and } \sum_{i,j=1}^{n} w_j p(u_i, x_i, y) \\ &\le \sum_{i,j=1}^{n} w_j c. \end{aligned} \tag{12}$$

□

Now, we know that $\sum_{j=1}^{n} w_j = 1$, then $\sum_{i,j=1}^{n} w_j p(u_i, x_i, y) \ge d$ and $\sum_{i,j=1}^{n} w_j p(u_i, x_i, y) \le c$.

$$\begin{aligned} d \le \sum_{i,j=1}^{n} w_j p(u_i, x_i, y) \le c &\Longrightarrow \min\{|x_i - y|\} \\ &\le \sum_{i,j=1}^{n} w_j p(u_i, x_i, y) \le \max\{|x_i - y|\}, \end{aligned} \tag{13}$$

and hence, $\min p\{x_i, y\} \le f(\langle u_1, x_1, y\rangle, \langle u_2, x_2, y\rangle, \ldots, \langle u_n, x_n, y\rangle) \le \max p\{x_i, y\}$.

Hence, it is proved.

Property 4 (idempotent). If f is a $P -$ IOWA operator having $p(x_i, y) = b$, for all i, then f is idempotent if

$$f(\langle u_1, x_1, y\rangle, \langle u_2, x_2, y\rangle, \ldots, \langle u_n, x_n, y\rangle) = b. \tag{14}$$

Proof 4. Suppose we get same penalty value for each i for a given penalty function.

That is, $p(x_i, y) = b$ for each $i = 1$ to n; then our operator can be defined as

$$\mathrm{P - IOWA}(u, x, y) = \sum_{i,j=1}^{n} w_j p(u_i, x_i, y) = \sum_{i,j=1}^{n} w_j b = b. \tag{15}$$

Hence, it is proved. □

Property 5 (nonnegativity). If f is a $P -$ IOWA operator, then according to the definition of penalty function, f is nonnegative if

$$f(\langle u_1, x_1, y\rangle, \langle u_2, x_2, y\rangle, \ldots, \langle u_n, x_n, y\rangle) \ge 0. \tag{16}$$

Proof 5. Now, for a penalty function $p(x_i, y) = |x_i - y|$, we have two possibilities,

$$\begin{aligned} |x_i - y| &= x_i - y, if\, x_i - y \ge 0 \\ &= -(x_i - y), if\, x_i - y < 0. \end{aligned} \tag{17}$$

□

In both the cases, $|x_i - y|$ is a nonnegative function. So, we can write

$$\begin{aligned} \mathrm{P - IOWA}(u, x, y) &= \sum_{i,j=1}^{n} w_j p(u_i, x_i, y) \\ &= \sum_{i,j=1}^{n} w_j |x_i - y| \ge 0, for\, x_i - y \ge 0 \text{ and } x_i - y < 0 \\ &\therefore f(\langle u_1, x_1, y\rangle, \langle u_2, x_2, y\rangle, \ldots, \langle u_n, x_n, y\rangle) \ge 0. \end{aligned} \tag{18}$$

Hence, it is proved.

Property 6 (reflexivity). If f is a $P -$ IOWA operator and if $x_i = y$, for all i, then f is reflexive if

$$f\left(\langle u_1, x_1, y\rangle, \langle u_2, x_2, y\rangle, \ldots, \langle u_n, x_n, y\rangle\right) = 0. \quad (19)$$

Proof 6. Suppose for a penalty function $p(x_i, y) = |x_i - y|$ we obtain identical fused values for each respective argument. In this instance, we receive a unanimous vote.

$$P - \mathrm{IOWA}(u, x, y) = \sum_{i,j=1}^{n} w_j p(u_i, x_i, y) = \sum_{i,j=1}^{n} w_j |x_i - y| = \sum_{i,j=1}^{n} w_j |y - y| = 0$$
$$\therefore f\left(\langle u_1, x_1, y\rangle, \langle u_2, x_2, y\rangle, \ldots, \langle u_n, x_n, y\rangle\right) = 0. \quad (20)$$

Hence, it is proved that f is reflexive. □

Note: In certain exceptional situations, the P-IOWA operator may not always satisfy the triangle inequality. Here, we can prove this by taking an example.

Example 3. Assume $x = (x_1, x_2, x_3) = (4, 2, 6)$ with $w = (0.2, 0.3, 0.5)$ and inducing variable $u = (4, 3, 7)$. Consider the penalty function $p(x, y) = |x_i - y|$. Here y is the mean (fused value).

The penalties are calculated as follows.

$p(x_1, y) = |4 - 4| = 0$, $p(x_2, y) = |2 - 4| = 2$, $p(x_3, y) = |6 - 4| = 2$, where $y = 4$ is a fused value.

$$\begin{aligned} \mathrm{P} - \mathrm{IOWA}(u_1, x_1, y) &= \mathrm{P} - \mathrm{IOWA}(4, 4, 4) = 0.3 \times 0 = 0, \\ \mathrm{P} - \mathrm{IOWA}(u_2, x_2, y) &= \mathrm{P} - \mathrm{IOWA}(3, 2, 4) = 0.5 \times 2 = 1, \\ \mathrm{P} - \mathrm{IOWA}(u_3, x_3, y) &= \mathrm{P} - \mathrm{IOWA}(7, 6, 4) = 0.2 \times 2 = 0.4. \end{aligned} \quad (21)$$

Here, we can see the following.

$\mathrm{P} - \mathrm{IOWA}(u_1, x_1, y) + \mathrm{P} - \mathrm{IOWA}(u_3, x_3, y) < \mathrm{P} - \mathrm{IOWA}(u_2, x_2, y)$ as $0 + 0.4 < 1$ and in another situation, it will be

$$\begin{aligned} &P - \mathrm{IOWA}(u_1, x_1, y) + \mathrm{P} - \mathrm{IOWA}(u_2, x_2, y) \\ &\quad > \mathrm{P} - \mathrm{IOWA}(u_3, x_3, y). \end{aligned} \quad (22)$$

As a result of the imposed inducing variable, it can be argued that the P-IOWA operator fails to satisfy the triangle inequality.

5.2. P-OWA and P-IOWA Operators With Geometric Mean.

When it comes to handling data that are more prone to volatility, the geometric mean is frequently favored over the arithmetic mean. The reason for this is that the arithmetic mean can be greatly impacted by extreme values or outliers within the data, thus diminishing its reliability for datasets that exhibit high volatility. The application of the geometric mean proves to be beneficial in the computation of ratios or rates of growth, such as compound annual growth rates or population growth rates. It has been noted in the past that researchers have increased the use of the OWA and IOWA operators by including the geometric mean in their literary works [14, 37, 38]. We may create the penalty-based ordered weighted geometric (P-OWG) averaging operator and penalty-based induced ordered weighted geometric (P-IOWG) averaging operator by extending the P-OWA and P-IOWA operators with the geometric mean.

Definition 6. A P-OWG averaging operator $P - \mathrm{OWG}: R^{n+1} \longrightarrow [0, \infty)$ is a mapping that has a corresponding weighting vector W with penalty function $p(x, y)$ having argument vector x and y as fused value such that

$$P - \mathrm{OWG}_W\left(x_{(i)}, y\right) = \prod_{i,j=1}^{n} p(x_i, y)^{w_j}, \quad (23)$$

where the penalties of x are arranged in a decreasing order.

Definition 7. A P-IOWG averaging operator $P-$ IOWG: $R^n \times R^n \times R \longrightarrow R^+$ is a mapping that has a corresponding weighting vector W with penalty function $p(x_i, y)$ having argument vector x_i and y as fused value such that

$$P - \mathrm{IOWG}_W(u_i, x_i, y) = \prod_{i,j=1}^{n} p(u_i, x_i, y)^{w_j}, \quad (24)$$

where the penalties of x are arranged as largest to lowest according to the inducing variable u_i for all i.

It is feasible to formulate the P-IOWA operator and P-OWA operator in a comparable manner by initially rearranging the arguments and subsequently computing the penalties. Assume g to be a measure defined as follows:

$$g\left(\langle u_1, x_1, y\rangle, \langle u_2, x_2, y\rangle, \ldots, \langle u_n, x_n, y\rangle\right) = \sum_{k=1}^{n} w_k c_k, \quad (25)$$

where c_k is the $p(x_k, y)$ value of the triplet $\langle u_i, x_i, y\rangle$ and x_k is the k^{th} argument variable of the set $x = (x_1, x_2, \ldots, x_n)$ and y is the fused value having the k^{th} largest u_k and u_k is the order-inducing variable. This measure is same as P-IOWA if reordering of both the operators is same.

Suppose f is the P-IOWA operator and g is the measure explained in equation (25). If $k = i$, then

$$\begin{aligned} &g(\langle u_1, x_1, y\rangle, \langle u_2, x_2, y\rangle, \ldots, \langle u_n, x_n, y\rangle) \\ &\quad = f(\langle u_1, x_1, y\rangle, \langle u_2, x_2, y\rangle, \ldots, \langle u_n, x_n, y\rangle). \end{aligned} \tag{26}$$

In the literature on ordered weighted averaging [11, 39, 40] operator, various measures have been employed for describing the weight vector. Addressing the behavior of the P-IOWA operator is of utmost importance. Implementing the approach of the OWA operator will enable us to define the measure for the penalty-based functions as given below:

$$\alpha(W) = \sum_{j=1}^{n} w_j^{*}\left(\frac{n-j}{n-1}\right), \tag{27}$$

where $\alpha(W)$ is the degree of orness defined by Yager.

Furthermore, there are alternative measures that could be explored such as entropy of dispersion, divergence of W, and the balance operator [41, 42], which are defined as follows.

The entropy of dispersion is

$$H(W) = -\sum_{j=1}^{n} w_j \ln\left(w_j\right). \tag{28}$$

For balance operator, it is defined as

$$\mathrm{BAL}(W) = \sum_{j=1}^{n}\left(\frac{n+1-2j}{n-1}\right)w_j. \tag{29}$$

For divergence of W, it is written as

$$\mathrm{DIV}(W) = \sum_{j=1}^{n} w_j\left(\frac{n-j}{n-1} - \alpha(W)\right)^2, \tag{30}$$

where w_j^{*} is the w_j weight of the P-IOWA aggregation ordered according to the values of the arguments $p(x_i, y)$.

A further factor that requires attention pertains to the challenge of inducing identical variables for two distinct arguments. This scenario presents a challenge in terms of ordering the arguments. Therefore, to address the issue of tied arguments, we suggest implementing the Yager and Filev policy [17] as a practical solution to obtain the average of such arguments.

5.3. Families of P-OWA and P-IOWA Operators. By considering a distinct weighted vector manifestation, various forms of P-OWA and P-IOWA operators [40, 41, 43] can be derived, such as the max-penalty, min-penalty, step P-OWA and step P-IOWA, olympic P-OWA and olympic P-IOWA, and many more.

Remark 1. By putting weights $w_1 = 1$ and $w_i = 0$, for all $i \neq 1$, in the operator P-OWA, we achieved the maximum or highest penalty value, and taking $w_n = 1$ and $w_i = 0$, for all $i \neq n$ weights, we get the min P-OWA or lowest penalty. Similarly, for P-IOWA, the maximum penalty is attained by replacing the weights $w_p = 1$ and $w_i = 0$, for all $i \neq p$, and $u_p = \max\{a_i\}$, and the minimum penalty is obtained by substituting $w_p = 1$ and $w_i = 0$, for all $i \neq p$, and $u_p = \min\{a_i\}$. The step P-OWA is formed when $w_k = 1$ and $w_i = 0$, for all $k \neq i$, and similar methodology is applied on step P-IOWA with inducing variable $u_k = a_k$. The olympic P-OWA operator is obtained when $w_1 = w_n = 0$ and $w_j = 1/(n-2)$, $j \neq 1$ or n, and the olympic P-IOWA is obtained when $w_1 = w_n = 0$ and for all $w_j = 1/(n-2)$ with inducing variable u_j, $j \neq 1$ or n.

Remark 2. Similarly, we can find window P-OWA when $w_j = 1/m$ for $k \leq j < k+m$ and $w_j = 0$ for $j \geq k+m$ and $j < k$ and window P-IOWA when $w_j = 1/m$ for $k \leq j < k+m$ and $w_j = 0$ for $j \geq k+m$ and $j < k$, with inducing variable u_j. Here k and m must be positive integers such that $k + m \leq n$.

6. P-OWA Operators With Generalized and Quasi-Arithmetic Mean

In this particular section, our objective is to enhance the P-OWA and P-IOWA operator by integrating the generalized and quasi-arithmetic mean. This combination helps to resolve complex situations effectively. Moreover, this section analyzes the particular instances of these operators.

6.1. GOWA Operator. Yager has introduced the operator known as GOWA and incorporated generalized means within the OWA operator, resulting in a more comprehensive approach.

Definition 8 (see [15]). A GOWA operator is an n-dimensional function GOWA: $R^n \longrightarrow R$ that has a weighting vector W such that $\sum_j w_j = 1$ and $w_j \in [0,1]$ and a parameter $\lambda \in (-\infty, \infty)$, according to the following formula:

$$\mathrm{GOWA}(a_1, a_2, \ldots, a_n) = \left(\sum_{j=1}^{n} w_j b_j^{\lambda}\right)^{(1/\lambda)}, \tag{31}$$

where b_j is the jth largest in a_i for all i = 1…n.

The induced generalized OWA (IGOWA) operator is derived by using classification approaches to the IOWA operator. It is defined as follows.

Definition 9 (see [44]). An IGOWA operator of dimension $2n$ is a mapping $IGOWA$: $R^n \times R^n \longrightarrow R$ such that it has a weighting vector $W = [w_1, w_2, \ldots w_n]^T$, where $w_i \in [0,1]$ and $\sum_j w_j = 1$, a set of ordered induced variables u_i, and a parameter $\lambda \in (-\infty, \infty)$, the resulting formula is

$$\text{IGOWA}(\langle u_1, a_1\rangle, \langle u_2, a_2\rangle, \ldots, \langle u_n, a_n\rangle) = \left(\sum_{j=1}^{n} w_j b_j^{\lambda}\right)^{(1/\lambda)}, \quad (32)$$

where b_j is the a_i value of the IOWA pair $\langle u_i, a_i\rangle$ having the jth largest value of the u_i.

6.2. P-OWA With Generalized Mean

Definition 10. A penalty-based GOWA is a mapping $P-$GOWA: $R^n \times R \longrightarrow R^+$ having $n+1$ dimension such that it has weighting vector W, a set of order-inducing variables u_i, and a parameter $\lambda \in (-\infty, \infty)$, the resulting formula is

$$P - \text{GOWA}(x_i, y) = \left(\sum_{i,j=1}^{n} w_j p(x_i, y)^{\lambda}\right)^{(1/\lambda)}, \quad x_i, y \in R, \quad (33)$$

where the weights are assigned to the penalties according to their respective penalty size by using the OWA function and also the penalties of x are arranged in a decreasing order.

Definition 11. A penalty-based IGOWA of dimension $(2n+1)$ is a function $P - IGOWA$: $R^n \times R^n \times R \longrightarrow R^+$, with corresponding weighting vector W, where $w_i \in [0,1]$ and $\sum_j w_j = 1$, for all $j = 1$ to n, and has a parameter $\lambda \in (-\infty, \infty)$; then accordingly, the formula is written as

$$P - \text{IGOWA}(u_i, x_i, y) = \left(\sum_{i,j=1}^{n} w_j p(u_i, x_i, y)^{\lambda}\right)^{(1/\lambda)}, \quad x_i, y \in R, \quad (34)$$

where the penalties of x are arranged in decreasing order according to the value of induced variable to u_i, for all i.

Taking different parameter values in equation (33) we get the families of P-GOWA operator such as P-OWG averaging operator for $\lambda \longrightarrow 0$, penalty-based ordered weighted quadratic averaging (P-OWQA) operator for $\lambda = 2$, penalty-based ordered weighted cubic averaging (P-OWCA) operator for $\lambda = 3$, and many more. Similarly, by employing the identical approach to the P-IGOWA operator, we can produce various categories of P-OWA operators.

The P-IGOWA also functions as an aggregation operator. As a result, it satisfies the commutativity, monotonicity, boundedness, and idempotent properties.

6.3. P-OWA With Quasi-Arithmetic Means. A more comprehensive extension of the OWA operator based on penalties can be achieved through the application of quasi-arithmetic means, in lieu of the arithmetic means. In the past, several authors have used quasi-arithmetic approaches to expand OWA and IOWA operators [16, 44]. In a way that reflects the earlier method, a similar concept like the quasi-penalty–based ordered weighted averaging (QP-OWA) operator, as well as the quasi-penalty–based induced ordered weighted averaging (QP-IOWA) operator, can be formulated. The advantage of these operators is valuable since they enable a broader application and provide a wide variety of specific scenarios. These operators are defined as follows.

Definition 12. A mapping QP − OWA: $R^{n+1} \longrightarrow R$ referred to as a QP-OWA operator of dimension $n+1$ can be characterized by a weighting vector W such that the summation of the weights equals to 1 and $w_j \in [0,1], \forall j = 1$ to n, and by a strictly monotonic continuous function $g(p(x_i, y))$, as follows:

$$\text{QP} - \text{OWA}(x, y) = g^{-1}\left(\sum_{i,j=1}^{n} w_j g(p(x_i, y))\right), \quad (35)$$

where the weights are assigned to the penalties according to their respective penalty size by using the OWA function and also the penalties of x are arranged in a decreasing order.

Definition 13. A QP-IOWA operator of dimension $2n+1$ is a mapping $QP - \text{IOWA}$: $R^n \times R^n \times R \longrightarrow R^+$ that has an n-dimensional weight vector W and has a strictly monotonic continuous function $g(p(u_j, x_j, y))$, which is defined as follows:

$$\text{QP} - \text{IOWA}(\langle u_1, x_1, y\rangle, \langle u_2, x_2, y\rangle, \ldots, \langle u_n, x_n, y\rangle) = g^{-1}\left(\sum_{i,j=1}^{n} w_j g\left(p\left(u_j, x_i, y\right)\right)\right), \quad (36)$$

where the penalties of x are ordered as largest to lowest in accordance with the value of the induced variable to u_i, for all i.

6.4. Particular Cases. In equations (35) and (36), by using different type of functions we are able to generate other operators, such as P-OWG operator, P-IOWG operator, the penalty-based induced ordered weighted harmonic averaging operator (P-IOWH) [15, 44], P-IOWA, penalty-based induced ordered weighted quadratic averaging operator (P-IOWQA), penalty-based induced ordered weighted cubic averaging operator (P-IOWCA) and many other operators. All the particular cases are depicted in Table 5.

In Table 5, we are taking $g(p)$ in place of $g(p(x_i, y))$ and $g(p(u_i, x_i, y))$ according to equations (35) and (36).

7. P-OWA Framework for Multinational Investment in South Asia

The main objective of this work is to apply the penalty-based operators as a novel aggregation operator for assisting

Table 5: Particular cases of the QP-OWA and QP-IOWA operators.

Functions	Particular cases			
	QP-OWA		QP-IOWA	
	Name	Formula	Name	Formula
$g(p) = 1$	P-OWG	$\prod_{i=1}^{n} p(x_i, y)^{w_j}$	P-IOWG	$\prod_{i=1}^{n} p(u_i, x_i, y)^{w_j}$
$g(p) = p$	P-OWA	$\sum_{i=1}^{n} w_j p(x_i, y)$	P-IOWA	$\sum_{i=1}^{n} w_j p(u_i, x_i, y)$
$g(p) = p^{-1}$	P-OWH	$\sum_{i=1}^{n} w_j / p(x_i, y)$	P-IOWH	$\sum_{i=1}^{n} w_j / p(u_i, x_i, y)$
$g(p) = p^2$	P-OWQA	$\sum_{i=1}^{n} w_j p(x_i, y)^2$	P-IOWQA	$\sum_{i=1}^{n} w_j p(u_i, x_i, y)^2$
$g(p) = p^3$	P-OWCA	$\sum_{i=1}^{n} w_j p(x_i, y)^3$	P-IOWCA	$\sum_{i=1}^{n} w_j p(u_i, x_i, y)^3$

various MNCs in selecting the most suitable investment options. The steps to use these new operators are as follows:

Step 1. Decide which part of the world MNCs wish to invest in or grow their business. Decision makers may take this into account when deciding which country's market to enter.

Step 2. The company needs an in-depth analysis of the economic, social, and political or governance (ESG) issues in each country. By considering ESG factors in investment decisions, it is possible to reduce risk and increase value for both investors and corporations, as ESG investment has the potential to influence the enduring sustainability and operational efficacy of a business enterprise. Pick out all the indicators you need for any company's investment.

Step 3. Construct the decision matrix having different countries as an alternatives and investment indicators as a criterion. Let $C = \{C_1, C_2, \ldots, C_n\}$ be the set of finite criteria and $A = \{A_1, A_2, \ldots, A_m\}$ be the set of finite alternatives for the decision matrix. Then the decision matrix is defined as follows:

$$\begin{array}{c} \\ A_1 \\ A_2 \\ \vdots \\ A_m \end{array} \begin{array}{c} \begin{array}{cccc} C_1 & C_2 & \cdots & C_n \end{array} \\ \begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ r_{21} & r_{22} & \cdots & r_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ r_{m1} & r_{m2} & \cdots & r_{mn} \end{bmatrix} \end{array}. \tag{37}$$

Here $\{r_{ij}\}_{m \times n}$, $i = 1, 2, \ldots m$ and $j = 1, 2, \ldots n$ represent the value of each criterion C_j for the respective country A_i.

Step 4. Find the fused value or ideal value for calculations of penalties. In the case of using present-year data and selecting ideal values from the subsequent year, decision makers should use slightly future-oriented ideal values for current data. It is a valid and forward-looking approach to decision making. It acknowledges that progress and improvement are ongoing processes and sets targets that encourage growth beyond current performance levels. The future year's ideal values likely represent the most recent aspirations and benchmarks available, incorporating the latest economic trends, policy, changes, and global developments.

Step 5. Determine the penalties for each criterion with suitable penalty function and then give weight to penalty according to their value. Here let $W = \{w_1, w_2, \ldots, w_n\}$ be a weighting vector such that $\sum_{j=1}^{n} w_j = 1$ and $w_j \in [0, 1]$.

Step 6. Calculate the order-inducing vector u to be used in the decision matrix for each alternative A_i and criteria C_j.

Step 7. Collect the required data for constructing decision matrix and then we aggregate data by different P-OWA operators. After getting aggregated criterion value of each country, we are ordering or ranking each country by aggregation operator's value. These rankings help in selection of optimal solution for given investment problem.

8. Case Study

In order to understand the new theoretical approaches presented in the previous sections, let us develop a case study. For doing so, we analyze an investment problem by employing penalty operators to demonstrate its applicability. First, we describe and analyze the case study. In continuation, we discuss the results providing useful information about the investment.

8.1. Overview of an Investment Problem. In this section, we will consider a case study to illustrate how to apply the P-OWA and P-IOWA operators to a problem involving the selection of a country for investment by MNCs.

Step 1. Let us consider any MNC that wants to do investment in South Asian countries for its business expansion. We take into consideration seven nations denoted as A_i as listed below:

1. A_1: Bangladesh
2. A_2: India
3. A_3: Maldives

4. A_4: Nepal
5. A_5: Pakistan
6. A_6: Sri Lanka
7. A_7: Bhutan

Step 2. The key indicators for an investment are environmental, social, and political or governance (ESG). In continuation, it can be categorized into nine major factors. The major factors which affect the foreign direct investment in any country according to literature review [45–47] are as follows:

1. C_1: GDP growth (annual %)
2. C_2: Exports of goods and services (% of GDP)
3. C_3: Imports of goods and services (% of GDP)
4. C_4: Inflation, consumer prices (annual %)
5. C_5: Labor force participation rate, total (% of total population ages 15+)
6. C_6: Wage and salaried workers, total (% of total employment)
7. C_7: Control of corruption—percentile rank, upper bound of 90% confidence interval
8. C_8: Government effectiveness—percentile rank, upper bound of 90% confidence interval
9. C_9: Political stability and absence of violence/terrorism—percentile rank, upper bound of 90% confidence interval

Step 3. Here in Table 6, we are using World Bank and Asian Development Bank data [48, 49] for each indicator of the countries. In this case, we took the roundoff value of each criterion. We have taken one year (2021) data of each country for prediction on investment. It should be noted that all the values of each criterion are in terms of percentage. The decision matrix of investment criteria is given in Table 6.

Using all these kinds of information, we construct the investment matrix which is defined in Table 6 according to Section 7 (Step 3). In this table, all criterion values are in terms of percentage. To understand this table, one can take the first row of decision investment matrix which is described as

- r_{11} denotes the value of GDP growth of country A_1
- r_{12} denote the value of exports of goods and services (% of GDP) of country A_1
- r_{13} denote the value of imports of goods and services (% of GDP) of country A_1
- r_{14} denote the value of inflation, consumer prices (annual %) of country A_1
- r_{15} denote the value of labor force participation rate, total (% of total population ages 15+) of country A_1
- r_{16} denote the value of wage and salaried workers, total (% of total employment) of country A_1
- r_{17} denote the value of control of corruption: percentile rank, upper bound of 90% confidence Interval of country A_1
- r_{18} denote the value of government effectiveness: Percentile rank, upper bound of 90% confidence interval of country A_1
- r_{19} denote the value of political stability and absence of violence/terrorism: percentile rank, upper bound of 90% confidence interval of country A_1

Step 4. In our investment table, it is evident that all criterion values fluctuate within a 0–100 scale, which makes it quite challenging for the decision maker to determine penalties in each criterion. So, first we normalize Table 6 using the min–max normalization method. Also, we use the inverted formula of the min–max normalization method for the criteria such as imports of goods and services (% of GDP) and inflation, consumer prices (annual %) as they are considered better with lower values among all the criteria. The normalized table of investment criteria is given in Table 7.

Step 5. After normalizing all criterion values, we find the desired values or ideal values which help in finding penalties. Here we are considering desired values from the year 2022 to calculate the penalties. The 2022 desired values are more reflective of the current market conditions and investment priorities, making the analysis more relevant and actionable. Basing the penalties on the 2022 desired values shifts the focus to future performance, rather than just looking at past data. This aligns better with the decision-making needs of MNCs evaluating investment opportunities. The table of ideal values for each criterion is given in Table 8.

In Table 8, all criterion values are normalized using the min–max and inverse min–max method.

Step 6. Using these ideal values, we can find penalties in each criterion using the penalty function as $p(x, y) = |x_i - y|$, where x_i is the criterion value C_i, and considering y as an ideal solution (fused value) [3, 50, 51]. So, we can calculate the penalties of each criterion which is mentioned in Table 9.

Step 7. Note that there are many methods for calculating weighting vector [40, 44]. Our main objective is to minimize penalties while aggregation. For this, we have to take our weighting vector such as it assigns higher weights to indicators with higher variability across the countries. We apply the normalized inverse penalty method that minimizes the overlap or redundancy between indicators, ensuring that each indicator contributes unique information to the overall assessment. This can be important when dealing with a set of correlated economic and governance metrics. By applying the method to the normalized data, we can derive a set of weights that reflect the relative importance of each indicator based on its variability and uniqueness. The weighting vector with $\sum_{j=1}^{9} w_j = 1$ and $w_j \in [0, 1]$ is given as follows:

$$w = \{0.1765, 0.1277, 0.0491, 0.0776, 0.2465, 0.0928, 0.0932, 0.0632, 0.0733\}, \tag{38}$$

Table 6: Decision matrix of investment criteria.

Alternatives	Criteria								
	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9
A_1	6.94	10.66	17.06	5.55	58.80	41.46	26.19	41.43	23.11
A_2	9.69	21.40	24.02	5.13	51.15	23.35	50.48	70.95	33.96
A_3	37.69	77.62	73.99	0.54	61.71	77.68	58.10	80.48	81.60
A_4	4.84	5.12	37.93	4.15	40.00	21.12	46.19	33.33	51.42
A_5	6.51	9.05	17.98	9.50	52.73	42.23	32.38	50.48	10.38
A_6	4.21	16.92	24.31	7.01	51.11	57.61	50.48	62.86	47.64
A_7	4.42	29.20	48.38	7.35	69.09	28.42	95.71	84.29	93.87

Note: Source: [48, 49].

Table 7: Normalized matrix of investment criteria.

Alternatives	Criteria								
	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9
A_1	0.08150	0.07645	0.99998	0.44133	0.17230	0.35970	0.00001	0.15892	0.15251
A_2	0.16367	0.22454	0.87768	0.48757	0.10221	0.03935	0.34934	0.73827	0.28246
A_3	0.99992	1.00003	−0.00005	0.99965	0.19902	0.99995	0.45894	0.92516	0.85308
A_4	0.01876	−0.00001	0.63333	0.59713	−0.00005	0.00002	0.28769	0.00007	0.49150
A_5	0.06881	0.05423	0.98386	0.00042	0.11669	0.37319	0.08905	0.33646	−0.00003
A_6	−0.00008	0.16278	0.87268	0.27737	0.10182	0.64513	0.34934	0.57942	0.44630
A_7	0.00631	0.33212	0.44979	0.24031	0.26666	0.12914	1.00006	0.99992	0.99998

Table 8: Criterion ideal values.

C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9
0.6415	0.7382	0.8648	0.8861	0.4000	0.1048	0.3938	0.4832	0.4516

Step 8. When choosing the inducing vector, it is essential to analyze the factors that impact our chosen criteria. Factors such as exports and remittances, foreign direct investment, human capital development, macroeconomic stability, trade policies, global market, social development, small market size, geographical isolation, legal system, limited infrastructure, cost of doing business, intellectual property protection, infrastructure and supply chain risks, trade barriers, and tariffs differ from one country to another. Considering these factors, experts in the respective fields determine the values of the inducing variables out of 10 for each criterion. The values of the inducing variable for each criterion are given in Table 10.

Step 9. In this step, we aggregate the data using different P-OWA and P-IOWA operators. Aggregated results are calculated using the formulas given in Sections 2, 3, and 4. To aggregate the data, we are using the values of Tables 9 and 10. The aggregated values of the different penalty operators are determined in Tables 11 and 12.

As we obtain the aggregated results for all operators subjected to penalties, we will considerately arrange the nations according to the least penalties incurred by each country for each operator. This is given in Table 13.

8.2. Comparative Analysis of Penalty-Based Operators. A thorough analysis is provided by systematically comparing the performance of various penalty OWA operators in selecting the top South Asian countries for MNC investment. This aids in grasping the advantages and drawbacks of each operator, facilitating the selection of the most suitable one for your decision making. After observing aggregated data from Tables 10 and 11, harmonic, geometric, and min-penalty operators are most risk-seeking operators. Here in the context of P-OWA operators, being "most risk-seeking" means that this operator tends to produce the lowest aggregate penalty value compared to other operators when applied to the same dataset. The MIN operator is the most effective in reducing the penalty. This makes sense as it always selects the minimum value, which in a penalty context would be

Table 9: Penalty values of each criterion.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9
A_1	0.5600	0.6618	0.1352	0.4447	0.2277	0.2549	0.3938	0.3243	0.2991
A_2	0.4778	0.5137	0.0129	0.3985	0.2978	0.0654	0.0444	0.2551	0.1692
A_3	0.3584	0.2618	0.8648	0.1136	0.2010	0.8952	0.0652	0.4420	0.4015
A_4	0.6227	0.7382	0.2315	0.2889	0.4000	0.1048	0.1061	0.4831	0.0399
A_5	0.5727	0.6840	0.1191	0.8857	0.2833	0.2684	0.3047	0.1467	0.4516
A_6	0.6416	0.5754	0.0079	0.6087	0.2982	0.5403	0.0444	0.0962	0.0053
A_7	0.6352	0.4061	0.4150	0.6458	0.1333	0.0243	0.6063	0.5167	0.5484

Table 10: Inducing variables for each criterion.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9
A_1	7.00	7.00	6.50	7.00	7.50	7.00	5.00	5.50	6.50
A_2	8.50	8.00	8.00	7.50	7.50	7.50	4.50	5.00	5.50
A_3	8.00	7.00	6.00	7.50	6.50	6.00	6.50	6.50	7.50
A_4	6.00	4.00	6.00	6.00	6.50	6.00	5.00	5.50	6.00
A_5	5.00	5.00	7.00	7.00	6.50	6.00	4.00	4.50	5.00
A_6	6.00	6.00	7.00	5.00	7.00	7.00	5.50	6.00	6.50
A_7	6.00	6.00	5.50	6.00	7.00	6.50	6.00	6.00	7.00

Table 11: Aggregated results-1.

	P-MIN	P-MAX	P-OWA	P-OWG	P-OWH	P-OWQA	P-OWCA
A_1	0.2951	0.4485	0.3965	0.3620	0.3270	0.4289	0.4579
A_2	0.1666	0.3346	0.2828	0.1974	0.0905	0.3311	0.3624
A_3	0.2816	0.5458	0.4646	0.3629	0.2606	0.5495	0.6148
A_4	0.2323	0.4520	0.3767	0.2864	0.1882	0.4439	0.4928
A_5	0.3079	0.5393	0.4513	0.3795	0.3151	0.5192	0.5760
A_6	0.1991	0.4286	0.3486	0.1804	0.0378	0.4264	0.4697
A_7	0.3269	0.5194	0.4814	0.3928	0.1956	0.5146	0.5325

Table 12: Aggregated results-2.

P-Iowa	P-IOWG	P-IOWH	P-IOWQA	P-IOWCA
0.3313	0.3033	0.2779	0.3607	0.3893
0.2961	0.2160	0.1073	0.3399	0.3672
0.3348	0.2592	0.1979	0.4149	0.4883
0.3193	0.2283	0.1494	0.3901	0.4397
0.4472	0.3641	0.2831	0.5135	0.5630
0.3344	0.1277	0.0246	0.4313	0.4799
0.4237	0.3445	0.2041	0.4647	0.4899

Table 13: Ordering of the countries for investments.

Operators	Ordering
P-MIN	$A_2 \succ A_6 \succ A_4 \succ A_3 \succ A_1 \succ A_5 \succ A_7$
P-MAX	$A_2 \succ A_6 \succ A_1 \succ A_4 \succ A_7 \succ A_5 \succ A_3$
P-OWA	$A_2 \succ A_6 \succ A_4 \succ A_1 \succ A_5 \succ A_3 \succ A_7$
P-OWG	$A_6 \succ A_2 \succ A_4 \succ A_1 \succ A_3 \succ A_5 \succ A_7$
P-OWH	$A_6 \succ A_2 \succ A_4 \succ A_7 \succ A_3 \succ A_5 \succ A_1$
P-OWQA	$A_2 \succ A_6 \succ A_1 \succ A_4 \succ A_7 \succ A_5 \succ A_3$
P-OWCA	$A_2 \succ A_1 \succ A_6 \succ A_4 \succ A_7 \succ A_5 \succ A_3$
P-IOWA	$A_2 \succ A_4 \succ A_1 \succ A_6 \succ A_3 \succ A_7 \succ A_5$
P-IOWG	$A_6 \succ A_2 \succ A_4 \succ A_3 \succ A_1 \succ A_7 \succ A_5$
P-IOWH	$A_6 \succ A_2 \succ A_4 \succ A_3 \succ A_7 \succ A_1 \succ A_5$
P-IOWQA	$A_2 \succ A_1 \succ A_4 \succ A_3 \succ A_6 \succ A_7 \succ A_5$
P-IOWCA	$A_2 \succ A_1 \succ A_4 \succ A_6 \succ A_3 \succ A_7 \succ A_5$

the lowest risk or cost. Among the more balanced operators, P-IOWH performs the best, followed closely by P-IOWG. The graph illustrating the comparison of penalty-based operators in relation to their aggregated values is presented in Figure 1. Furthermore, their analysis can be found in Table 14.

8.3. Results and Discussion. Employing various P-OWA operators within the framework of country selection for investment enables a more detailed and refined analysis. Each operator contributes a distinct viewpoint regarding the aggregation of risk and performance metrics. This diversity enables a more thorough analysis, covering various aspects

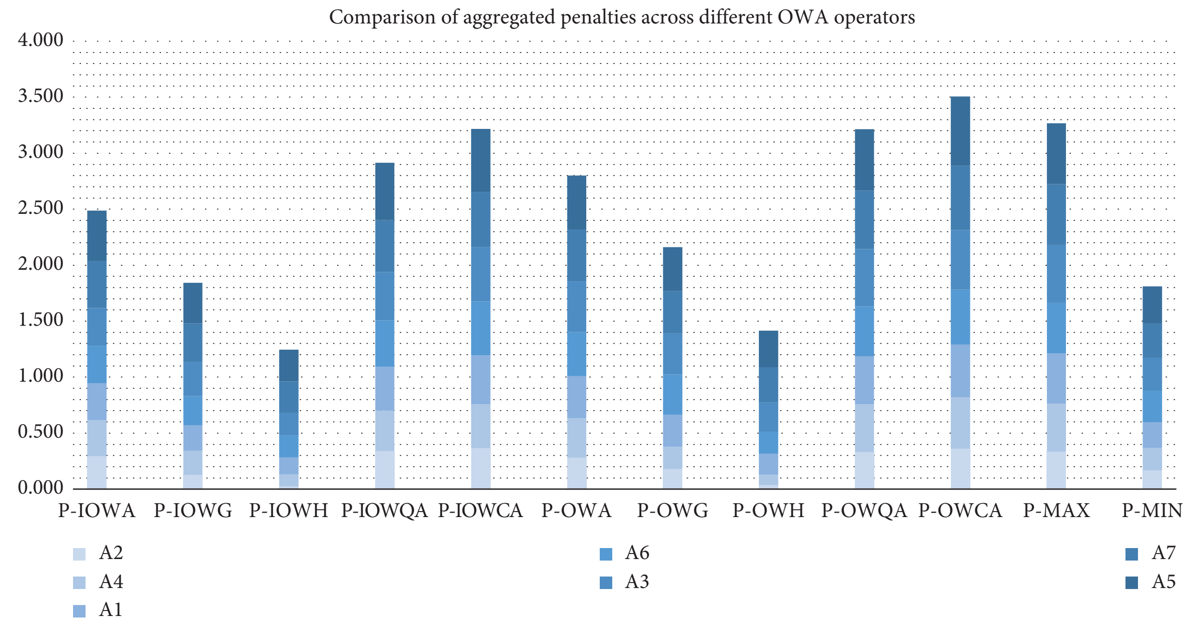

Figure 1: Comparative analysis.

of countries risk profiles. From Table 10, it is observed that with the given weights, the P-OWA scores range from 0.283 to 0.481, indicating that this operator provides a middle-ground approach.

Observations:

- A2 and A6 consistently rank as the top performers across most operators.
- A3, A5, and A7 tend to be the lower performers across most operators.
- A1 and A4 are generally mid-range performers.

Based on the analysis, we can identify which countries need improvement and in which areas:

A1: Needs improvement in C2 (exports), C1 (GDP growth), and C4 (inflation)

A2: Performs well overall, but could improve C2 (exports) and C1 (GDP growth)

A3: Needs significant improvement in C3 (imports), C6 (wage and salaried workers), and C8 (government effectiveness)

A4: Should focus on improving C2 (exports), C1 (GDP growth), and C8 (government effectiveness)

A5: Requires major improvements in C4 (inflation), C2 (exports), and C1 (GDP growth)

A6: Performs well overall, but should address C4 (inflation) and C1 (GDP growth)

A7: Needs to improve C4 (inflation), C7 (control of corruption), and C9 (political stability)

Countries needing the most improvement are A3, A5, and A7. These countries consistently rank lower across different operators and have high penalties in multiple criteria. It has been observed that country A3 has good high scores in some criteria (C3, C6, and C8) but likely has very low scores in others. This uneven performance can lead to a poor overall ranking, especially if the low-scoring criteria are weighted heavily in the aggregation process. Different aggregation penalty methods (P-OWA, P-OWG, P-OWH, P-OWQA, and P-OWCA) may treat outliers or extreme values differently.

Based on the given data and analysis, we can identify several strengths and weaknesses of this research. Here, we have described them in Table 15.

To strengthen the research, we might consider

- Incorporating more years of data for trend analysis
- Including qualitative assessments to complement the quantitative analysis
- Conducting sensitivity analyses to understand how results change with different weightings or methodologies
- Providing more context on the socioeconomic and political situations in each country
- Exploring causal relationships between different factors

Table 14: Penalty-based aggregation analysis.

Analysis	Absolute	Based on arithmetic	Based on geometric	Based on harmonic	Based on quadratic	Based on cubic
Mathematical approach	Absolute max or min	Simple average	Nth root of product	Inverse of mean of inverses	Square root of mean of squares	Cube root of mean of cubes
Behavior	Most sensitive to the smallest or largest penalties	Equally weights all values	Balances between small and large values, but still more sensitive to smaller ones	Gives more weight to smaller values	Gives more weight to larger values	Gives more weight to larger values than OWQA
Penalty value range variation	Both P-MAX and P-MIN give extreme values	Both P-OWA and P-IOWA are giving stability in penalties	P-IOWG reduced individual risk while P-OWG is stable in variation	P-IOWH is effective in risk reduction as well as variation	Though P-OWQA reduces penalties more than P-IOWQA, P-IOWQA gives less variation	Same as P-OWQA
Risk interpretation	P-MIN is most risk-seeking	Risk-neutral	Most risk-seeking	Most risk-seeking	Moderately risk-averse	Most risk-averse
Suitable data distribution	Can be used for best and worst scenarios	Evenly distributed data	More low values than high, but some balance	Many low values, few high outliers	More high values than low	High values are critical and should not be understated

Table 15: Strengths and limitations of the proposed work.

Strengths	Weaknesses
1. Comprehensive criteria: Uses 9 diverse indicators covering economic and governance aspects.	1. Limited time frame: Possibly based on single-year data, missing long-term trends which can be updated in future research.
2. Advanced analytical techniques: Employs various penalty-based OWA operators for sophisticated analysis.	2. Data availability and quality: Relying on potentially inconsistent data.
3. Comparative framework: Allows direct comparison between South Asian countries.	3. Complexity: Advanced techniques may be difficult to interpret for nonexperts.
4. Normalization methods: Uses the min–max method and base 0-1 scaling for comparability.	4. Limited sample size: Only seven countries, limiting statistical power.
5. Multidimensional analysis: Considers both economic and governance factors.	5. It is very difficult for decision makers to find inducing variable.
6. Flexibility in aggregation: Multiple penalty-based OWA methods allow for different perspectives.	—

These strengths and weaknesses provide a framework for understanding the value of the project and areas where it could potentially be improved or expanded.

9. Conclusion

This study has provided an introduction to the P-IOWA operator as well as some of its primary extensions. It is an aggregation operator that integrates both the IOWA and the penalty function into the same formulation. The unique feature of P-IOWA, as compared to other classical OWA operators, is its assignment of weights to the penalty value of each argument rather than the argument itself. Therefore, decision makers are now offered more flexibility when it comes to adjusting penalties in order to align them with their specific needs, which enables them to identify optimal solutions based on their unique requirements.

Our study comprises a comprehensive analysis of the essential characteristics and classes of the topic, encompassing the maximum penalty, minimum penalty, step P-IOWA, olympic P-IOWA, and window P-IOWA. The P-IOWA operator is characterized by its ability to quantify the variability of data points around their fused values. This operator provides a numerical measure of the degree to which individual data points differ from the expected outcome. Penalty-based induced aggregation allows for relative comparisons between different groups, categories, or datasets. It provides insights into which group or category exhibits greater variability compared to others. Additionally, we have initiated a discussion concerning specific cases of P-IGOWA and QP-IOWA with the aim of decreasing or amplifying the aforementioned features.

In recent years, it has been observed that the living standards of individuals have been on a steady rise. This has resulted in an increase in the number of MNCs establishing themselves in developing nations. Consequently, in order for a multinational corporation to expand its reach, it must first assess all essential indicators of the country in question. Our study endeavors to tackle this matter by constructing a theoretical model of the aforementioned issue. This study has provided a comprehensive analysis of seven South Asian countries' economic and governance performance using multiple criteria and advanced analytical techniques. By employing various P-OWA and P-IOWA operators, we were able to aggregate and compare complex data across diverse indicators such as GDP growth, trade metrics, labor force participation, and governance effectiveness. Our analysis demonstrates the utility of multicriteria decision-making approaches in evaluating national performance, while also highlighting the sensitivity of results to different aggregation methods. The findings underscore the importance of considering both economic and governance factors in assessing a country's overall standing. However, it is crucial to note that these quantitative measures should be complemented with qualitative insights for a more holistic understanding. This research provides valuable benchmarks for policymakers and researchers, offering a foundation for identifying areas of strength and opportunities for improvement in each country. Future studies could benefit from incorporating longitudinal data and exploring causal relationships between the various factors influencing national performance in the South Asian context.

In conclusion, this research not only contributes to our understanding of South Asian development but also demonstrates the utility and limitations of advanced multicriteria decision-making techniques in comparative country analysis. The findings can serve as a valuable resource for policymakers, researchers, and international organizations working toward regional development and improved governance in South Asia. However, they should be interpreted with an awareness of the methodological constraints and the complex realities that numerical indicators alone cannot fully capture.

The challenge of this proposed work is interpreting aggregated results that are based on the P-IOWA operator without taking into account more context or information, thus sometimes making the problem more complex. Drawing meaningful conclusions from the aggregated data can be difficult if one does not comprehend the underlying factors that contribute to variability. In the literature, many other approaches could be used such as aggregation operators [52] including the use of probabilities, moving averages [53], logarithms [54], and Bonferroni means [55]. But in this paper, we focus on penalty-based operators.

Our study has taken on a decision-making problem that has wide-ranging applications and can be of great benefit to those looking to make sense of complex data scenarios. By leveraging innovative operators, as explained in this paper, we have made significant strides toward addressing this problem. However, note that many other operators can be developed in the future by using a wide range of techniques including weighted averages, logarithms, moving averages, and Bonferroni means. Additionally, we plan to expand the scope of our research by investigating the potential of penalty operators in fields such as artificial intelligence, finance, the stock market, and risk management [56]. It is noteworthy that our approach could be extended to various real-life decision-making problems, like electronic waste management and renewable energy source selection [57, 58]. Also, different P-OWA operators possess remarkable potential in a variety of domains. Within control systems, they can significantly improve adaptive, robust, and decentralized control by effectively managing the trade-offs between competing objectives [59, 60]. In the realm of fuzzy theory, they can enhance both aggregation and real-time fuzzy inference through the dynamic weighting of inputs [61, 62]. In neural networks, these penalty-based operators can optimize the training process, increase explainability, and serve as regularization mechanisms for nonlinear challenges [63]. For nonlinear systems, they can facilitate stability analysis, nonlinear aggregation, and optimization tasks in chaotic or intricate settings [64]. Future research may delve into their integration with AI, increased computational efficiency, and applications across different domains, such as robotics, bioinformatics, and intelligent systems.

Data Availability Statement

The dataset is publicly available at https://data.worldbank.org/indicator and https://kidb.adb.org/.

Conflicts of Interest

The authors declare no conflicts of interest.

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