Endogenous Inferential Expectations

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Submitted in partial fulfilment of the requirements for the degree of

Bachelor of Business (Honours)

School of Finance and Economics

Supervisor: Dr. Gordon Menzies

17th November 2011

Certificate of Authorship

I certify that this thesis has not already been submitted for any degree and is not being submitted as part of candidature for any other degree.

I also certify that this thesis has been written by me and that any help that I have received in preparing this thesis, and all sources used, have been acknowledged within this thesis.

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17th November 2011

Acknowledgements

It is an enormous privilege to thank those who contributed to this thesis; primarily my supervisor, Dr. Gordon Menzies, whose knowledge, counsel and time was fundamental to its successful completion. Above all, I wish to thank Gordon for his enthusiasm and generosity as a mentor.

I am appreciative of the suggestions provided to me by Associate Professor David Goldbaum, Professor Susan Thorp and Professor Ron Bird; all which helped enrich this final product. I am grateful toward my fellow honours students for the camaraderie and banter that made this year memorable. For their unconditional support, I am indebted to my family and friends.

Finally, I would like to express my gratitude to the academic staff at the University of Technology, Sydney and the State University of New York at New Paltz, who have made my undergraduate years intellectually stimulating and immensely rewarding.

Abstract

Previous research on inferential expectations (IE) (Menzies and Zizzo, 2009) has only considered a test statistic that is exogenous, based on time. This thesis examines the theory of IE for a test statistic that is endogenously determined, and incorporates IE into the standard cobweb model. Three applications are developed; an IE cobweb model nested in adaptive expectations, IE employed to estimate the value of a new parameter, and an IE model which generalises econometric learning. Under the latter, it is shown that belief conservatism results in greater forecast errors, even in a model where equilibrium outcomes are dependent on expectations.

Contents

1. Introduction

The decisions made by individual economic agents play a crucial role in contemporary macroeconomics, and their forward-looking beliefs, or expectations as they are called, influence any variable that is dependent on predictions. Economic outcomes, in turn, influence future expectations, raising the likelihood of 'self-fulfilling' economic fluctuations (Blanchard, 2000). Given their significance, an understanding of how expectations are formed, and change, is necessary for analysis in economics, and should be given pride of place in economic research. Indeed, the awarding of the 2011 Nobel Prize in Economics to Thomas Sargent and Christopher Sims – both of whom have made significant contributions to the field of expectations modelling – recognises the significance of expectations research in the development of modern-day economic theory[1].

Sargent's work, in particular, was influential in the widespread adoption of rational expectations (RE), which has become the standard methodology for modelling expectations in contemporary macroeconomics. Muth (1961) was the first to explicitly formulate the notion of RE, observing that "*expectations, since they are informed predictions of future events, are essentially the same as the predictions of the relevant economic theory*". Such model-consistent, or 'rational', expectations hypothesise that predictions of the future value of an unknown variable are a 'best guess' that uses all available information. As such, RE

[1] Formally, the Sveriges Riksbank Prize in Economic Sciences in Memory of Alfred Nobel 2011 was awarded jointly to Sargent and Sims "for their empirical research on cause and effect in the macroeconomy… expectations regarding the future are primary aspects of this interplay" (The Royal Swedish Academy of Sciences, 2011).

does not differ systematically or predictably from market equilibrium results. Any deviations from predictions therefore must be random, representing shocks or other influences only revealed beyond the current time period. Sheffrin (1983) summarises the RE hypothesis in two statistical lemmas. First, agents' subjective expectations coincide with the model's conditional expectations. Thus, the conditional expectation of the forecast error has to be zero. Second, agents' forecast errors are uncorrelated with any information available in the period in which the forecast is made. Additionally, rationality implies that when new information is made available, agents update their beliefs correctly. The RE hypothesis was widely adopted in the 1970s, beginning with Lucas (1972) and Sargent (1973), and has since become the benchmark for identifying multiple equilibrium problems in dynamic general equilibrium models. In international macroeconomics, the seminal application of RE is the Dornbusch (1976) model of exchange rate determination.

Although being widely accepted, RE has drawn a considerable amount of criticism. Opponents of RE point out that its stringent assumptions tend to ignore what John Maynard Keynes (1936) coined "animal spirits" – downplaying the role of human inefficiency, venality and ignorance. Empirical evidence has also suggested that economic agents do not form expectations according to the definition of full rationality. This has led to development of a number of alternative theories of expectation formation, which modify the assumptions set forth under RE. Sargent and Sims have made key advancements in the study of the limits of rationality, and have contributed, in particular, toward adaptive learning models in their more recent research.

This paper focuses on one such alternative: the theory of inferential expectations (IE), proposed in Menzies and Zizzo (2009). Using statistical hypothesis tests as the basis for belief formation, IE is able to provide an explanation for any mismatch between the *gradual* erosion of economic fundamentals and a *rapid* decline in asset prices, which are supposed to reflect the underlying fundamentals. Such dramatic changes in forecasts, as a result of small increments of information over time, are otherwise inconsistent with RE[2]. Menzies and Zizzo (2009) introduce IE through its application to the Dornbusch (1976) model of exchange rates; this thesis provides a more general framework for IE, though it is illustrated with an application to the cobweb model. It also extends IE to allow for the test statistic agents use in a hypothesis test to be endogenously determined. Three original models of expectation formation are developed and simulated, with the concept of 'self-fulfilling' expectations discussed inside a cobweb model context. As such, this paper contributes to the research agenda on expectations, and in particular, the growing literature on alternatives to RE. It further explores the theory originally presented by Menzies and Zizzo (2009), and allows for a broader application of IE to the cobweb model, which is well-known in economics and has been widely used in previous literature on expectation formation.

The format of this thesis is as follows: Section 2 examines the arguments against RE, both theoretical and empirical. Section 3 provides an overview of a number of alternative theories of expectation formation, including a more detailed examination of IE. Together, Section 2 and 3 serve as a broad review of the literature most relevant to this paper. Section 4 introduces the simple cobweb model under the most common expectations regimes. Section 5 follows with three cobweb model developed under IE, along with selected results of

[2] Macroeconomists have done creative work in modifying and extending RE in ways that allow for an understanding of bubbles and crashes in terms of optimism and pessimism. However, these results emerge by allowing for small deviations from RE. An influential example of such work is Harrison and Kreps (1978), which assumes heterogeneous agents; a set of investors with complete information can arrive at different subjective assessments. This is otherwise inconsistent with the first lemma of RE (Sheffrin, 1983).

simulations run under these new models. Included in this section is an investigation into the notion of self-fulfilling expectations within the cobweb model. Finally, Section 6 provides concluding remarks.

2. The Case Against Rational Expectations

2.1 Theoretical Arguments Against RE

A number of theoretical arguments against rational expectations are summarised in Menzies and Zizzo (2008). It is argued that expectations can only be ‘rational’ if collecting and processing information is costless, in terms of money, time and cognitive effort. If information is costly to obtain and process, then forecasts will not be strictly rational, and will induce shortcuts in the use of information. To put it another way, if there is a positive marginal cost to information and a decreasing marginal benefit, the information collected will fall short of the complete information required to have full knowledge of the system.

On this issue, Arrow (1986) remarks that all knowledge is costly, even the knowledge of prices, hence preventing the complete exploitation of information. In addition, he notes that for RE to be feasible, it requires some strong supplementary assumptions. The most prevalent assumption is homogeneity across individual agents, which is needed in order to reduce the informational requirements. However, it can be problematic to assume all individuals have the same utility functions – a fundamental tenet of much economic analysis is the existence of differences between agents; it is their differing beliefs about the future values of commodities that induce trade. Arrow’s contention has been countered with a widely-held belief that by providing the market’s expectation, RE is providing an approximation that, on aggregate, captures the micro-level diversity amongst its participants (Muth, 1961). However, is has been argued that this 'escape clause' renders incoherent the very notion of a rational market – if it were populated by participants who made use of different forecasting strategies, they would each be considered irrational, in the sense that they would ignore systemic forecast errors, and thereby endlessly forego seemingly obvious profit opportunities (Lucas, 2001; Frydman and Goldberg, 2010).

Similar criticism has been drawn toward the implicit RE assumption that agents have the capacity to be able to develop the correct underlying model of the stochastic process or economic variable they are attempting to forecast. One can observe that this does not apply in reality, as demonstrated by the fact that there is often disagreement amongst economists and financial analysts as to the structural form and parameter values of economic models. Furthermore, it is often the case that multiple models are used in policymaking and forecasting – all which leads to different forecasts of the future. Strictly speaking, there would be no market transactions due to differences of information, utility-preferences or model-inconsistency in a completely rational market. RE, then, is severely strained by information-gathering and computing abilities of agents, and Arrow (1986) remarks that it is incompatible with the limits of the human being, even augmented with artificial aids.

A potential logical inconsistency of RE was also foreshadowed as early as Morgenstern (1935), who argued against a stochastic form of perfect foresight. In prescient articles, Merton (1957), Grunberg and Modigliani (1954) and Simon (1957) all suggest that forecasts could be self-denying or self-fulfilling; that is, the existence of the forecast would alter behaviour as to make an otherwise false forecast true, or cause a correct one to be false. From this mechanism, it would seem that RE could not be denied. However, individuals would require not only the necessary first-order knowledge, but also the knowledge that every other individual is also rational, and in turn, that those individuals know that every other individual is rational. From a general equilibrium point of view, this results in the informational and computational complexities of RE becoming further compounded.

Perhaps the best argument in favour of RE is one which proposes that agents who do not hold RE will be driven out of the market by more rational agents, who are able to bankrupt them through the superior use of information and arbitrage (Alchian, 1950; Friedman, 1953). This 'survival of the smartest' is commonly referred to as the evolutionary dynamics argument.

However, the evidence supporting evolutionary dynamics in economics has been underwhelming. Noise trader models, such as Shleifer (2000) and De Long et al. (1990) have demonstrated how less-than-rational agents may survive contemporaneously with rational agents. A considerable amount of research has been dedicated to the limits to arbitrage, which have been shown to be both risky, and costly (Barberis and Thaler, 2003). Arbitrageurs are faced with the potential of making losses; there is, in all likelihood, the possibility that future price movements for a mispriced security can further increase the mispricing. Hedging against systematic risk has been to shown to be imperfect – finding a perfect substitute to short-sell is rare, and still does not eliminate idiosyncratic risk. A further problem is that even if a perfect substitute does exist, it may itself be mispriced (Froot and Dabora, 1999). Implementation costs also make arbitrage less attractive. The cost of finding and exploiting arbitrage opportunities may be considerable, given that noisy trader demand can cause large and persistent mispricing, with very little predictability in returns, as to be virtually undetectable (Merton, 1987). Furthermore, Grossman and Stiglitz (1980) argue that even if agents had the cognitive ability to form RE, they would be unable to profit from the resultant information since their actions would then reveal their information to others.

There has also been great difficulty in disproving the RE hypothesis. This is due to what Fama (1970) termed the "joint hypothesis problem". Tests of RE are also necessarily tests of the models they are embedded in, since they are *model*-consistent by definition. Hence, the notion of RE cannot not be rejected without an accompanying rejection of the underlying model of market equilibrium. Additionally, there lies the difficulty in identifying RE relative to alternatives in regression analysis, such as the regression test of uncovered interest parity (Menzies and Zizzo, 2009). Failures of rational expectations are often blamed on other auxiliary assumptions, though the problem may lie in the model of expectation formation.

2.2 Empirical Arguments Against RE

Apart from its theoretical shortcomings, detractors of RE have also pointed to systematic empirical failure of RE in experimental settings. Whilst the perfectly rational agent (*homo economicus*) is assumed to make model-consistent forecasts and update their beliefs correctly, behavioural psychologists have discovered that this differs from how agents form expectations in practice. Indeed, the entire field of behavioural economics is built around the assumption that particular phenomenon are better explained using models in which some agents are not treated as fully rational.

These experiments have proven to be so influential that it is worth reviewing some of their main features. Barberis and Thaler (2003) discuss a number of biases that can lead to the failure of the RE hypothesis. Self-attribution bias and hindsight bias have been shown to result in the overconfidence of future forecasts. Confidence intervals assigned by agents on their estimates have been shown to be far too narrow (Alpert and Raiffa, 1982) and individuals have been shown to be poor at estimating probabilities (Fischloff, Slovic and Lichtenstein, 1977). Such 'irrational' overconfidence can be partly attributed to the finding that agents are overly optimistic about their abilities and prospects. Weinstein (1980) notes that over 90% of people surveyed believed themselves to be above average. Perhaps as a result of their optimism, human beings also commonly display a systematic planning fallacy, grossly underestimating the amount of time required to complete premeditated tasks [3] (Buehler, Griffin and Ross, 1994). The latter is particularly relevant, since it may imply a 'short-changing' of time devoted to the in-depth calculations required by RE.

Behavioural experiments also find that agents are imperfect in computing and updating probability values. In attempting to determine the probability that a data set was generated by a particular model, agents often use the representative heuristic. When attempting to determine whether data set A was generated by the stochastic process B, agents make the incorrect assumption that

$$P(A|B) = P(B|A),$$

that is, they equate inverse probabilities. This results in agents commonly putting too much weight on the conditional probability, $P(B|A)$, which captures representativeness, whilst failing to consider (or placing too little weight on) the base rate, $P(B)$, which is the prior probability unconditioned on feature evidence (Kahneman and Tversky, 1974)[4]. The correct

[3] Such as writing thesis papers.

[4] For example, consider a school with 60% male and 40% female students. The female students wear trousers and skirts in equal numbers, whereas the boys all wear trousers. When determining whether a student, observed

calculation uses Baye's rule to take into account the probabilities of both A and B, and is given as

$$P(A|B) = \frac{P(B|A)}{P(B)} P(A).$$

There are also instances in which agents fail to take into account the sample size when determining probabilities, and tend to infer the underlying data-generating process too quickly, based on too few data points. Conversely, the opposite can arise where agents over-emphasise base rates relative to sample evidence. This conservatism leads people to react too little to new information, and rely too much on their prior beliefs (Edwards, 1968). Such 'coarseness' between belief categories was modelled in Mullainathan (2001); rather than updating continuously, updates are made only when agents see enough data to suggest an alternative category is a better fit. Since changes in categories are discrete, small amounts of information can trigger a large change in expectations, even though Bayesian decision making would dictate only the proportionate small change. Hence, this approach can account for both the under-reaction (when information does not lead to a change in category) and over-reaction (when categories change) phenomenon[5].

A related finding has been belief perseverance. Lord, Ross and Lepper (1979) comment that agents may be reluctant to search for evidence that is contrary to their prior, and may ignore (or treat with excessive skepticism) any such evidence, if found. Confirmation bias may also result in individuals misinterpreting contrary evidence to fit their

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wearing trousers, is female, the representative heuristic only considers the probability of wearing trousers *given* the student is female (equating the probabilities), which is 50%. However, this fails to account for the independent probabilities that the student is female and that a randomly selected student is wearing trousers, regardless of any other information. The correct calculation, using Baye's rule, gives the probability as 25%.

[5] As we will see, this is remarkably similar to how expectations evolve under IE, in which the null and alternative hypotheses can be treated as two different categories.

prior belief, such as to avoid realising immediate losses. Most commonly, in situations where agents begin with an initial prior and adjust away from it, experimental evidence has shown that the adjustment is often insufficient (Kahneman and Tversky, 1974), and that those studied "anchored" too much of the initial value.

Difficulty exists in eliminating such biases and sub-optimal behaviour. It could be believed that agents will "learn" their way out of biases; however, such learning is often muted by errors of application (Barberis and Thaler, 2003). Even when a bias is explained and understood, individuals have been shown to immediately violate it again in specific application. The belief that experts in a field, such as financial advisers or fund managers, will make fewer errors is also debatable; experts have been found to exhibit more overconfidence than the average individual (van de Venter and Michayluk, 2008). On the belief that stronger incentives can reduce bias, Camerer and Hogarth (1999) conclude that whilst incentives can reduce the effect of certain biases, no replicated study has made rationality violations disappear purely by raising incentives.

When interpreting these results, RE predictions are not rejected as null hypotheses in some contexts, but rather the most common outcome is that individuals do not hold rational expectations. Some recent behavioural contributions have relied more on a narrative mode of analysis, allowing for richer descriptions of market behaviour than models of rational or irrational behaviour can deliver (Akerlof and Shiller, 2009). The next step in expectations research involves investigating how the insights delivered by behavioural economics can be incorporated into fully-specified economic models. For the most part, however, behavioural economics remains a piecemeal solution to particular market anomalies, rather than a unified theory.

3. Alternatives to Rational Expectations

3.1 Alternative Literature

The theoretical and empirical shortcomings of RE have led to the development of numerous new theories of expectation formation in recent decades which assume less-than-fully rational agents. An early theory was a model of 'near rationality'. Near rational behaviour is defined as that which is sub-optimal, but imposes very small individual losses on its practitioners relative to the consequences of optimal, rational behaviour. In the monopolistic-competition, efficiency-wage model presented by Akerlof and Yellen (1985), firms that adjust prices and wages slowly are said to be behaving 'sub-optimally,' and may suffer small losses from their failure to optimise – that is, exhibit inertial wage-price behaviour.

Model uncertainty and robustness (Hansen and Sargent, 2001) allows for the models developed by agents to be misspecified from the underlying process. Agents are assumed not to know the underlying models, but are able to form approximations. They can then develop alternative models by adjusting their approximations using robust control theory[6]. These alternative models originate from varying an agent's approximation to allow its shocks to feed back onto state variables arbitrarily. This allows the approximating model to miss functional forms, the serial correlation between shocks and exogenous variables, and the relationship between those exogenous variables and endogenous state variables. Given a large

[6] Robust control is an approach for confronting model uncertainty, and aims at finding decision making rules which perform well across a range of alternative models. This typically leads to a minimax approach, where the robust decision rules minimises the worst-case outcome from the possible set (Williams, 2008).

sample of time-series observations, a set of alternative models is shown to be statistically indifferent from the original approximating model.

Optimal expectations (Brunnermeier and Parker, 2005) provides a framework which incorporates the overconfidence of agents uncovered in behavioural experiments such as those of Weinstein (1980) and Buehler, Griffith and Ross (1984). The theory suggests agents consistently err in their forecasts, particularly in overestimating the probability of favourable outcomes. Agents are assumed to optimise according to utility maximisation, however, Brunnermeier and Parker (2005) allow agents' subjective probabilities of the likelihood of outcomes to differ from the objective probabilities. Such a subconscious bias implies that agents are unable to figure out the true probabilities from the model, and hence make poorer decisions than a rational agent would. Interestingly, it is shown that under this framework, agents can be better off (utility-wise) *not* receiving information – despite the benefits of better decision making – and may take more risk, which may lead to an evolutionary advantage.

Alternate theories exist which address the criticism of the informational complexities of RE. Such theories include rational inattention (Sims, 2003) and infrequent RE (Carroll, 2003). Rational inattention adds information processing costs to the standard optimisation problem, and considers the erratic response of individuals arising from information capacity constraints. The effects on information processing, such as the emphasis and presentation of economic information in a newspaper, are shown to influence the behavioural response to data. Similarly, infrequent RE proposes a model in which people obtain their macroeconomic views from the news media, and encounter such information on an infrequent basis (the probability of which is determined by an absorption parameter). The forecasts available in the media are assumed to be provided by professional forecasters, who themselves hold RE.

Individuals who do not encounter such information simply continue to believe the last forecast they read about. This model is shown to be adept at capturing variations of survey measures of inflation and unemployment information (Carroll, 2003), and there lies potential for broader applicability if the determination of the absorption parameter can be modelled[7].

A considerable amount of literature has sought to relax the first statistical lemma of RE, that is, the formulation of model consistent forecasts. 'Bounded rationality' (Townsend 1983; Bray and Kreps, 1987) focuses on models of rational learning, in which agents are assumed to know the reduced form equations of the actual model but are unsure of the associated parameter values. Bray (1982) also previously demonstrated that expectations under such learning, in the context of general, cobweb-type models, will converge over time to their RE value. These models still require greater insight and prior knowledge on the part of agents, which may be unrealistic in the practical sense (Tan, 2006). This had led to the development of bounded rationality frameworks with less stringent assumptions, notably Frydman (1982), Bray and Savin (1986) and Sargent (1993).

Sargent (1993) proposes a specific form of bounded rationality in which agents act as statisticians or econometricians when making forecasts. The agents' utility-maximisation approach is retained, with rational expectations replaced by least-squares learning. As new information becomes available over time, agents incorporate this data into their econometric model, and use the output as their revised forecast. This approach is called adaptive (or econometric) learning. Evans and Honkapohja (2001) focus on a form of bounded rationality

[7] An IE model in which the alternative hypothesis belief is RE can serve as one method of modelling the absorption parameter. The fraction of agents doing a rational calculation then becomes the fraction of agents who reject the null hypothesis, which itself is dependent on individual agents' degree of belief conservatism.

in which expectations affect the time path of the economy, and the forecast functions are updated in response to new information. This corresponds to a form of econometric learning under which the parameters are updated by recursive least squares. Agents who fail to take the feedback of expectations into account therefore end up estimating a misspecified model.

3.2 Inferential Expectations

Menzies and Zizzo, 2009 offer another boundedly rational model of belief formation, which is embedded within the adaptive learning research agenda. The theory of IE behaves as a form of imperfect (or delayed) learning, suggesting that evidence is noticed by agents as they form expectations, but they do not change their forecasts until *enough* evidence has been observed. More formally, IE proposes that the formation of beliefs be treated as statistical hypothesis tests, with agents noticing information without changing their expectations until the rejection region is reached; upon which, their forecasts can be dramatically updated. As such, the theory has the ability to deliver rapid changes in beliefs with small increments of new information. IE can be considered as a 'fast and frugal heuristic' of bounded-rational belief formation, characterised by information-gathering and processing costs. The justification behind IE lie in the findings of belief conservatism in behavioural experiments, and the theory is in close spirit to categorical inferences (Mullainathan, 2001), inattention (Carroll, 2003) and also, as will be discussed further in this thesis, encompasses econometric learning (Sargent, 1993; Evans and Honkapohja, 2001).

It is important to identify the key elements in the IE framework: a cognitive target, hypothesis tests and the test size, α. The cognitive target is defined as the variable(s), or parameter(s), in the model about which agents conduct these hypothesis tests. In other words, it is the parameter(s) 'of inference'. Each application of IE must have at least one cognitive target, and it should be noted that the variable being forecast itself need not be the cognitive target. It may be the case that agents conduct inference on parameter(s) in the underlying forecast equation, in order to then form their expectation.

Under IE, agents must also have a null hypothesis (H_0) about the cognitive target, which they maintain unless enough evidence exists such that they can reject it for an alternative hypothesis (H_1). The test size, or significance level attributed to the hypothesis test, denoted α, determines the critical value upon which the null hypothesis is rejected. As such, α can be seen the degree of agents' belief conservatism. When $\alpha = 1$, agents hold no belief conservatism, and never fail to reject the null hypothesis; when $\alpha = 0$, they are completely unresponsive to evidence and always fail to reject the null hypothesis, never updating their beliefs.

In the Dornbusch-style model of exchange rates presented by Menzies and Zizzo (2009), agents are assumed to switch to the underlying model of RE upon rejecting the null hypothesis. In the period that this occurs, IE is able to deliver sharp movements in the exchange rate consistent with the delayed overshooting puzzle, which is otherwise unexplainable using the original model formulated solely under RE. IE is also shown to be consistent with the empirical finding of downward bias in uncovered interest rate parity and the term structure of interest rates (Menzies and Zizzo, 2009). A further application of IE has been to the Barro-Gordon model of monetary policy (Henckel et al., 2011). It is demonstrated that when agents hold IE, the enforceable range of the equilibrium inflation rate narrows; when the private sector is belief-conservative, the monetary authority is able to cheat for several periods before being 'caught'. In both applications of IE, the original models –

Dornbusch (1976) and Barro and Gordon (1983) – are nested within the more general IE framework. Both become a special case of the IE model, with the test size equalling unity; that is to say, if agents are untainted by belief conservatism, the original and IE model are identical.

In both models of IE, the test statistic is dependent on time, which is exogenously determined. In Menzies and Zizzo (2009), the null hypothesis is rejected after agents witness a monetary expansion that has been sustained for a sufficient number of periods. Similarly, in Henckel et al. (2011) the information criterion is the number of periods on which inflation exceeds the monetary authority's policy rule. The magnitude of such a time-dependent test statistic does not have any feedback effects upon the underlying model[8].

This thesis develops a cobweb model under IE in which the test statistic is *endogenously* determined. A state-dependent test statistic is therefore dependent on the past data series, and has feedback effects that influence future data points. The time path of model variables and the speed of convergence to rationality (Bray, 1982) then become dependent on the size of the test statistic and agents' degree of belief conservatism.

4. The Cobweb Model

4.1 Formulation

To illustrate some of the different approaches to modelling expectations, we will employ a basic cobweb model of cyclical supply and demand, first under the most common

[8] In Menzies and Zizzo (2009) the moment of rejection (t^*) is calculated by equating the p-value to the test size and solving ($e^{-t^*} = \alpha$). Similarly, n Henckel et al. (2011), H_0 is rejected at $t^* \geq 1 - \frac{\ln \alpha}{\ln 2}$. Each test statistic is therefore exogenously determined, and the moment of rejection is independent of the evolution of the system.

expectations regimes, and then under IE. The cobweb model is commonly used to illustrate expectations due to its familiarity and simplicity. The following version of the cobweb model is similar to that employed by Nerlove (1958), Bray and Savin (1986), Evans and Honkapohja (2001) and Muth's (1961) classic formulation of RE.

We begin by considering a single good competitive market, in which there is a one-period time lag in production[9]. Production is completely determined by firms' response to price, and the price is determined by available supply. The log-demand and supply schedules, d_t and s_t respectively, are given by

$$d_t = \bar{d} - d_p p_t + \nu_{1t}$$

$$s_t = \bar{s} + s_p p_t^e + s_w w_{t-1} + \nu_{2t}$$

where d_t depends negatively on price elasticity d_p, s_t depends positively on price elasticity s_p, and $d_p, s_p > 0$. $\bar{d}$ and $\bar{s}$ represent the intercepts, and ν_{1t} and ν_{2t} are assumed to be unobserved, i.i.d. white noise shocks. Supply in period t depends on the expected price at time *t*, using information available at time $t-1$, and the observable exogenous variable w_{t-1}. As in the previous literature, we make the representative agent assumption that all agents hold the same expectation.

Assuming the market clears, $s_t = d_t$, giving the reduced form solution

$$p_t = \gamma - \beta_1 p_t^e - \beta_2 w_{t-1} + \eta_t \tag{4.1}$$

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[9] This makes the implicit assumption that the entire output of a period is placed on the market, with no part held in storage. This is appropriate in situations where the commodity is perishable or no inventory is kept by the supplier, such as in agricultural markets (Ezekiel, 1938).

where $\gamma = \frac{\bar{d}-\bar{s}}{d_p}$, $\beta_1 = \frac{s_p}{d_p}$, $\beta_2 = \frac{s_w}{d_p}$ and $\eta_t = \frac{v_{1t}-v_{2t}}{d_p}$. It can also be seen from (4.1) that the equilibrium market price p_t is dependent on its expectation, lending credence to the idea of self-fulfilling expectations.

4.2 Static Expectations

Under naïve or static expectations, which were widely used in early literature, firms believe the price in any given period is equal to the price in the previous period. The price estimate in the cobweb model takes the form

$$p_t^e = p_{t-1}$$

Substituting this into equation (4.1) yields

$$p_t = \gamma - \beta_1 p_{t-1} - \beta_2 w_{t-1} + \eta_t$$

which converges to a stationary stochastic process if $|\beta_1| < 1$. This condition is satisfied when the price elasticity of demand exceeds that of supply; that is, when $d_p > s_p$.

4.3 Adaptive Expectations

The adaptive expectations hypothesis was introduced in the 1950s by Cagan (1956), Friedman (1957) and Nerlove (1958). It assumes that agents form their expectations about the future based on past events, and revise past expectations that have been incorrect for the future. In the terms of the price estimate, the hypothesis takes the form

$$p_t^e = p_{t-1}^e + \lambda(p_{t-1} - p_{t-1}^e) \qquad (4.2)$$

in which p_{t-1}^e is known to firms as their price estimate formed last period and λ represents the proportional adjustment in the difference, or the degree of 'correction' from last period. It should be noted that $\lambda = 1$ is a special case of adaptive expectations that gives static expectations; that is, agents fully correct their estimate such that it is equal to the market price in the previous period. Conversely, $\lambda = 0$ represents a case where agents refuse to adapt and do not apply any correction to their estimate. Therefore, their price estimate remains unchanged from the previous period and through time.

In the context of the cobweb model, substituting the reduced form solution of p_t from (4.1), lagged one period, into (4.2), we obtain the system

$$p_t^e = p_{t-1}^e + \lambda(\gamma - \beta_1 p_{t-1}^e - \beta_2 w_{t-2} + \eta_{t-1} - p_{t-1}^e)$$

which, when simplified, gives

$$p_t^e = (1 - \lambda(1 + \beta_1))p_{t-1}^e + \lambda\gamma - \lambda\beta_2 w_{t-2} + \lambda\eta_{t-1}.$$

Substituting this into equation (4.1) yields the price equation

$$p_t = \gamma(1 + \beta_1\lambda) - \beta_1 p_{t-1}^e\big(1 - \lambda(1 + \beta_1)\big) - \beta_2 w_{t-1} - \lambda\beta_1\beta_2 w_{t-2} + \beta_1\lambda\eta_{t-1} + \eta_t.$$

Adaptive expectations were widely used in macroeconomics literature in the decades following its introduction. For example, inflation expectations were often modelled adaptively in the analysis of the expectations-augmented Phillips curve (Evans and Honkapohja, 2001).

4.4 Rational Expectations

Under RE, the price estimate is given as

$$p_t^e = E_{t-1}p_t$$

where $E_{t-1}p_t$ denotes the mathematical expectation of p_t conditional on information at time $t - 1$. For the cobweb model, substituting this into the reduced form equation (4.1) gives

$$p_t = \gamma - \beta_1 E_{t-1}p_t - \beta_2 w_{t-1} + \eta_t. \tag{4.3}$$

Taking conditional expectations E_{t-1} of both sides of (4.3) yields

$$E_{t-1}p_t = \gamma - \beta_1 E_{t-1}p_t - \beta_2 w_{t-1},$$

which, when rearranged, gives the expectation as

$$p_t^e \equiv E_{t-1}p_t = (1+\beta_1)^{-1}\gamma - (1+\beta_1)^{-1}\beta_2 w_{t-1}. \tag{4.4}$$

Substituting (4.4) into (4.1) and simplifying yields the price market as

$$p_t = (1+\beta_1)^{-1}\gamma - (1+\beta_1)^{-1}\beta_2 w_{t-1} + \eta_t. \tag{4.5}$$

From (4.5), the Rational Expectations Equilibrium (REE) can therefore be expressed as

$$p_t = a_{RE} + b_{RE}w_{t-1} + \eta_t \tag{4.6}$$

where $a_{RE} = (1+\beta_1)^{-1}\gamma$ and $b_{RE} = -(1+\beta_1)^{-1}\beta_2 w_{t-1}$. It can be observed that this is a unique REE, as p_t does not depend on expected future prices; that is, it is not conditional upon p_t^e.

It can also be seen that the non-error component of the right hand side of (4.5) is in fact the price estimate given in (4.4). Hence, under RE

$$p_t = p_t^e + \eta_t,$$

which can be alternatively expressed as

$$p_t - p_t^e = \eta_t.$$

This confirms that, under RE, any deviations from perfect foresight are i.i.d random errors.

4.5 Econometric Learning

Suppose firms believe prices follow the process specified by the REE in equation (4.6), that is

$$p_t = a_{RE} + b_{RE}w_{t-1} + \eta_t,$$

but the values of a_{RE} and b_{RE} are unknown to them. As such, they must come up with estimates for a_{RE} and b_{RE}, $\hat{a}$ and $\hat{b}$ respectively, and forecast according to 'feasible RE' based on

$$p_t = \hat{a} + \hat{b}w_{t-1} + \eta_t.$$ [10]

Let us assume that data on periods $i = 0, \ldots, t-1$ is known. Thus, at time $t-1$, the information set available to each representative firm can be denoted $\{p_i, w_i\}_{i=0}^{t-1}$. Under so-called Econometric Learning (EL) (Sargent, 1993; Evans and Honkapohja, 2001), firms act like econometricians, estimating $\hat{a}$ and $\hat{b}$ by a least squares regression of p_i on w_i and an intercept[11]. The forecast at time $t-1$, of the price next period, is therefore

$$\hat{p}_t = \hat{a}_{EL,t-1} + \hat{b}_{EL,t-1}w_{t-1} \tag{4.7}$$

where $\hat{p}_t$ is the econometrically-learnt estimate of p_t^e, and $\hat{a}_{EL,t-1}$, $\hat{b}_{EL,t-1}$ denote the OLS estimates through time $t-1$, based on the information set $\{p_i, w_i\}_{i=0}^{t-1}$.

These estimates are updated each period as new $\{p, w\}$ couples are collected. Given the price estimate from (4.7), w_{t-1} and the random draw for η_t, the price at time t is subsequently determined by equation (4.1). Thus, at time t, (p_t, w_t) is added to the data, and the information set is now $\{p_i, w_i\}_{i=0}^{t}$. Re-estimating $\hat{a}$ and $\hat{b}$, the forecast at time t of the price next period is

$$\hat{p}_{t+1} = \hat{a}_{EL,t} + \hat{b}_{EL,t}w_t. \tag{4.8}$$

where $\hat{p}_{t+1}$ is now the econometrically-learnt estimate of p_{t+1}^e, and $\hat{a}_{EL,t}$, $\hat{b}_{EL,t}$ denote the OLS estimates through time t, based on the information set $\{p_i, w_i\}_{i=0}^{t}$. This process is continued over time, with firms updating the parameters in (4.8) via least squares estimation each period.

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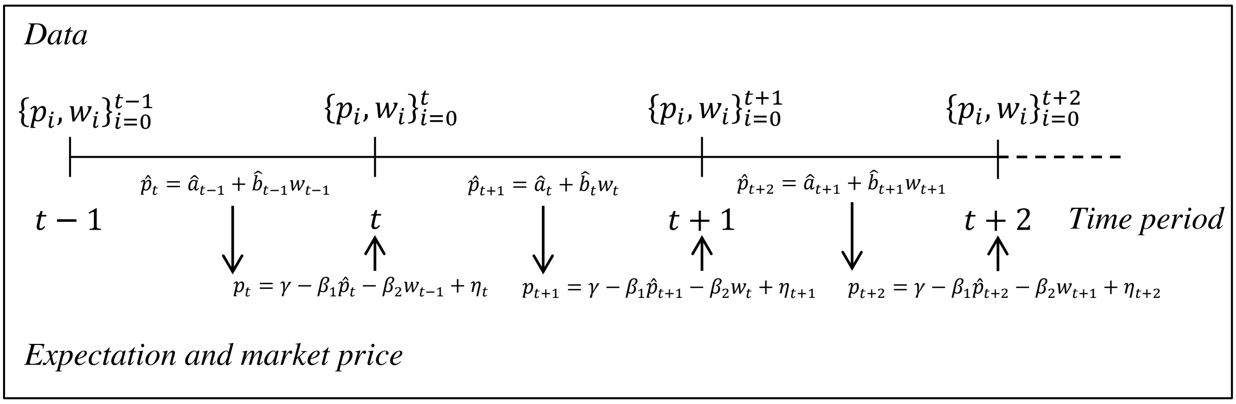
[10] Given the arguments against RE in Section 2, it is considered that perfect rationality is unattainable. Feasible RE represents the best possible use of information in forming expectations.

[11] In some applications, learning is accomplished by a modified form of least squares, where the so-called ‘gain’ in recursive least squares is calibrated. For analysis and discussion of the dynamics of these systems, including stability, see Evans and Honkapohja (2001).

The timing of parameter estimation, expectation formation and the subsequent influence on price is represented in Figure 1.

Figure 1 – Econometric Learning



It is important to note that the expectations and econometric forecast functions influence future data points, and this self-referential feature makes these systems nonstandard.

5. Inferential Expectations Cobweb Models

5.1 Model Variants

We will now refine the cobweb model by endowing suppliers (firms) with IE in forming their price expectations. In the first application that follows in Section 5.2, IE is applied to the adaptive expectations cobweb model, where the cognitive target is the adaptively-formed price estimate itself. In Section 5.4, we examine a scenario where the cognitive target is an unknown demand shock parameter, whilst the other parameters in the cobweb model are assumed to be known. Finally, in Section 5.6, we consider an application of IE to a cobweb model under EL (called Generalised Econometric Learning, or GEL), where the cognitive targets are the parameters of the econometrically-estimated forecast equation, namely $\hat{a}$ and $\hat{b}$.

Each model is simulated following its introduction and detail, with select results included in this thesis. Note that the inclusion of simulations is primarily to assist the reader in understanding the operation and dynamics of these models. It is also important to acknowledge that, as equations are in log-specification, the value of shocks induced on the system are often substantial, and beyond regularly observed phenomenon in economics. These are utilised in order to illustrate the subsequent impact on model variables, which prove useful and consistent for analysis.

5.2 Adaptive IE

We will first consider an IE cobweb model that is nested in the adaptive expectations framework. It is seen that under the adaptive expectations cobweb model, firms correct, or 'adapt,' their forecast every period (assuming $\lambda \neq 0$). By introducing IE into the adaptive expectations framework, we can allow firms to consider the accuracy of their previous price estimate *before* determining whether a correction is necessary. The cognitive target in such an adaptive IE model is therefore the previous period's price estimator, or p_{t-1}^e.

Assume that in the current time period t, firms have data on previous price estimates and market prices. Market prices as formed given in equation (4.1), and firms forecast prices adaptively, as given in equation (4.2). They seek to forecast the price at time t, that is, estimate p_t^e. Under 'adaptive IE', prior to forming this forecast, firms perform a hypothesis test on the accuracy of their estimate last period; that is, how good of an estimator p_{t-1}^e was of the market price last period p_{t-1}. This is tested using a standard hypothesis test to determine if the two values were statistically similar, given as

$$H_0: p_{t-1}^e = p_{t-1}$$

$$H_1: p_{t-1}^e \neq p_{t-1}.$$

As the variance of p_t from equation (4.1) is given by

$$\frac{\sigma_\eta^2}{d_p^2} = \frac{\sigma_{v_1}^2 + \sigma_{v_2}^2}{d_p^2},$$

it follows that the test statistic – under the null hypothesis – is calculated as

$$\frac{p_{t-1} - p_t^e}{\sqrt{\frac{\sigma_{v_1}^2 + \sigma_{v_2}^2}{d_p^2}}} \sim Z \tag{5.1}$$

where the critical value is dependent on α, the significance level attributed to the test, and a measure of firms' belief conservatism. It is assumed that the denominator in (5.1) is observable to firms from past volatility in the market price.

If firms fail to reject the null hypothesis, it is believed that $p_{t-1}^e$was a good estimator of the market price p_{t-1}, and as such, no correction is required for the current estimate of p_t. Thus, under the null hypothesis, $p_t^e = p_{t-1}^e$. Conversely, if the null hypothesis is rejected, it is believed that the market price $p_{t-1}$was sufficiently different from the estimate p_{t-1}^e to the extent that a correction is required for the current period. Under the alternate hypothesis, the price estimate at time t is then given by the adaptive expectations formula from (4.2), that is

$$p_t^e = p_{t-1}^e + \lambda(p_{t-1} - p_{t-1}^e)$$

where λ represents the proportion adjustment of the difference or the degree of correction.

The equilibrium market price p_t is consequently formed, conditional on the outcome of the hypothesis test and as per equation (4.1). A similar hypothesis test is then performed at the beginning of the following period $t+1$, where the accuracy of the estimate p_t^e is tested against the resultant market price p_t, and the price estimate p_{t+1}^e formed conditional on the outcome, as before. This process is repeated in each subsequent period, as t increases.

In the general case, the hypothesis performed at the beginning of period n is

$$H_0: p_{n-L}^e = p_{n-1}$$

$$H_1: p_{n-L}^e \neq p_{n-1}$$

where and L is the number of periods since the null hypothesis was last rejected. The price estimate formed under the null-hypothesis belief

$$p_n^e = p_{n-L}^e$$

under which there is no change in the cognitive target p_n^e from the period since null hypothesis was last rejected. However, if the null hypothesis is rejected, the cognitive target is assumed to be updated in line with adaptive expectations, so that

$$p_n^e = p_{n-L}^e + \lambda(p_{n-1} - p_{n-L}^e).$$

Adaptive IE therefore allows agents to consider the standard error of the price estimate, instead of bluntly updating their forecast in each period. However, it is important to note that under adaptive IE, agents ignore deviations of the market price from their price estimate that they consider 'too small' and as such, this information is not considered in the price estimate in the following period. The amount of deviation required for the price estimate to be updated depends on the test size. This lies in contrast to adaptive expectations, where whereby all movements of price from its estimate are considered in forming the expectation in the following period (although whether the full adjustment is made depends on the value of the correction term λ).

Therefore, adaptive expectations remains a more complete use of information in this framework, and represents a special case of adaptive IE where the test size is unity; that is, when agents are unhindered by belief conservatism, adaptive IE is indistinguishable from adaptive expectations. Adaptive IE therefore provides a more general framework that encompasses adaptive expectations, and includes a more versatile treatment of belief conservatism, that is otherwise treated crudely under adaptive expectations.

5.3 Adaptive IE Simulation

We will now simulate a cobweb model under which expectations are formed using adaptive IE. The model specified is identical to the cobweb model formulated in Section 4.1. For simplicity, we can omit the exogenous variable w_t, without a loss of generality[12]. The reduced form solution is therefore given as

$$p_t = \gamma - \beta_1 p_t^e + \eta_t \tag{5.2}$$

and $\eta_t = \sigma z$, where $z \sim N(0,1)$. To allow for comparison, expectations will be formed under both adaptive expectations as detailed in Section 4.3 and adaptive IE as detailed in Section 5.2, above. The equilibrium market price is formed as per equation (5.2).The price estimate p_t^e and the equilibrium market price p_t are observed for 100 consecutive periods of time t.

Figure 2 (below) illustrates the time path of the price expectation p_t^e, under both adaptive expectations and adaptive IE. The corresponding time paths of the equilibrium

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[12] That is to say, $\beta_2 = 0$. It can be noted that a model in which $\beta \neq 0$ behaves similarly.

market price are shown in Figure 3 (below). The parameter values specified in the simulated model are also provided.

It can be observed from Figure 2 that, unlike adaptive expectations, adaptive IE exhibits periods where (under the null hypothesis) the price expectation is not updated. Time periods where the expectation is revised follow periods with large fluctuations in the market price in Figure 3. This is as the deviations of the market price from the previous price estimate are consider significantly large, and as such, the new information is factored into the price estimate in the following period The frequency of updating is dependent on α, the degree of agents' belief conservatism. As previously discussed in Section 5.2, the adaptive IE solution approaches the adaptive expectations solution as $\alpha \rightarrow 1$; that is, as the degree of belief conservatism declines.

It should also be noted from Figure 3 that the time path of market prices does not appear to vary significantly between adaptive IE and adaptive expectations. In the models presented, movements in the market price are largely driven by fluctuations in the error term η_t. It is worth investigating model behaviour when price volatility is low, in order to develop generalisations and to allow for broader analysis[13]. In this instance, we will examine the effect that volatility has on the frequency of updating under adaptive IE.

Table I (below) outlines the behaviour of similarly specified models for a range of values for σ and α. The frequency with which the null hypothesis is rejected, over 100 simulated data points, is collated for these ranges of volatility and belief conservatism.

[13] The impact of the error term on equilibrium price outcomes is investigated in more detail in Section 5.7.

Figure 2: Price Expectations Under Adaptive IE and Adaptive Expectations

γ = 5, β1 = 0.75, λ = 0.5, σ = 1, α = 0.05

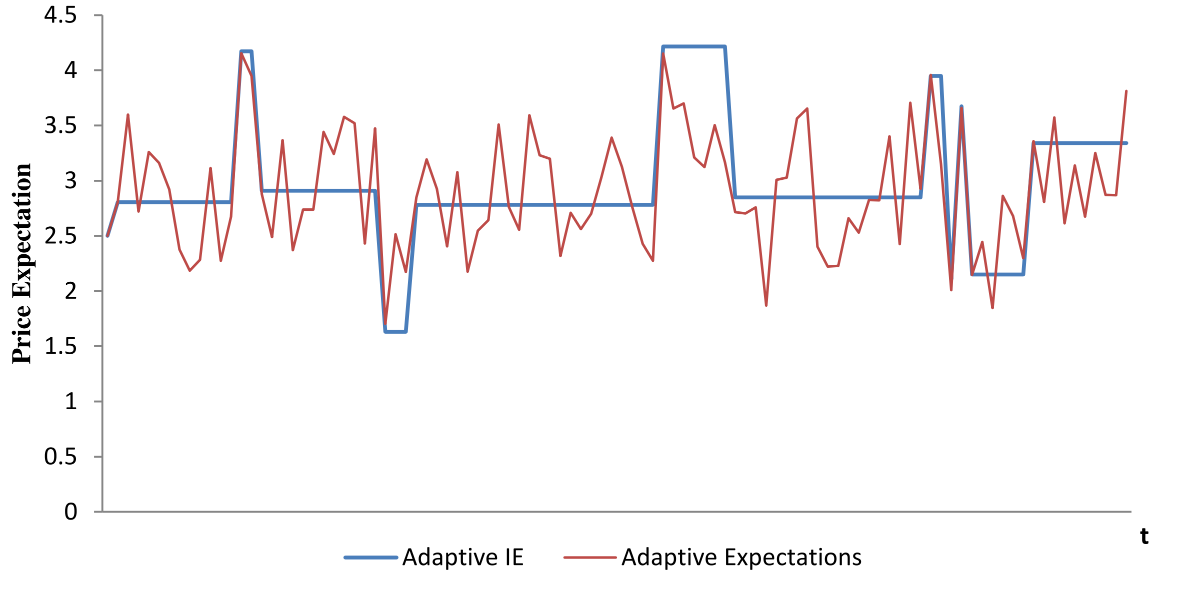


Figure 3: Equilibrium Market Prices Under Adaptive IE and Adaptive Expectations

γ = 5, β1 = 0.75, λ = 0.5, σ = 1, α = 0.05

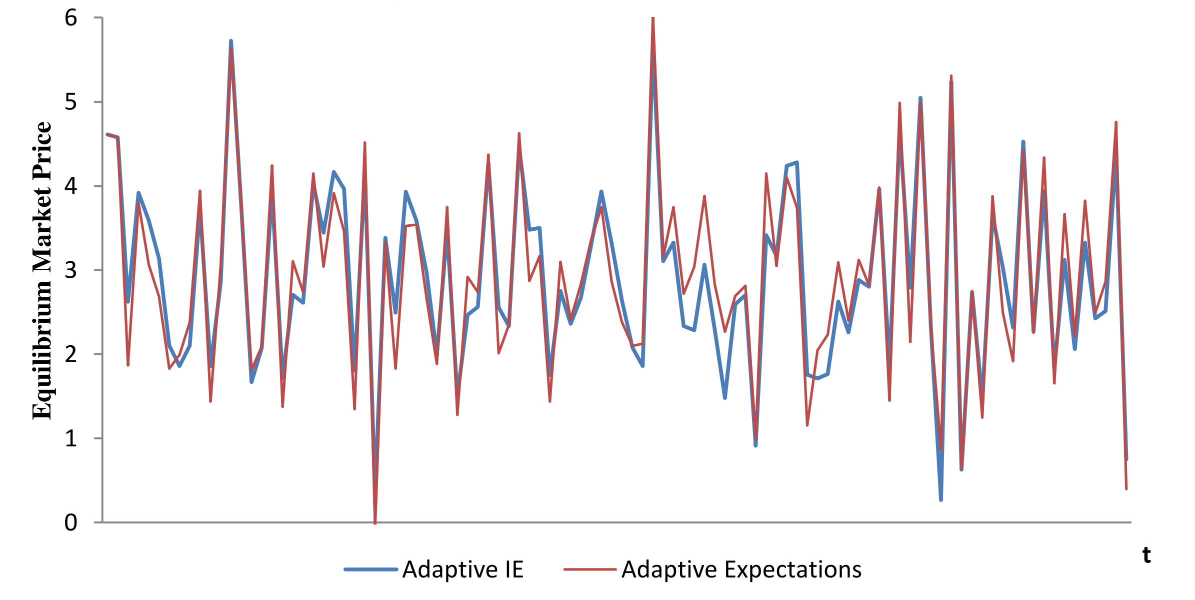


Table I

Effect of Price Volatility and Belief Conservatism on Frequency of Updating Under Adaptive IE

Table I presents the results of a simulated cobweb model under which expectations are formed under adaptive IE. A range of values for the variance of the error term σ are provided along with the frequency of updating for adaptive IE for different values of α. Total of 100 simulated consecutive observations. A period in which the null hypothesis is rejected is counted as one update. Cobweb model specified as in Section 5.3 above. Adaptive IE operates as detailed in Section 5.2. Adaptive expectations operate as detailed in Section 4.3. Parameter values used in the simulation are as follows: $\gamma = 5$, $\beta_1 = 0.75$, $\lambda = 0.5$.

σ	$\alpha = 0$	$\alpha = 0.01$	$\alpha = 0.1$	$\alpha = 0.5$	$\alpha = 1$
0.01	0	9	15	62	99
0.1	0	2	15	59	99
0.5	0	0	13	60	99
1	0	0	11	61	99
1.5	0	0	11	61	99
10	0	2	11	61	99
100	0	2	11	61	99

We can interpret the results in an attempt to form a narrative of the interplay between α, σ and the rate at which forecasts are revised. As can be expected, for a given level of volatility, forecasts are updated with greater frequency as the level of belief conservatism declines. Naïvely, one may also expect a less volatile market to require fewer revisions of agents' expectations; however, under adaptive IE, Table I shows that lower volatility results in agents updating more frequently (or no less) across the range of α's. This can be explained by the endogeneity of the test statistic. Mathematically, as seen in equation (5.1), the denominator of the test statistic is dependent on price volatility; lower volatility results in larger test statistics, and consequently more frequent rejection of the null hypothesis. Intuitively, this can be interpreted as agents in less volatile markets having lower tolerance of forecast errors, and hence being more sensitive (that is, likely to update) as a result of deviations. This effect is dampened as volatility increases – in a market with wild fluctuations, large forecast errors are 'expected', and this is accounted for in establishing when revisions are necessary. This can be attenuated (or exaggerated) by the size of α, which influences the critical value which the test statistic must exceed for H_0 to be rejected. Hence, the relative sizes of *both* α and σ determine the frequency of belief updating under adaptive IE.[14]

It is also important to consider the stability conditions for adaptive IE, and to determine the range of parameter values for which the model is stable. The stability of the classic cobweb model has been discussed extensively in literature. Nerlove (1958) demonstrates that, for a cobweb model as formulated in Section 4.1, where expectations are formed adaptively as given in equation (4.2), that is

$$p_t^e = p_{t-1}^e + \lambda(p_{t-1} - p_{t-1}^e),$$

a return to equilibrium will be achieved as t increases if, and only if,

$$|\,(-\beta_1 - 1)\lambda + 1\,| < 1 \tag{5.3}$$

[14] Chebychev's inequality (McClarke and Dietrich, 1989) demonstrates that there is an inverse relationship between the two, given as $p\left(\left|\frac{x-\mu}{\sigma}\right| > \frac{\sigma}{\sqrt{\alpha}}\right) \leq \alpha$. It follows that for any given increase in σ, there is a proportional decrease in α that gives the same result.

where β_1 is the coefficient of p_t^e from the reduced form equation (4.1), and $\beta_1 = \frac{s_p}{d_p}$. As before, λ represents the proportional adjustment from the previous period. It can be seen that when $\lambda = 1$, equation (5.3) is reduced to

$$|-\beta_1| < 1$$

which represents the stability condition for a cobweb model under such static expectations. Thus, (5.3) is a generally sufficient and necessary condition under which the adaptive expectations cobweb model is stable.

Given that the time path of prices under adaptive IE is dependent on the outcome of hypothesis tests, it is problematic to solve for an equivalent, closed-form solution for a return to equilibrium. Instead, we will analyse stability using the adaptive IE cobweb model simulated earlier, and examine the time path of prices for values of β_1 and λ that are consistent with the stability condition as given equation (5.3), and similarly for values of the same parameters that are not.

Figure 4 (below) illustrates the time path of the equilibrium market price, under which the adaptive expectations stability condition, as given in equation (5.3), is satisfied. Figure 5 that follows, illustrates the time path of the equilibrium market price under which the adaptive expectations stability condition is no longer satisfied. The parameter values used in each case are also provided. It is observed from Figure 4 that, when equation (5.3) is satisfied, both models achieve a return to equilibrium as t increases. This is not the case in

Figure 4: Adaptive Expectations and Adaptive IE Market Price

Time path of p_t when the stability condition (5.3) is satisfied.

$\gamma = 10$, $\beta 1 = 0.9$, $\lambda = 1$, $\sigma = 1$, $\alpha = 0.05$

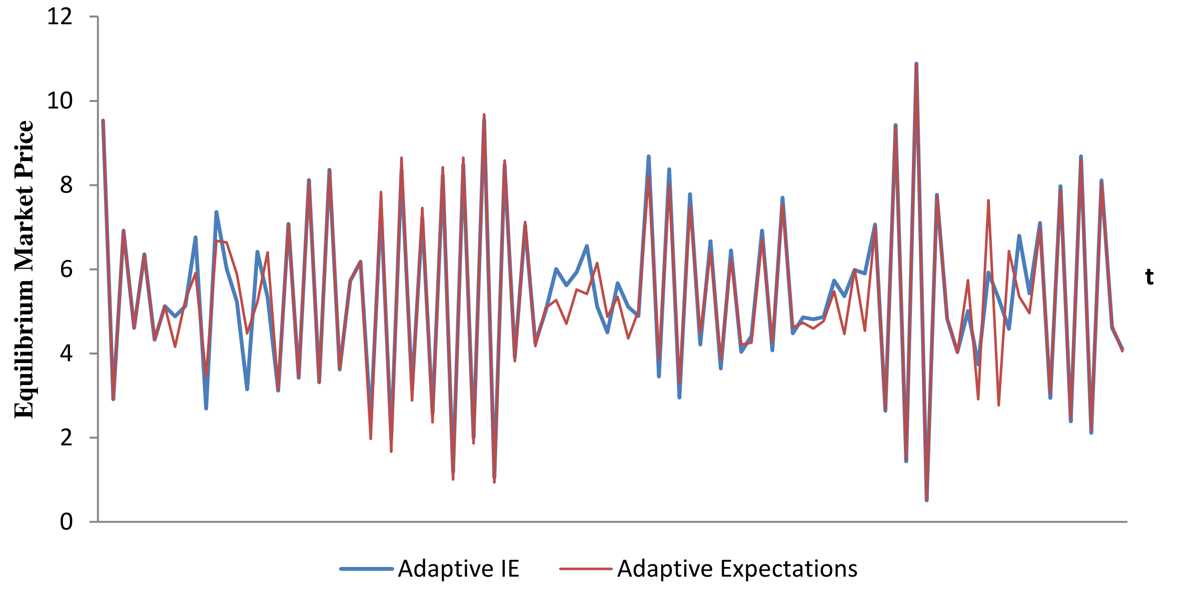


Figure 5: Adaptive Expectations and Adaptive IE Market Price

Time path of p_t when the stability condition (5.3) is not satisfied.

$\gamma = 10$, $\beta 1 = 1.1$, $\lambda = 1$, $\sigma = 1$, $\alpha = 0.05$

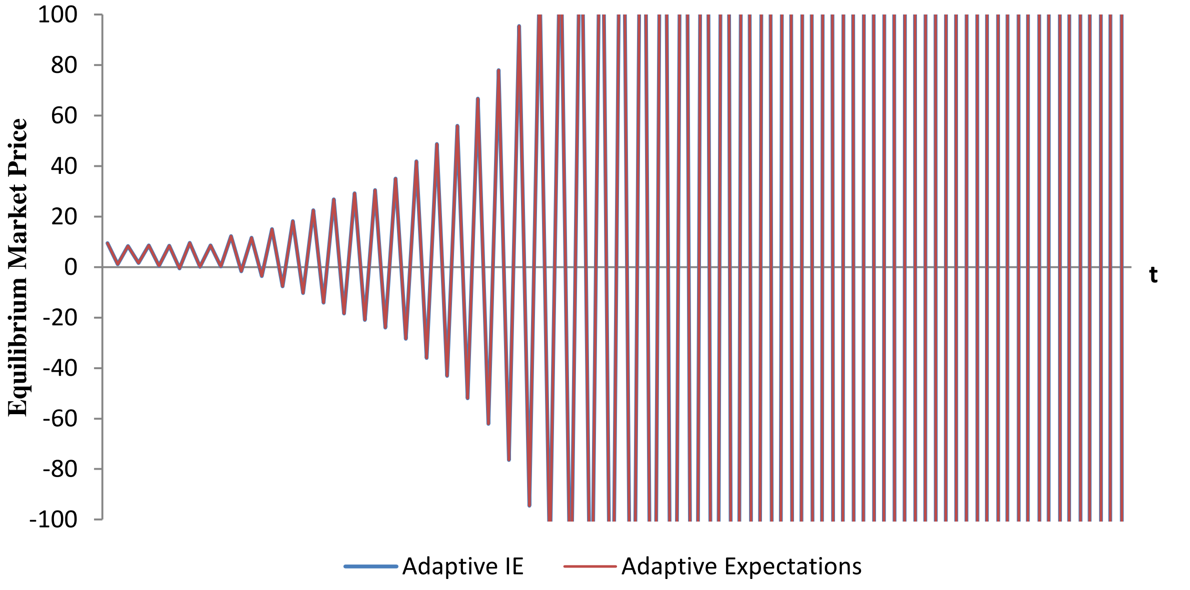
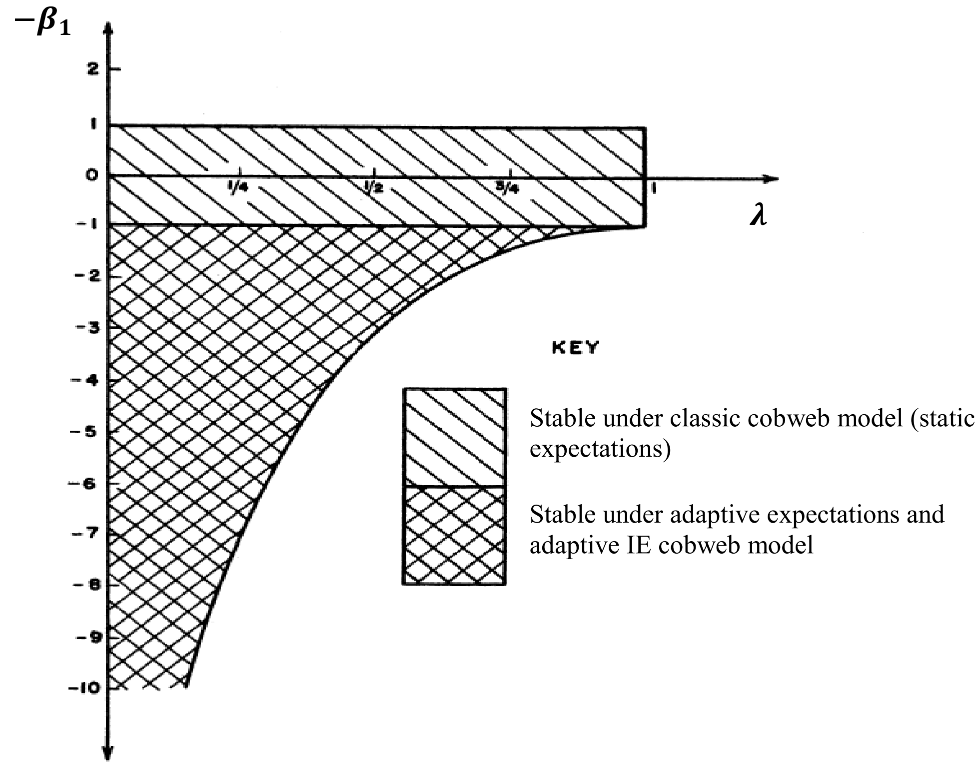


Figure 5; both adaptive expectations and adaptive IE are explosively oscillating when (5.3) is violated[15].

Thus, (5.3) appears to be a generally sufficient and necessary condition under which the adaptive IE cobweb model is stable, and this stability condition is identical to that under adaptive expectations. The range of β_1 and λ compatible with stability are shown in Figure 6 below. It can be noted that the permissible range of β_1 widens as λ falls; almost any positive elasticity of supply and negative elasticity of demand is compatible with stability when the coefficient of proportional adjustment is sufficiently low (Nerlove, 1958).

Figure 6 – Range of β_1 and λ Consistent with Stability under Static Expectations, Adaptive Expectations and Adaptive IE



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[15] For robustness and certainty, this was modelled across a range of simulated values of β_1, λ, σ and α. The results consistently demonstrated that stability was achieved strictly when condition (5.3) was satisfied. Due to the large volume of output, these results have been not been included in this thesis or subsequent appendices.

5.4 Parameter IE

We can also consider a specific application of IE to the cobweb model, whereby suppliers can use IE to learn the value of a new parameter after a structural change to the underlying model. As before, a cobweb model omitting the exogenous variable w_t can be employed, without a loss of generality. The respective log-demand and supply equations are given as

$$d_t = \bar{d} - d_p p_t + v_{1t}$$

$$s_t = \bar{s} + s_p P_t^e + v_{2t}$$

where the price variable and parameters are consistent with those in the previous formulation in Section 4.1. We will assume that suppliers have observed this market for a sufficient number of periods to know the true value of the parameters in the model, and have also been able to compute the variance of the error term in the demand and supply equations: $\sigma_{v_1}^2$ and $\sigma_{v_2}^2$ respectively, both $\sim N(0, \sigma_v^2)$. Thus, firms initially forecast prices such that

$$p_n^e = E_{n-1} p_n$$

where n can be any given time period, prior to the structural change to the model. This formulation of price expectations is consistent with that under RE, discussed in Section 4.4.

To demonstrate how IE can be used to make inference the value of a new parameter in this model, we allow the system to experience an unexpected, permanent and exogenous change to demand at time t. This could be due to new government regulation or a change in consumer preferences, for example. The cognitive target then becomes the unknown shift parameter, Ψ, which affects only the intercept of the demand equation. The new system, following the shock, is therefore

$$d_t = \bar{d} + \Psi - d_p p_t + v_{1t}$$

$$s_t = \bar{s} + s_p p_t^e + v_{2t}$$

in which the parameters $\bar{d}$, d_p, $\bar{s}$, s_p and s_w are known to suppliers. Firms adopt parameter IE to infer the value of Ψ. Setting $d_t = s_t$ as before yields the reduced form solution

$$p_t = \frac{-\bar{s} + \bar{d} - s_p P_t^e + \Psi}{d_p} + \frac{\eta_t}{d_p} \tag{5.4}$$

where $\eta_t = v_{1t} - v_{2t}$.

Since the shift in demand was unexpected, it is assumed that the price estimate for time t (the period in which the shock occurred) is calculated 'rationally' by suppliers as before; that is, without the knowledge that the additional shift parameter exists. This is the equivalent of firms estimating the expectation of p_t at time $t-1$, conditional on the shift parameter being zero, that is

$$p_t^e = E_{t-1} p_t \mid \Psi_t = 0. \tag{5.5}$$

Substituting (5.5) into (5.4) gives

$$p_t = \frac{-\bar{s} + \bar{d} - s_p E_{t-1} p_t}{d_p} + \frac{\eta_t}{d_p}.$$

Taking expectations of both sides and rearranging, given $E_{t-1}\left(\frac{\eta_t}{d_p}\right) = 0$, yields

$$E_{t-1} p_t + \frac{s_p E_{t-1} p_t}{d_p} = \frac{-\bar{s} + \bar{d}}{d_p}$$

$$E_{t-1} p_t \left(1 + \frac{s_p}{d_p}\right) = \frac{-\bar{s} + \bar{d}}{d_p}$$

which can be simplified to give the price estimate at time t as

$$p_t^e \equiv E_{t-1}p_t = \left(\frac{-\bar{s}+\bar{d}}{d_p}\right)\left(\frac{1+s_p}{d_p}\right)^{-1}. \tag{5.6}$$

The equilibrium market price at time t is therefore given by substituting the price estimate for time t, given by equation (5.6), into the reduced form equation (5.4). This gives the price at time t as

$$p_t = \left(-\bar{s}+\bar{d}-s_p\left(\frac{-\bar{s}+\bar{d}}{d_p}\right)\left(\frac{1+s_p}{d_p}\right)^{-1}+\Psi\right)\left(d_p\right)^{-1}+\frac{\eta_t}{d_p}. \tag{5.7}$$

In the *following* period, $t+1$, prior to the equilibrium of supply and demand, firms are able to observe the resultant market price that occurred in the previous period, p_t. Firms observing fluctuations in the market price (influenced by Ψ) infer the possible presence of the new shift parameter. Prior to determining their supply, they attempt to estimate its value. The initial estimate is formed by rearranging (5.7) gives

$$\Psi+\eta_t = d_p p_t+\bar{s}-\bar{d}+s_p\left(\frac{-\bar{s}+\bar{d}}{d_p}\right)\left(\frac{1+s_p}{d_p}\right)^{-1}$$

and given that $E(\eta_t)=0$, we assume IE firms naively solve for Ψ_t setting $\eta_t=0$, that is

$$\widehat{\Psi}_t = d_p p_t+\bar{s}-\bar{d}+s_p\left(\frac{-\bar{s}+\bar{d}}{d_p}\right)\left(\frac{1+s_p}{d_p}\right)^{-1}, \tag{5.8}$$

which serves as their alternate belief of the true value of Ψ_t, if the null hypothesis described presently is rejected.

Under parameter IE, the formation of beliefs are treated as statistical hypothesis tests, and firms test their belief of the true value of Ψ under the hypotheses

H_0: $\Psi_t = 0$

H_1: H_0 is false, and Ψ_t is given by equation (5.8)

Under the null hypothesis, firms hold the belief that the departure of the market price in (5.7) from their expected price as given in (5.6) is attributable entirely to noise, or in other words, that the true value of Ψ_t is zero. As the variance of p_t from equation (5.7) is given by

$$\frac{\sigma_\eta^2}{d_p^2} = \frac{\sigma_{v_1}^2 + \sigma_{v_2}^2}{d_p^2},$$

it follows that the test statistic – under the null hypothesis – is calculated as

$$\frac{p_t - p_t^e}{\sqrt{\frac{\sigma_{v_1}^2 + \sigma_{v_2}^2}{d_p^2}}} \sim Z$$

where the critical value for is dependent on α, the significance level attributed to the test, and a measure of firms' belief conservatism. As is standard in statistical hypothesis tests, the null hypothesis is rejected if the absolute value of the test statistic is greater than the critical value, and is failed to be rejected if otherwise.

The price expectation at time $t + 1$ is then formed, conditional on the outcome of the above hypothesis test. If firms fail to reject the null hypothesis, they assume any deviations of the market price p_t from the price expectation p_t^e were random, unsystematic shocks, and not attributable to any new shift parameter. Therefore, equation (5.4) is estimated in the following period, $t + 1$, conditional upon the null hypothesis belief

$$p_{t+1}^e = E_t p_{t+1} \mid \Psi_t = 0,$$

that is to say, the price expectation in the following period is calculated as before.

However, if firms reject the null hypothesis, the price expectation at time $t + 1$ is calculated using equation (5.4), conditional upon

$$p_{t+1}^e = E_t p_{t+1} \mid \Psi_t = d_p p_t + \bar{s} - \bar{d} + s_p \left(\frac{-\bar{s} + \bar{d}}{d_p} \right) \left(\frac{1 + s_p}{d_p} \right)^{-1}.$$

Hence, the estimated value of Ψ is now incorporated into the price forecast, as firms have witnessed a significant enough deviation in price (from their expectation) to infer an exogenous shock parameter exists.

Following the outcome of the hypothesis test, the market price is formed using the resultant price estimate, with similar calculations of Ψ_{t+1}, Ψ_{t+2},… and accompanying hypothesis tests performed in each subsequent period. Note that under rejection of the null hypothesis belief in period $t + 1$, the hypothesis test performed in period $t + 2$ becomes

$$H_0: \Psi = \widehat{\Psi}_t$$

$$H_1: \Psi = \widehat{\Psi}_{t+1}.$$

where $\widehat{\Psi}_t$ is the value of the shift parameter estimated previously, given in equation (5.8), whereas $\widehat{\Psi}_{t+1}$ is the *updated* value of the shift parameter, solved using the conditional expectation of Ψ as given in equation (5.8). p_{t+2}^e is then formed using (5.4), conditional on the value of Ψ determined by the outcome of the hypothesis test.

Put generally, the hypothesis tests performed in subsequent periods similarly test if the true value of the shift parameter is equal to its most recent estimate, versus the estimate since the period when the null hypothesis was last rejected. That is

$$H_0: \Psi = \widehat{\Psi}_{n-L}$$

$$H_1: \Psi = \widehat{\Psi}_{n-1}$$

where n is the current period, and L represents the number of periods since the null hypothesis was last rejected. Used recursively in this manner, parameter IE can be utilised to gradually learn the value of the shift parameter, and of any other new parameters assigned as cognitive targets in the cobweb model. It can also be noted that parameter IE can also serve as a temporary model of expectation formation until sufficient information is available to perform signal extraction.

5.5 Parameter IE Simulation

We will now illustrate such a model of parameter IE, in which the cognitive target is a new shift parameter in the cobweb model. The model is specified as outlined in Section 5.4, with prices following the reduced form solution given in equation (5.4)

$$p_t = \frac{-\bar{s} + \bar{d} - s_p P_t^e + \Psi}{d_p} + \frac{\eta_t}{d_p}$$

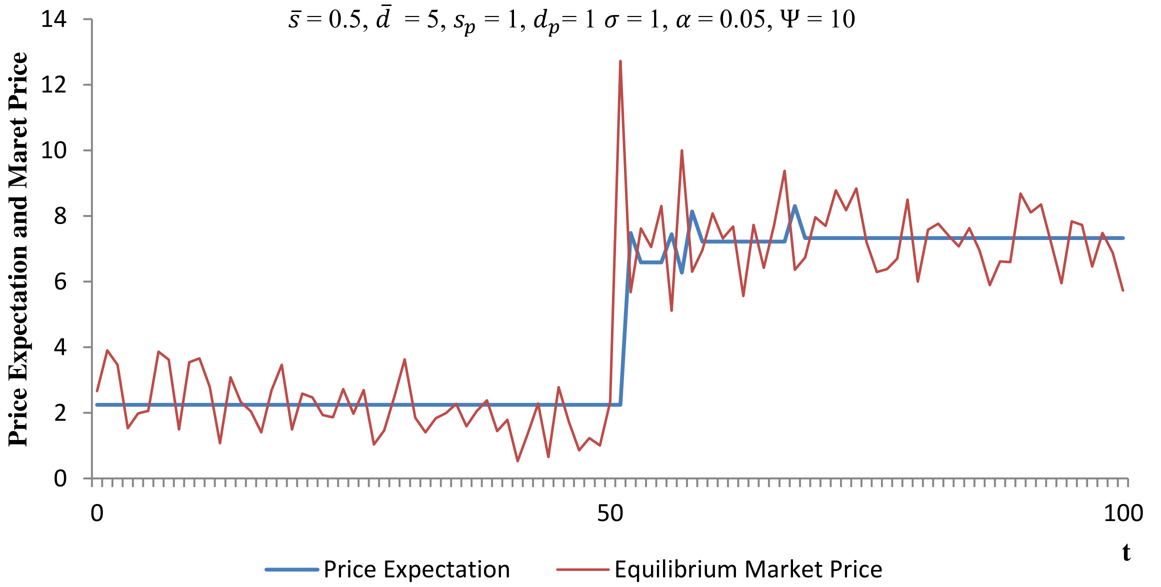
where $\bar{s} = 0.5$, $\bar{d} = 5$, and demand and supply are both unitary elastic, that is $s_p = d_p = 1$. It is also assumed that $\eta_t = \sigma z$, where $z \sim N(0, \sigma_{v_1}^2 + \sigma_{v_2}^2)$ and $\sigma = 1$. $\sigma_{v_1}^2$ represents the variance of the error term in the demand equation, $v_1 \sim N(0, 0.5)$. $\sigma_{v_2}^2$ represents the variance of the error term in the supply equation, $v_2 \sim N(0, 0.25)$. Therefore, $z \sim N(0, 0.75)$. The model consists of 100 observations of consecutive time periods t.

Initially, at $t = 0$, to period $t = 50$, the value of Ψ is assumed to be zero. The system undergoes an unexpected shock at time $t = 51$, whereby there is an exogenous increase in demand such that $\Psi = 10$. The shock is permanent, and lasts until $t = 100$. Initially, firms are assumed to know the value of all the parameters in the model, and hence forecast prices rationally. Following the exogenous shock to demand, firms employ parameter IE to infer the value of the shift parameter Ψ.

The time path of suppliers' price expectation p_t^e and the equilibrium market price p_t under the model specified is shown in Figure 7 (below). It can be observed that, since the price forecast is formulated rationally until the period in which the shock to demand takes place, there is no change in suppliers' beliefs over this time. There is a significant shortfall in supply at period $t = 51$ due to the unexpected increase in demand, which results in a sharp increase in the market price at that point in time.

Figure 7: Time Path of Prices under IE

$\bar{s} = 0.5$, $\bar{d} = 5$, $s_p = 1$, $d_p = 1$ $\sigma = 1$, $\alpha = 0.05$, $\Psi = 10$

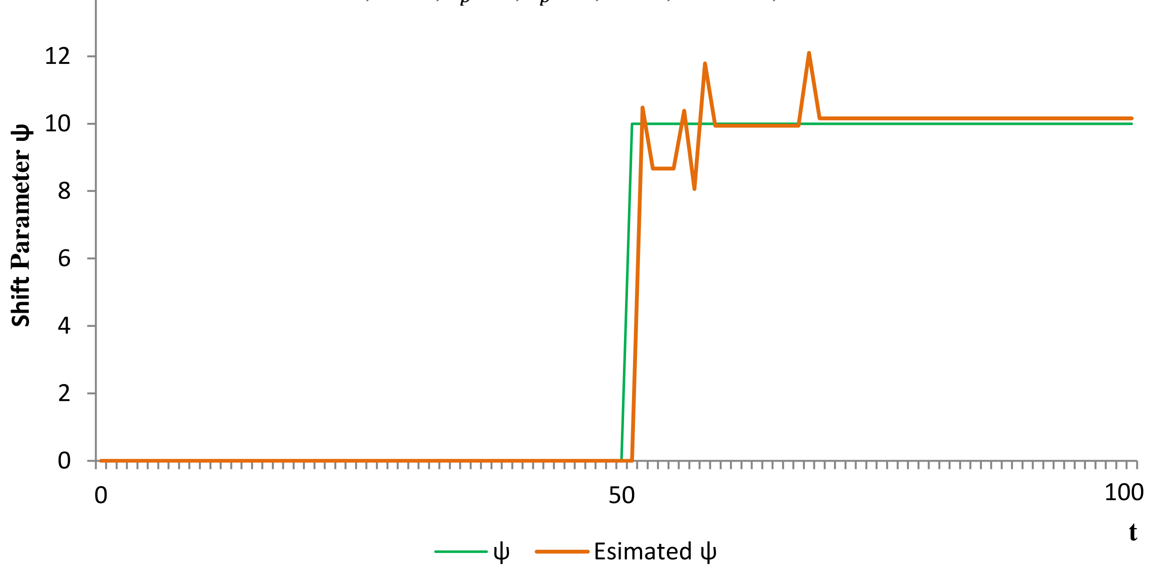


Given suppliers are assumed to know the variance of the market price; this volatile price change alerts suppliers to an unknown shift in demand. The value of the cognitive target Ψ is therefore estimated using parameter IE in subsequent periods.

Figure 8 (below) presents the estimate of the cognitive target Ψ over time. The effects of differing parameter values on the model and cognitive target are modelled in the Appendix, and discussed presently. The degree to which firms are able to successfully estimate the value of the shift parameter is primarily dependent on two parameters in the model, σ and α. As σ increases, the market price increases in volatility, and it becomes increasingly difficult for firms to distinguish large fluctuations in price due to the presence of an unknown shift parameter, from noise. Hence, a number of periods may pass before firms notice a structural change in the model (Figure A23 in Appendix). As $\sigma \to \infty$, the price volatility is so large, that firms may always fail to reject the null hypothesis, and the presence of the shift parameter could possibly go by unnoticed (for reasonable values of Ψ). Conversely, as σ decreases, firms are able to more easily distinguish structural changes to the model (due to the low levels of noise), allowing for 'faster' and more accurate estimates of Ψ.

Figure 8: True and Estimated Values of ψ

$\bar{s} = 0.5$, $\bar{d} = 5$, $s_p = 1$, $s_p = 1$, $\sigma = 1$, $\alpha = 0.05$, $\Psi = 10$



As in prior formulations of IE, α can be interpreted as a metric for firms' belief conservatism. As $\alpha \to 0$, firms become reluctant to update their prior beliefs, and require significantly larger deviations in price from expectations in order to do so. Hence, for smaller values of α, there may exist a time lag between the shock to supply, and the time in which firms are sufficiently convinced to incorporate Ψ into their forecast (this is similar to a large σ). It follows that firms also update their estimate of the cognitive target less frequently when this is the case. The opposite holds true as α approaches unity – firms are more sensitive to fluctuations in price, and are alerted to the presence of the shift parameter sooner; they also update their estimate of the shift parameter more frequently in subsequent periods.

The relative size of σ and α therefore affect the time path of the price estimate and market prices in tandem. Suppliers are poorer at forecasting Ψ as volatility increases and belief conservatism increases, or as σ increases and α decreases. Their forecasting ability is greatly improved as volatility and belief conservatism decrease, or as σ decreases and α increases. However, problem arises in extreme cases of either, under which firms naïvely adjust the cognitive target due to variations from noise. This results unstable and erroneous estimates of Ψ, illustrated in Figures A19, A20, A24 and A25 in the Appendix.

5.6 Generalised Econometric Learning

In this section, it will be shown that a cobweb model under which the parameters are estimated using EL is nested in the IE framework, and IE is therefore a natural extension to EL. We label the application of IE to EL as Generalised Econometric Learning (GEL). The cognitive targets – the EL parameters – are assumed to be estimated using least squares; however, under GEL, they are only updated each period if new information suggests that the newly estimated parameters are significantly different from those estimated in the previous period. How 'different' the parameters need to be depends on the degree of believe conservatism on the part of the forecaster(s).

Beginning at time $t-1$, the initial price estimate of p_t is assumed to be estimated similarly to the first feasible EL forecast. Firms believe the prices follow the process represented in equation (4.6) (REE) but, as under EL, are unsure of the values of a and b. With the information set in the current period given as $\{p_i, w_i\}_{i=0}^{t-1}$, firms estimate $\hat{a}$ and $\hat{b}$ by a least squares regression of p_i on w_i and an intercept. The forecast at $t-1$ of the price at time t is therefore

$$\hat{p}_t = \hat{a}_{GEL,t-1} + \hat{b}_{GEL,t-1} w_{t-1} \quad (5.9)$$

where $\hat{p}_t$ is the expected price at time t, $\hat{a}_{GEL,t-1}$ is the estimated intercept, and $\hat{b}_{GEL,t-1}$ is the estimated slope coefficient, and w_{t-1} represents the value of the observable exogenous variable. Both GEL parameters are estimated using the same method and data set available at time $t-1$ as EL, and as such $\hat{a}_{GEL,t-1} = \hat{a}_{EL,t-1}$ and $\hat{b}_{GEL,t-1} = \hat{b}_{EL,t-1}$. Hence, the forecast equation and price estimator $\hat{p}_t$ is equal to that under EL. Put generally; the initial parameter values and price estimate are always identical under GEL and EL, assuming the expectations are formed in the same time period.

The forecasts are not necessarily identical in the following periods, however, as GEL allows for belief conservatism on the part of firms, which is not considered under EL. In the context of the cobweb model, the coefficients $\hat{a}_{GEL,t-1}$ and $\hat{b}_{GEL,t-1}$ represent the null-hypothesis coefficient beliefs. These estimates are overturned only if firms obtain *enough* evidence to reject that the coefficients estimated in subsequent periods are equal to $\hat{a}_{IE,t-1}$ and $\hat{\beta}_{IE,t-1}$, and if so, will update their forecast. Unless this occurs, firms notice new information (p_t, w_t) without changing their estimated coefficients.

The price estimate given by (5.9) is then used by firms to determine supply in the following period, which results in a market price as specified by the original reduced form equation (4.1). Subsequently, in period t, (p_t, w_t) is added to the data, and the information set available to firms is now $\{p_i, w_i\}_{i=0}^{t}$. To forecast prices in period $t+1$, firms again run a least squares regression of p_i on w_i and an intercept, which yields

$$\hat{p}_{t+1} = \hat{a}_{GEL,t} + \hat{b}_{GEL,t} w_t. \quad (5.10)$$

However, under GEL, firms will update their parameter forecasts only if the new parameters estimated in (5.10), at time t, are statistically significantly different to those of the previous model (5.9), estimated at time $t-1$. Firms form their price expectations according to the hypotheses

$$H_0: a = \hat{a}_{GEL,t-1}, b = \hat{b}_{GEL,t-1}$$

$$H_1: a = \hat{a}_{GEL,t}, b = \hat{b}_{GEL,t}$$

This test is performed by estimating a restricted model, with the null hypothesis coefficient beliefs, $\hat{a}_{GEL,t-1}$ and $\hat{b}_{GEL,t-1}$, imposed on the current data set $\{p_i, w_i\}_{i=0}^{t}$. The fit of the restricted model is compared with the most recently estimated, unrestricted model, as estimated in (5.10). This is operationalised by performing an F-test, with the F-statistic is calculated as

$$\frac{\left[\Sigma\left(p_i - \hat{a}_{GEL,t-1} - \hat{b}_{GEL,t-1} w_i\right)^2 - \Sigma\left(p_i - \hat{a}_{GEL,t} - \hat{b}_{GEL,t} w_i\right)^2\right] / J}{\Sigma\left(p_i - \hat{a}_{GEL,t} - \hat{b}_{GEL,t} w_i\right)^2 / (T-K)} \sim F_{J,T-K}$$

where J is the number of restrictions (equivalent to the number of cognitive target(s); in this case, two), K is the total number of parameters in the model (in this case $\hat{a}$ and $\hat{b}$), and T represents the number of observations in $\{p_i, w_i\}_{i=0}^{t}$ (t). Note that if the cognitive target is all of the parameters in the forecast equation, then $J = K$. The critical value for the hypothesis test, $F_{critical}$, will depend on α, the significance level attributed to the test, and a measure of firms' belief conservatism. The critical value is obtained from the standard $F_{J,T-K}$ distribution.

If $F_{J,T-K} > F_{critical,}$, the null hypothesis is rejected, and the updated parameters are used in the forecast equation of $\hat{p}_{t+1}$, as per equation (5.10). If firms fail to reject the null hypothesis, the parameters in the original model estimated at time $t-1$ are used, such that

$$\hat{p}_{t+1} = \hat{a}_{GEL,t-1} + \hat{b}_{GEL,t-1}w_t. \qquad (5.11)$$

Thus, it can be observed that GEL takes the estimated standard errors for the regression (and their correlations) into consideration when making forecasts, whereas these are ignored under EL. If the residual sum of squares of the restricted model is too large (as determined by the F-statistic), the updated parameters are considered a poor fit for the data set, and the null hypothesis is rejected. Conversely, the residual sum of squares of the restricted model is not large enough, belief-conservative firms consider the new parameters as being 'no better,' and the previous values are taken instead.

The price estimate at time $t+1$ is subsequently calculated, using either (5.9) if the null hypothesis is accepted, or (5.10) if the null hypothesis is rejected. The process is continued through time, with the information set and estimated parameter values in following periods.

In the general case, firms forecast according to the hypotheses

$$H_0: a = \hat{a}_{GEL,n-L}, b = \hat{b}_{GEL,n-L}$$

$$H_1: a = \hat{a}_{GEL,n}, b = \hat{b}_{GEL,n}$$

where n represents the current time period (with $n-1$ data points available for regression), and L represents the period since H_0 was last rejected.

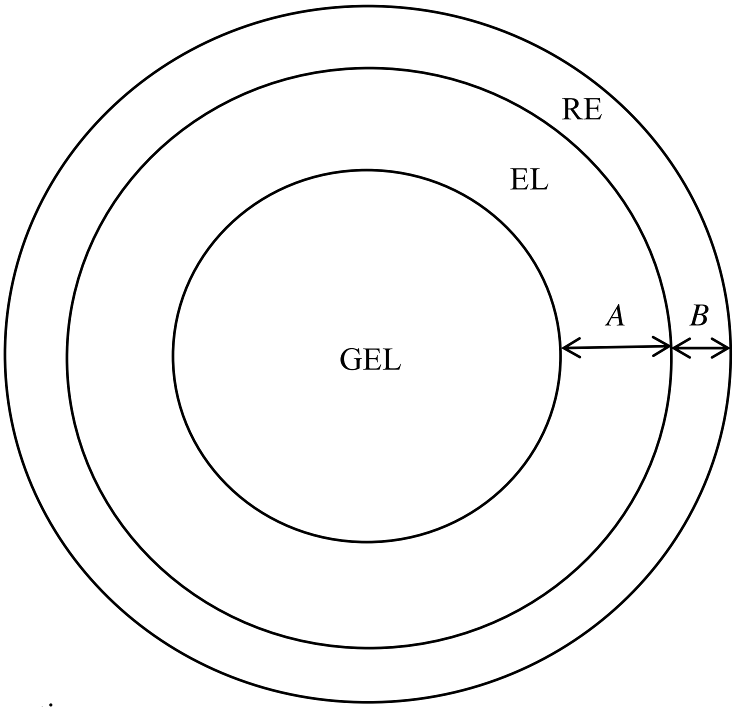
It should be noted that when the null hypothesis is rejected, firms switch to a model of expectation formation that is identically estimated as EL, although the value of the cognitive target(s) and forecast price may not be equal due to differences in the value of *past* prices. Thus, if the test size is unity, GEL is indistinguishable from EL; within the GEL paradigm, EL represents a form of feasible RE. GEL therefore encompasses EL, and provides a more general framework for expectation formation that takes into consideration the belief conservatism that has been shown to exist amongst individual agents. This is otherwise ignored in the standard formulation of EL. It should also be noted that, as past data points are used in calculating the F-statistic, belief formation (and as a result, the equilibrium time path of the market) is endogenously determined.

Figure 9 (below) summarises agents' possible use of information in forming expectations in the context of the cobweb model. It is highlighted that RE is the best possible use of information and incorporates all the relevant data in forming expectations by definition. However, given the informational and computational demands of RE, EL provides the next best alternative in estimating the REE, and is hence considered feasible RE. GEL forecasts represent a step down in information use from EL, as information can be ignored, conditional on the extent of agents' belief conservatism[16].

It can also be noted that the GEL model developed in this thesis differs from, and is perhaps more reasonable than, the Dornbusch (1976)-style application of IE in Menzies and Zizzo (2009), where upon rejecting the null hypothesis, agents were assumed to form their expectations according to the underlying model of RE. EL as a form of feasible RE is

[16] Under endogenous IE models, it is recognised that up-to-date information is used in calculating the test statistic and determining the outcome of the hypothesis test, and only subsequent to this is new information discarded (under the null hypothesis). Therefore, it can be argued that IE models, including GEL, have equal, if not more, informational and computational demands than feasible RE. From this perspective, IE represents a framework for belief *conservation*, rather than economising on the calculation, and may be permissible if updating beliefs is costly. Regardless, IE forecasts remain bias and conservative.

Figure 9 - Use of Information in the Cobweb Model under GEL, EL and RE



A: Belief conservatism
B: Informational and computational demands

arguably a more realistic method of expectation formation under the assumption of no belief conservatism, given the criticisms that have been previously weighed against RE.

5.7 Generalised Econometric Learning Simulation

In order to perform further analysis, we will now simulate a cobweb model under which expectations are formed using RE, EL and GEL. The model simulated is identical to that formulated in Section 4.1, and the reduced form is therefore given as in equation (4.1); that is

$$p_t = \gamma - \beta_1 p_t^e - \beta_2 w_{t-1} = n_t$$

where the parameters are defined as before. The price estimate p_t^e and the equilibrium market price p_t are observed for 100 consecutive periods of time t. It is assumed that $\eta_t = \sigma z$, where $z{\sim}N(0,1)$. We will also assume w_t initially is initially equal to zero, and an exogenous shock to the system occurs such that $w_t = 1$ beginning in period $t = 51$ and takes this value in each subsequent period.

Figure 10 to Figure 13 (below) illustrate the time paths of price expectations p_t^e and the resultant equilibrium market price p_t from this simulation, under RE, EL and GEL. The parameter values specified in the simulated model are also provided, and are consistent across expectations regimes. Differing values of the test size α are used in the GEL models for comparison, and it can be noted that Figure 11 represents a special case of GEL where the test size equals unity; that is GEL is identical to EL. Conversely, Figure 12 illustrates a model of GEL where $\alpha = 0$, and hence agents' are unwilling to update their prior belief.

It has been shown in Section 4.4, equation (4.4) that the price estimate under RE is independent of the variance of the error term, which is determined by σ. It can be seen from Figure 11, Figure 12 and Figure 13 that for the GEL cobweb model with the parameters specified, there does not appear to be significant variation in the time path of the equilibrium market price. This observation suggests that the volatility inherent in p_t is largely driven by the inherent volatility of p_t (a function of the error), rather than expectation regime employed. It may also suggest that the method of expectation formation has little impact on forecast accuracy in this particular model. This raises two broader questions in relation to GEL worth investigating:

1) To what extent does the volatility of the market affect GEL, and

Figure 10: Time Path of Prices Under RE

$\gamma = 5, \beta_1 = 0.5, \beta_2 = 0.9, \sigma = 1$

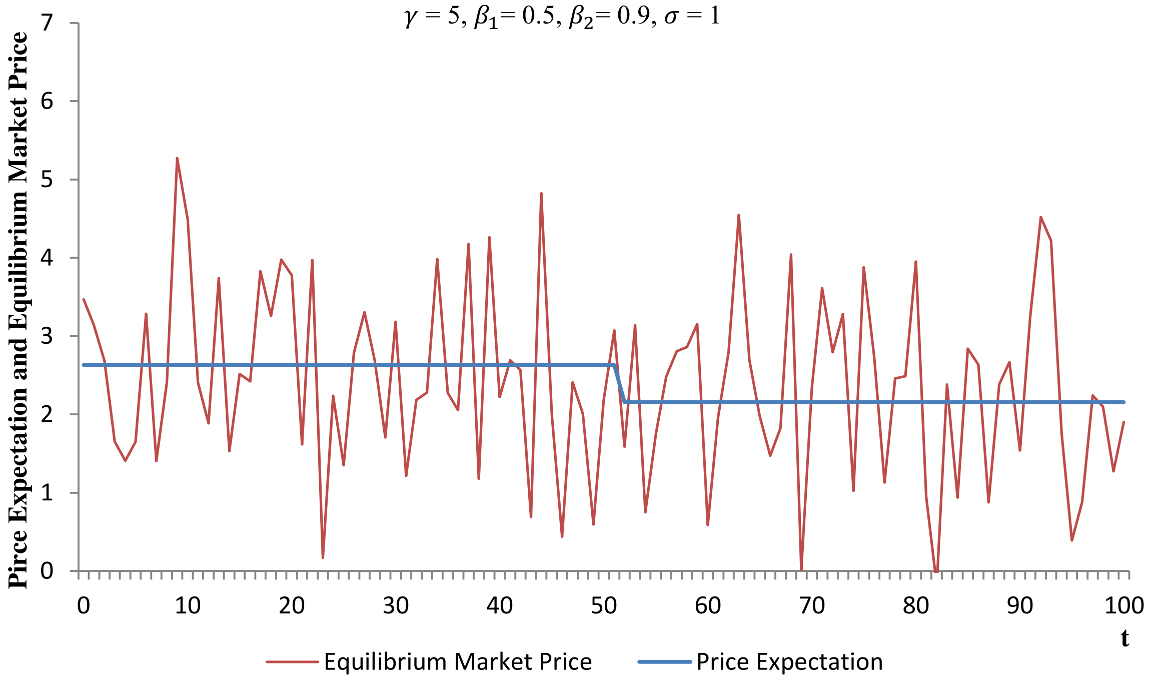
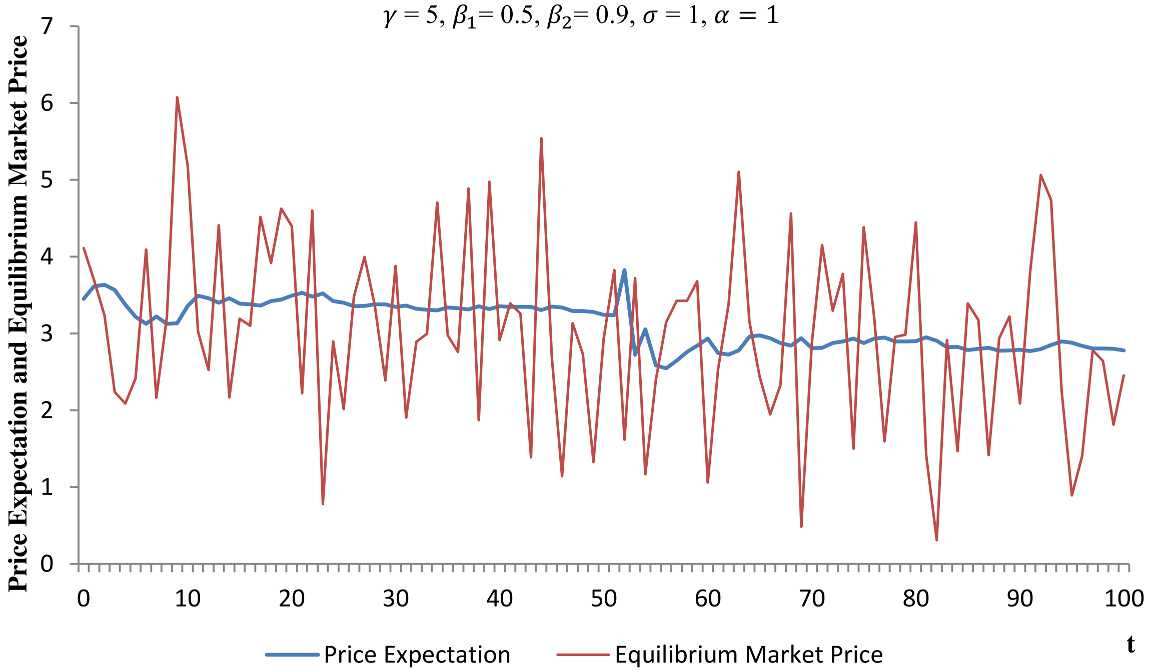


Figure 11: Time Path of Prices Under GEL (EL)

$\gamma = 5, \beta_1 = 0.5, \beta_2 = 0.9, \sigma = 1, \alpha = 1$



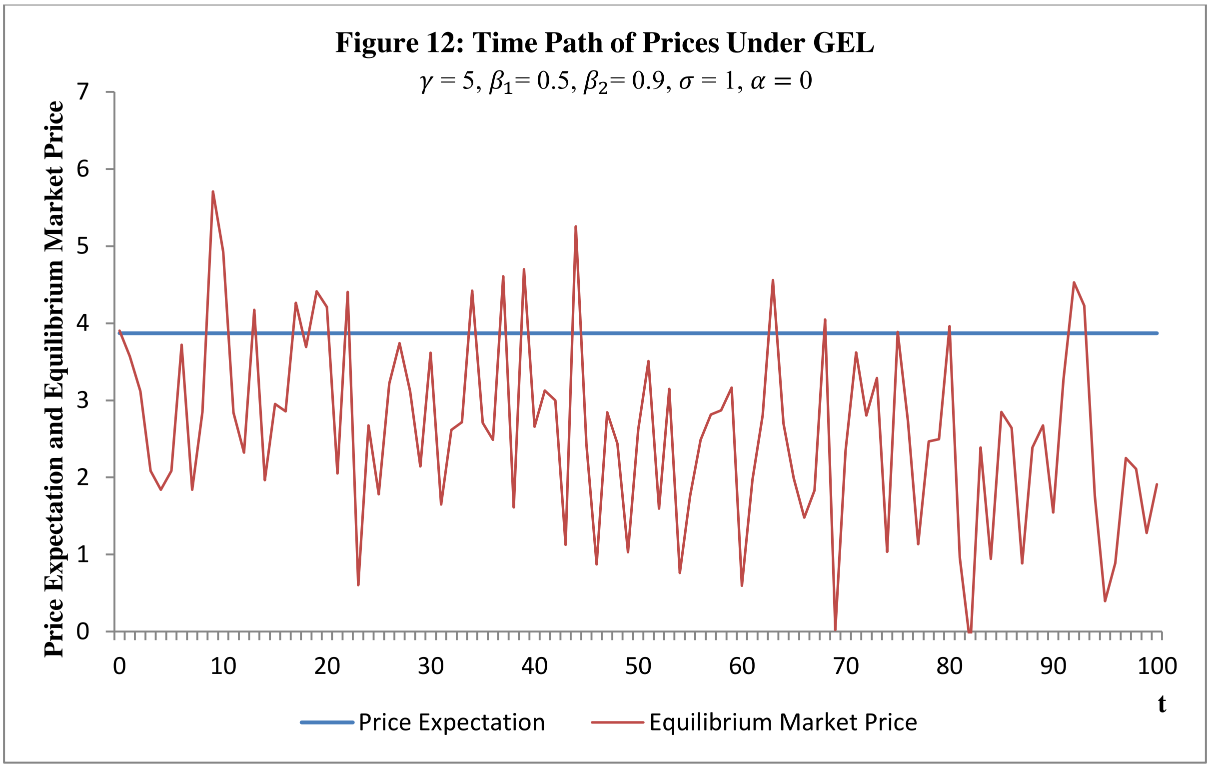


Figure 12: Time Path of Prices Under GEL
$\gamma = 5$, $\beta_1 = 0.5$, $\beta_2 = 0.9$, $\sigma = 1$, $\alpha = 0$
Price Expectation and Equilibrium Market Price
0
1
2
3
4
5
6
7
0
10
20
30
40
50
60
70
80
90
100
t
Price Expectation
Equilibrium Market Price

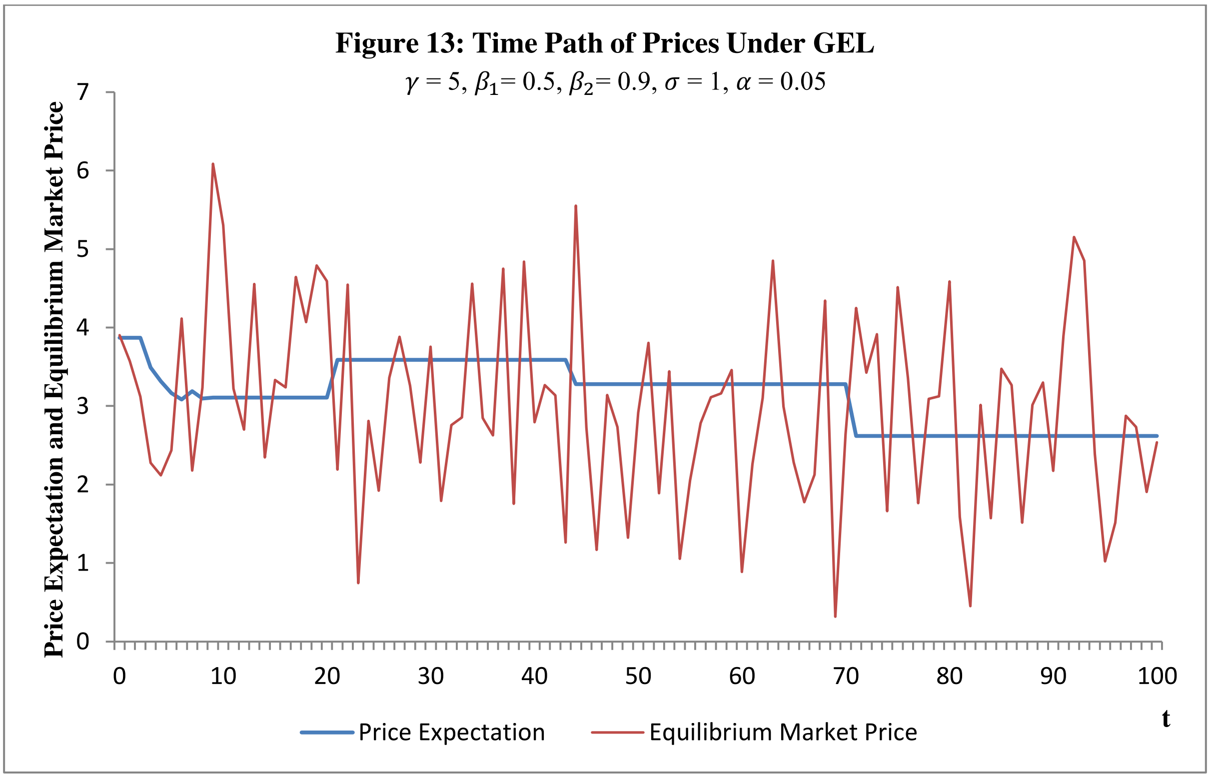


Figure 13: Time Path of Prices Under GEL
$\gamma = 5$, $\beta_1 = 0.5$, $\beta_2 = 0.9$, $\sigma = 1$, $\alpha = 0.05$
Price Expectation and Equilibrium Market Price
0
1
2
3
4
5
6
7
0
10
20
30
40
50
60
70
80
90
100
t
Price Expectation
Equilibrium Market Price

2) Does belief conservatism affect forecast accuracy in models where prices are endogenously determined?

Question 2 arises naturally from the theory due to the fact, under a cobweb model as specified in equation (4.1), equilibrium price outcomes are dependent on the price expectation for that period. It therefore may be the case that expectations are 'self-fulfilling'; that is, regardless if price expectations are formed sub-optimally, the amount by which agents are subsequently mistaken may not differ as greatly as they would otherwise, given that *the resultant market price is itself influenced by the expectation*[17].

The difference between the actual and predicted value of a time series is most commonly measured using the forecast error. Similarly, we will utilise the in-sample forecast error (ISFE) in the three models as a metric for forecast accuracy. The ISFE is calculated in each of the 100 observed periods as

$$FE_t = | \, p_t - p_t^e \, |$$

where FE_t is the ISFE and p_t is the equilibrium market price, each in period t. p_t^e represents price expectation of p_t formulated at time $t-1$. It should also be observed that the absolute value of the difference between p_t and p_t^e is used as we are interested exclusively in the magnitude of the error. Taking the absolute value also allows for a meaningful calculation of the mean ISFE, which is subsequently calculated as an indicator of average forecast accuracy. The mean ISFE is subsequently calculated for each of the three simulated models (RE, EL and GEL). A fourth model, in which the price expectations are generated randomly is also

[17] Specifically, sub-optimal expectations refer to expectations formed on any basis other than the best use of all available information. In the context of the present cobweb model, sub-optimal expectations are those formed using a framework other than RE or feasible RE (EL). GEL can be considered a form of sub-optimal expectation formation, as belief conservatism inhibits agents from using all available information in forming their forecast of future prices.

included, as 'control' in determining whether expectations are self-fulfilling[18]. The mean ISFE for all four models is presented in Table II (below), for a range of values of price volatility σ. It is known that as $\alpha \rightarrow 1$, GEL converges to EL, and hence the forecast errors between these two will similarly converge. The model illustrated is assumed for $\alpha = 0.05$, as this is common significance level in hypothesis testing and a conceivable level of belief conservatism.

Table II

In-Sample Forecast Errors for RE, EL and GEL

Table II presents the results of a simulated cobweb model under which expectations are formed under RE, EL, GEL and a model in which price expectations are randomly generated. A range of values for the variance of the error term σ are provided, and the in-sample forecast is calculated each period. The mean in-sample forecast error is computed over a total of 100 consecutive observations. Cobweb model specified as in Section 5.7 above. RE operates as detailed in Section 4.4. Econometric learning operates as detailed in Section 4.5. GEL operates as detailed in Section 5.6. Parameter values used in the simulation are as follows: $\gamma = 5$, $\beta_1 = 0.5$, $\beta_2 = 0.9$ and $\alpha = 0.05$.

σ	**Mean ISFE RE**	**Mean ISFE EL**	**Mean ISFE GEL**	**Mean ISFE Random**
0.01	0.000093	0.037112	0.089348	3.802218
0.1	0.0093	0.0402	0.0914	3.8022
0.5	0.2331	0.2509	0.3051	3.8014
1	0.9324	0.9698	0.9549	3.7989
1.5	2.10	2.17	2.16	3.94
10	93.24	96.10	94.46	93.06
100	9,323.81	9,609.99	9,455.48	9,323.63

[18] Price expectations in this model were obtained each period using a random number generator.

It is important to note that, due to the endogenous determination of prices within each model, the type of expectation regime employed influences future data points. Hence, the time path of the price expectation and equilibrium market price are not identical between the four models. It is therefore inappropriate to infer that the ISFEs can be directly compared as a metric for forecast accuracy, and hence a lower ISFE does not imply more accurate forecasting; each ISFE is calculated for a *different* set of data points.

However, two overarching comments can be made based on the results in Table II. First is the observation that, as the variance of the error term σ increases, the ISFE increases proportionately. This is consistent with the formulation of the model, where $\eta_t = \sigma z$, where $z \sim N(0,1)$. For the models under RE, EL and GEL, the ISFE $\approx \sigma^2$. This result is intuitively pleasing (and perhaps obvious), given that one would expect greater forecast errors in a model with greater volatility, as prices are harder to predict.

Second, it does not immediately appear that expectations are self-fulfilling based on these results. For models where price expectations play a greater role in determining the equilibrium market price (that is, where σ is small and white noise is therefore minimised), there appears to be diversity amongst ISFEs. For $\sigma = 0.5$, it can be seen that the ISFEs of RE, EL and GEL are similar, however the forecast error for the random model is approximately four times larger. This suggests that the expectation regime does influence equilibrium price outcomes, as a model under which price expectations are randomly drawn is an inferior predictor of the market price. Only when the market is significantly volatile do similarities appear across ISFEs. This is as the influence of expectations dampens as volatility increases, and the majority of variation in market prices – and therefore forecast error – is increasingly sourced from noise; for models where $\sigma = 10$ and $\sigma = 100$, it can be seen that EL and GEL are no better at predicting market prices than the randomly generated model.

It is expected that self-fulfilling expectations manifests itself in the form of invariant ISFEs across different expectations regimes. We can test the second observation more definitively by determining if the estimated ISFEs are significantly different between a model in which price expectations are formed sub-optimally (GEL), and models in which it is formed with the best use of information (RE and EL). As a control, we can similarly compare the ISFEs from random expectations model against those of RE and EL. This can be performed via a standard OLS regression, with four models specified as

$$(p_{RE,t} - p^e_{RE,t}) = \rho_1(p_{GEL,t} - p^e_{GEL,t}) \tag{5.12}$$

$$(p_{EL,t} - p^e_{EL,t}) = \rho_2(p_{GEL,t} - p^e_{GEL,t}) \tag{5.13}$$

$$(p_{RE,t} - p^e_{RE,t}) = \rho_3(p_{X,t} - p^e_{X,t}) \tag{5.14}$$

$$(p_{EL,t} - p^e_{EL,t}) = \rho_4(p_{X,t} - p^e_{X,t}) \tag{5.15}$$

where $(p_{RE,t} - p^e_{RE,t})$ represents the ISFEs from the RE model, $(p_{EL,t} - p^e_{EL,t})$ represents the ISFREs from the EL model, $(p_{GEL,t} - p^e_{GEL,t})$ represents the ISFEs from the GEL model, and $(p_{X,t} - p^e_{X,t})$ represents the ISFEs from the random expectations model. The models (5.12), (5.13), (5.14) and (5.15) are each estimated for a range of values of the price volatility σ.

To determine if the ISFEs are significantly different between the selected models, a hypothesis test is performed on the coefficient ρ in each case. The null and alternate hypotheses for the test are given as

$$H_0: \rho = 1$$

$$H_1: \rho \neq 1$$

where under the null hypothesis, the ISFEs between the two models are statistically identical. It follows from this that

$\rho_1 = 1$ implies that GEL forecasts do not different significantly from RE,

$\rho_2 = 1$ implies that GEL forecasts do not different significantly from EL,

$\rho_3 = 1$ implies that the random expectations forecasts do not different significantly from RE, and

$\rho_4 = 1$ implies that the random expectations forecasts do not different significantly from EL.

The OLS estimates of ρ and outcomes of the hypothesis test for all four models are collated in Table III (below). As before, the model illustrated is for $\alpha = 0.05$, as this is common significance level in hypothesis testing and a conceivable level of belief conservatism. Sensitivity analysis around α results in the GEL ISFEs converging to the EL ISFEs as $\alpha \rightarrow 1$.

These results largely dismiss the notion of self-fulfilling expectations. For markets with low volatility ($0.01 \leq \sigma \leq 0.5$) it can be seen that the null hypothesis is rejected across all four model specifications. In these models, where expectations are largely driving the time path of market prices, the ISFEs are statistically different between RE, EL, GEL and random expectations. Therefore, differences in the method of expectation formation lead to significant differences in forecast accuracy. This result suggests that there lies a cost (in terms of incorrect supply decisions based on inferior price forecasts) in forming expectations sub-optimally. Bias forecasts arising from agents' belief conservatism are simulated to result in greater forecast errors than if their expectations were formed using RE or feasible RE.

In a model where $\sigma = 1$ and $\sigma = 1.5$, it can be seen that the coefficient in model (5.13) is not significantly different from unity at the 1% significance level. This implies that,

Table III

OLS Regression of In-Sample Forecast Errors Between RE, EL and GEL

Table III presents the results of a simulated cobweb model under which expectations are formed under RE, EL, GEL and a model in which price expectations are randomly generated. A range of values for the variance of the error term σ are provided, and the in-sample forecast error is calculated each period. The in-sample forecast errors observed over a total of 100 consecutive observations are regressed as per (5.12), (5.13), (5.14), (5.15). Parameter values used in the simulation are as follows: $\gamma = 5$, $\beta_1 = 0.5$, $\beta_2 = 0.9$ and $\alpha = 0.05$. Coefficient values of ρ are provided along with the standard error in parentheses. Significance levels for failure to reject the null hypothesis (i.e. $\rho = 1$) indicated: * 10%, ** 5%, *** 1%.

σ	ρ_1 Model (5.12)	ρ_2 Model (5.13)	ρ_3 Model (5.14)	ρ_4 Model (5.15)
0.01	0.00 (0.00)	0.34 (0.03)	0.00 (0.00)	0.01 (0.00)
0.1	0.01 (0.00)	0.34 (0.03)	0.00 (0.00)	0.01 (0.00)
0.5	0.65 (0.03)	0.77 (0.03)	0.06 (0.00)	0.06 (0.00)
1	0.89 (0.03)	0.95*** (0.03)	0.22 (0.02)	0.22 (0.02)
1.5	0.92 (0.02)	0.97*** (0.02)	0.41 (0.03)	0.40 (0.04)
10	0.97* (0.01)	0.99*** (0.02)	0.99*** (0.00)	1.02*** (0.02)
100	0.97* (0.01)	1.00*** (0.02)	1.00*** (0.00)	1.03*** (0.02)

in markets where expectations and volatility both play a role in forming equilibrium price outcomes, agents are able to get away with sub-optimal forecasts, as market outcomes are less dependent on their expectations. Simulated agents forecasting under GEL see no significant difference in their forecast errors than agents simulated to be forecasting under EL, although the time path of equilibrium market prices is unique between the two. However, their forecast errors are still found to differ from a rational set of forecasters, which proposes that an advantage does exist for agents who are able to overcome the informational and computational demands of RE, in the form of greater forecast accuracy. The fact that ρ_3 and ρ_4 are found to differ from unity also counters the self-fulfilling expectations argument, as it is observed that the ISFEs are not attenuated by the influence of p_t^e on p_t, when forecasts are randomly generated.

It is only for the larger values of σ specified that similarities between the ISFEs exist. GEL forecasts are seen to be indifferent from RE forecasts at the 10% significance level, and virtually identical to feasible RE, with the coefficient in model (5.13) at the 1% significance level, when $\sigma = 10$ and $\sigma = 100$. It can be safely concluded that this supposed similarity in forecast accuracy is independent of the expectations regime, by examining the output from models (5.14) and (5.15). The randomly generated expectations models have statistically identical forecast errors as RE and EL, although the latter two methods employ far more sophisticated techniques in determining their forecasts.

The finding that expectations are *not* self-fulfilling is significant in the context of GEL, and indeed, in the wider application of cobweb-type models. Even when the differing time paths resulting from endogenously determined market outcomes are accounted for, it is shown that belief conservatism leads to inferior forecasts when compared to approaches that make the best available use of information. The feedback of expectations on prices does not significantly attenuate ISFEs, and, in markets in which outcomes are more weighted on expectations in particular, GEL may still pose significant costs for belief conservative agents. Switching expectations regimes, or 'shedding' belief conservatism, such that $\alpha \to 1$, can allow agents to bring their forecast accuracy closer to RE or feasible RE.

6. Conclusion

This thesis has extended the theory of IE to allow for an endogenously determined test statistic, and has allowed for broader applicability of IE through application to the simple cobweb model. Three cobweb models variants under IE have been developed and simulated. It is shown that adaptive IE allows for agents to consider the accuracy of past forecasts prior to updating without sacrificing stability. Parameter IE allows agents to estimate the value of new parameters introduced into a fully-specified model. Finally, GEL is shown to encompass EL, and can be used as an approach for incorporating belief conservatism into least squares estimation. Upon further investigation, it is determined that, for a simulated model under RE, EL and GEL, the method of expectation formation affects forecast accuracy, and that the data suggests expectations are not self-fulfilling, at least in a cobweb model context.

The modelling and results discussed in this thesis also provide scope for further research and more extensive applications of IE. Future research can be broadly divided into two streams; first, the finding that model predictions under IE are absent of self-fulfilling expectations addresses the identification problem that otherwise deemed RE logically inconsistent by Merton (1957) and others. Using econometric analysis, it may be possible to uncover RE in empirical data. The second route involves seeking evidence for wider formulations of IE; primarily confirmation of agents not changing their expectations upon receipt of information, such as in the experiment of Menzies and Zizzo (2011).

Other work that could be undertaken involves extending endogenous IE outside of the cobweb model, or loosening the representative agent assumption. It is possible to model agents with varying degrees of belief conservatism in an IE model by allowing for a distribution of α's. Cobweb models with heterogeneous agents, such as Brock and Hommes (1997), have provided interesting insight into the behaviour of expectations and equilibria time paths that may prove fertile ground for IE.

References

Akerlof, George and Janet Yellen, 1985, A near-rational model of the business cycle with wage and price inertia, *Quarterly Journal of Economics* 100, Supplement (1985), 823-838.

Akerlof, George and Robert Shiller, 2009, *Animal Spirits: How Human Psychology Drives the Economy and Why It Matters for Global Capitalism*, (Princeton University Press, Princeton).

Alchian, Armen, 1950, Uncertainty, evolution and economic theory, *Journal of Political Economy* 58, 211-221.

Alpert, Marc and Howard Raiffa, 1982, A progress report on the training of probability assessors, in Daniel Kahneman, Paul Slovic and Amos Tversky eds.: *Judgement Under Uncertainty: Heuristics and Biases*, (Cambridge University Press, Cambridge).

Arrow, Kenneth, 1986, Rationality of self and others in an economic system, *The Journal of Business* 59, 385-399.

Barberis, Nicholas and Richard Thaler, 2003, A survey of behavioural finance, *Handbook of the Economics of Finance*, 1051-1121.

Blanchard, Olivier, 2000, What do we know about macroeconomics that Fisher and Wicksell did not?, *Quarterly Journal of Economics* 115, 1375-1409.

Bray, Margaret and David Kreps, 1987, Rational learning and rational expectations, in George Feiwel ed.: *Arrow and the Ascent of Modern Economic Theory*, (New York University Press, New York).

Bray, Margaret and Nathan Savin, 1986, Rational Expectations Equilibria, Learning and Model Specification, *Econometrica* 54, 1129-1160.

Bray, Margaret, 1982, Learning Estimation, and the Stability of Rational Expectations Equilibria, *Journal of Economic Theory*, 26, 318-339.

Brock, Williams and Hommes, Cars, 1997, A rational route to randomness, *Econometrica* 65, 1059-1095.

Buehler Roger, Dale Griffin and Michael Ross, 1994, Exploring the planning fallacy: why people underestimate their task completion times, *Journal of Personality and Social Psychology* 67, 366-381.

Cagan, Phillip, 1956, The monetary dynamics of hyper-inflation, in Milton Friedman ed.: *Studies in the Quantity Theory of Money*, (University of Chicago Press, Chicago).

Camerer, Colin and Robin Hogarth, 1999, The effects of financial incentives in experiments: a review and capital-labor production framework, *Journal of Risk and Uncertainty* 19, 7-42.

De Long, James, Andrei Shleifer, Lawrence Summers and Robert Waldmann, 1990, Noise trader risk in financial markets, *Journal of Political Economy* 98, 703-738.

Dornbusch, Rudi, 1976, Expectations and exchange rate dynamics, *Journal of Political Economy* 84, 1161-1176.

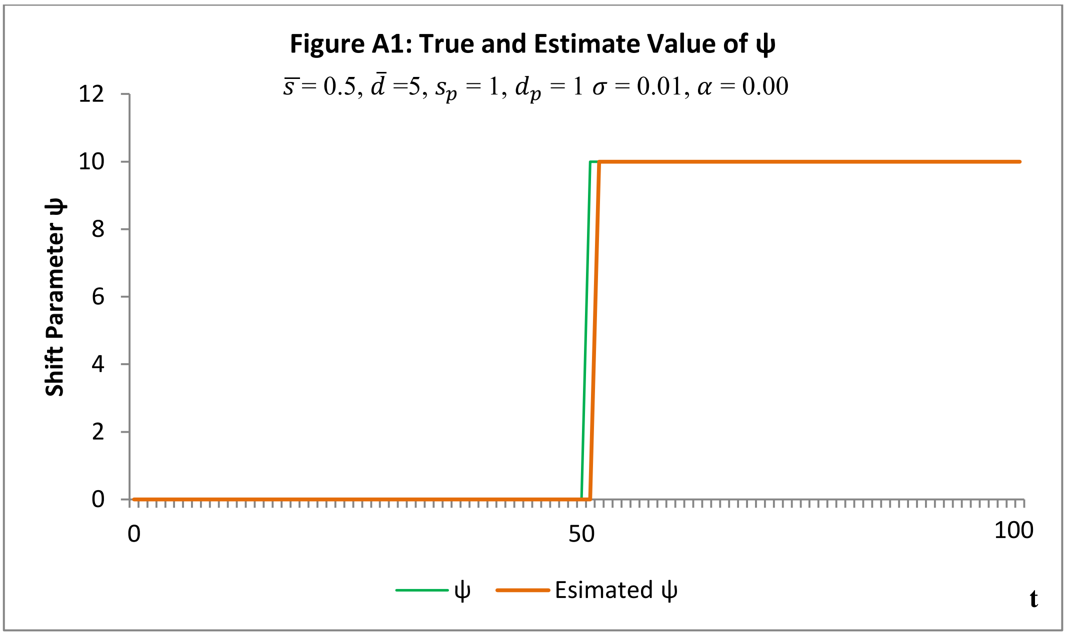
Edwards, Ward, 1968, Conservatism in human information processing, in Benjamin Kleinmutz, ed.: *Formal Representation of Human Judgement* (Wiley, New York).
Ezekiel, Mordecai, 1938, The cobweb theorem, *The Quarterly Journal of Economics* 52, 255-280.
Fama, Eugene, 1970, Efficient capital markets: a review of theory and empirical work, *Journal of Finance* 25, 383-417.
Fischoff, Baruch, Paul Slovic and Sarah Lichtenstein, 1977, Knowing with certainty: the appropriateness of extreme confidence, *Journal of Experimental Psychology: Human Perception and Performance* 3, 552-564.
Friedman, Milton, 1953, The case for flexible exchange rates, in: *Essays in Positive Economics* (University of Chicago Press, USA).
Friedman, Milton, 1957, *Theory of the Consumption Function*, (Princeton University Press, Princeton).
Froot, Kenneth and Emil Dabora, 1999, How are stock prices affected by the location of trade?, *Journal of Financial Economics* 53, 189-216.
Frydman, Roman and Michael Goldberg, 2010, Life after rational expectations? Imperfect knowledge, behavioural insights and the social context, in *Proceedings of the Inaugural Conference of the Institute for New Economic Thinking* (Cambridge University Press, Cambridge).
Frydman, Roman, 1982, Towards an Understanding of Market Processes: Individual Expectations, Learning, and Convergence to Rational Expectations Equilibrium, *American Economic Review* 72, 652-68.
Grossman, Sanford and Joseph Stiglitz, 1980, On the impossibility of informationally efficient markets, *The American Economic Review* 70, 393-408.
Grunberg, Elin and Franco Modigliani, 1954, The predictability of social events, *Journal of Political Economy* 62, 465-478
Hansen, Peter and Thomas Sargent, 2001, Robust control and model uncertainty, *American Economic Review* 91,, 60–66.
Harrison, Michael and David Kreps, 1978, Speculative investor behaviour in a stock market with Heterogeneous expectations, *The Quarterly Journal of Economics* 92, 323-336.
Henckel, Timo, Gordon Menzies, Nick Prokhovnik and Daniel Zizzo, 2011, Barro-Gordon revisited: reputational equilibria with inferential expectations, *Economic Letters* 112, 114-147.
Kahneman, Daniel and Amos Tversky, 1974, Judgement under uncertainty: heuristics and biases, *Science* 185, 1125-1131.
Keynes, John, 1936, *The General Theory of Employment, Interest, and Money*, (Harcourt, Brace & Company, New York).
Lord, Charles, Lee Ross and Mark Lepper, 1979, Biased assimilation and attitude polarization: the effects of prior theories on subsequently considered evidence, *Journal of Personality and Social Psychology* 37, 2098-2109.
Lucas, Robert, 1972, Expectations and the neutrality of money, *Journal of Economic Theory*

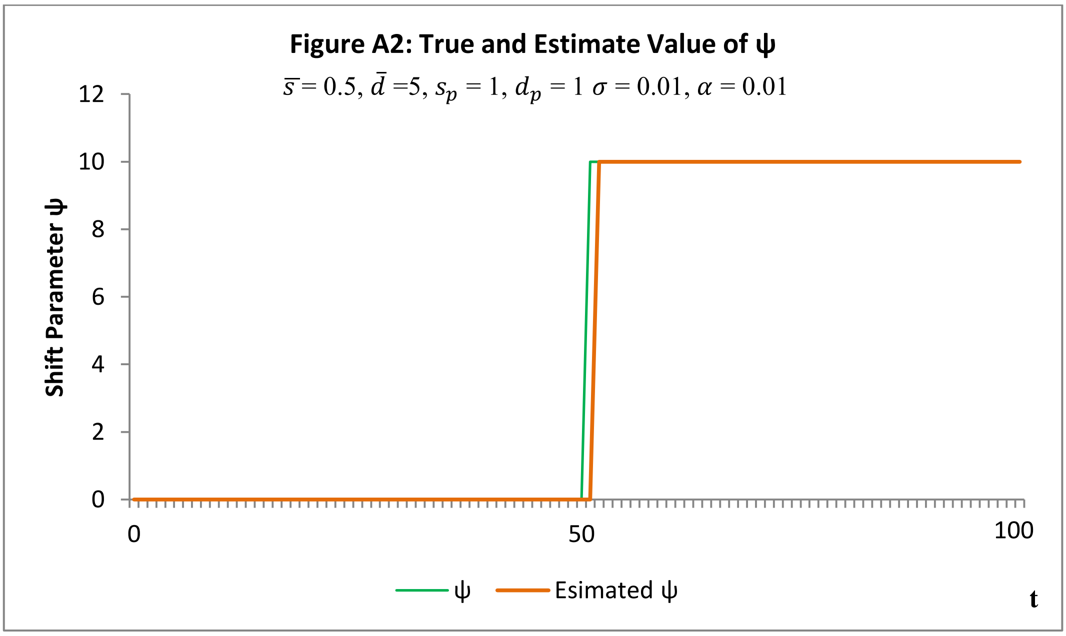
4, 103-24.
Lucas, Robert, 2001, Professional memoir, in *A Lecture Given in the Nobel Economists Lecture Series at Trinity University*, (Mimeo, San Antonio).
McClarke, James and Dietrich, Frank, 1989, *First Course in Statistics,* (Pellen Publishing, San Francisco).
Menzies, Gordon and Daniel Zizzo, 2008, Rational expectations, in William A. Darity, ed.: *International Encyclopedia of the Social Sciences*, (Macmillan Reference, Detroit).
Menzies, Gordon and Daniel Zizzo, 2009, Inferential expectations, *The B.E. Journal of Macroeconomics* 9, 1-25.
Menzies, Gordon and Daniel Zizzo, 2011, Monetary policy and inferential expectations of exchange rates, *Journal of International Financial Markets, Institutions and Money*, forthcoming.
Merton, Robert, 1957, The self-fulfilling prophecy, in *Social Theory and Social Structure*, (Free Press, New York).
Merton, Robert, 1987, A simple model of capital market equilibrium with incomplete information, *Journal of Finance* 42, 483-510.
Morgenstern, Oskar, 1935, Vollkommene voraussicht und wirtschaftliches gleichgewicht, *Zeitschrift für Nationalökonomie* 6, 337-357.
Mullainathan, Sendhil, 2002, Thinking through categories, Working paper, Massachusetts Institute of Technology and National Bureau of Economic Research.
Muth, John, 1961, Rational expectations and the theory of price movements, *Econometrica* 29, 315-335.
Nerlove, Marc, 1958, Adaptive expectations and cobweb phenomena, *The Quarterly Journal of Economics* 72, 227-240.
Sargent, Thomas, 1973, Rational expectations, the real rate of interest and the natural rate of unemployment, *Brookings Papers on Economic Activity* 2, 429-72.
Sargent, Thomas, 1993, *Bounded Rationality in Macroeconomics*, (Oxford University Press, Oxford).
Sheffrin, Steven, 1983, Rational expectations, (Cambridge University Press, New York).
Shleifer, Andrei, 2000, *Inefficient Markets: An Introduction to Behavioural Finance*, (Oxford University Press, Oxford)
Simon, Herbert, 1957, *Models of Man*, (Wiley, New York).
Sims, Christopher, 2003, Implications of rational inattention, *Journal of Monetary Economics* 50, 665-690.
Tan, Kang, 2006, Essays on Learning in international macroeconomics, unpublished doctoral thesis, Australian National University.
The Royal Swedish Academy of Sciences, 2011, *Press Release 2011-10-10: The Prize in Economic Sciences 2011*, Stockholm, Sweden.
Townsend, Robert, 1983, Forecasting the forecasts of others, *Journal of Political Economy* 91, 546-588.
Van de Venter, Gerhard and David Michayluk, 2008, An insight into overconfidence in the

forecasting abilities of financial advisors, *Australian Journal of Management* 32, 545-558.
Weinstein, Neil, 1980, Unrealistic optimism about future life events, *Journal of Personality and Social Psychology* 39, 806-820.
Williams, Noah, 2008, Robust control, in Steven Durlauf and Lawrence Blume, eds.: *The New Palgrave Dictionary of Economics*, (Palgrave Macmillan, New York).

Appendix

This appendix contains a parameter IE model specified in Section 5.5 and operating as defined in Section 5.4. The output below represents the true and estimated values of the cognitive target (shift parameter Ψ) for different values of volatility σ and belief conservatism α. The parameter values used in each case are also provided.





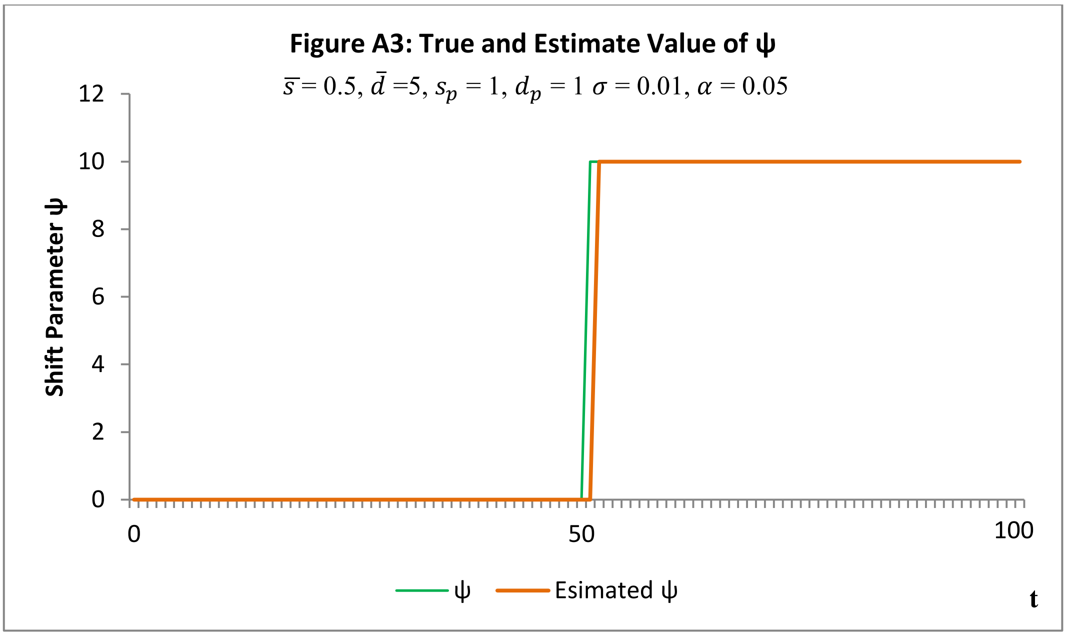


Figure A3: True and Estimate Value of ψ
$\bar{s}$= 0.5, $\bar{d}$ =5, s_p = 1, d_p = 1 σ = 0.01, α = 0.05
Shift Parameter ψ
12
10
8
6
4
2
0
0
50
100
ψ
Esimated ψ
t

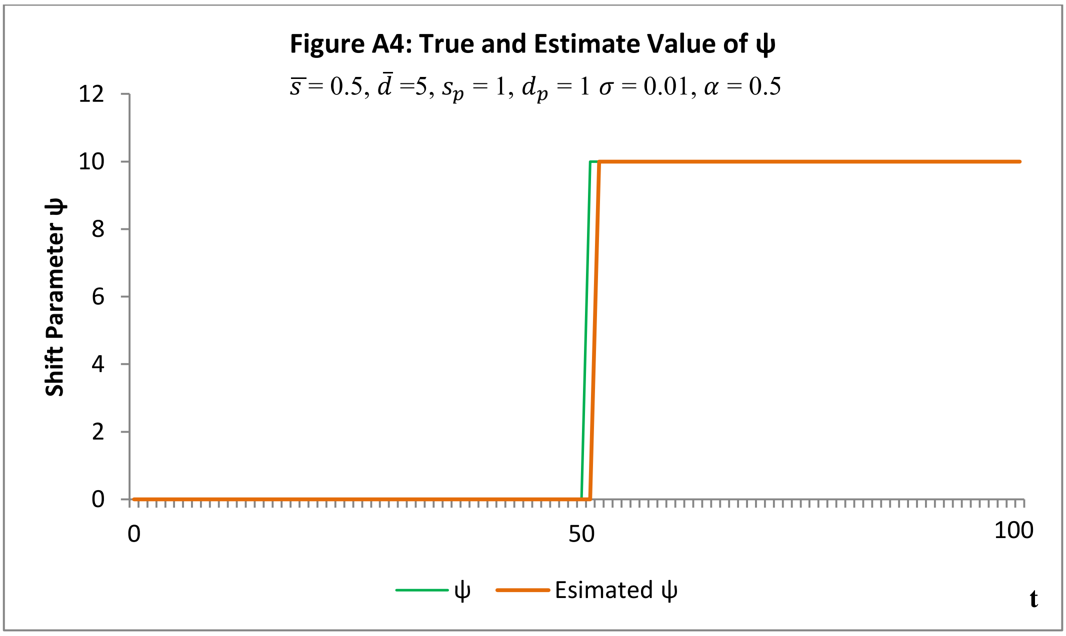


Figure A4: True and Estimate Value of ψ
$\bar{s}$= 0.5, $\bar{d}$ =5, s_p = 1, d_p = 1 σ = 0.01, α = 0.5
Shift Parameter ψ
12
10
8
6
4
2
0
0
50
100
ψ
Esimated ψ
t

Figure A5: True and Estimate Value of ψ

$\bar{s} = 0.5$, $\bar{d} = 5$, $s_p = 1$, $d_p = 1$ $\sigma = 0.01$, $\alpha = 1$

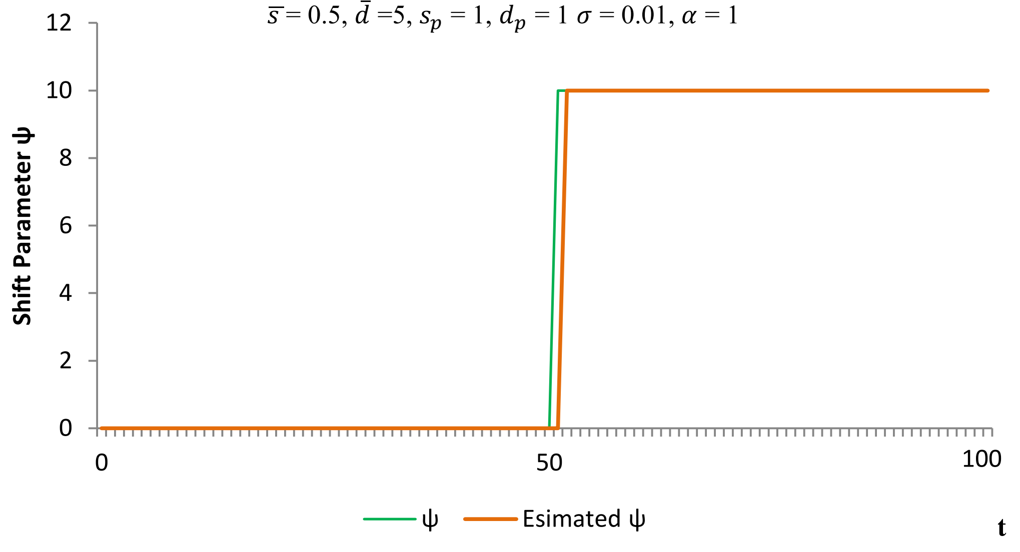
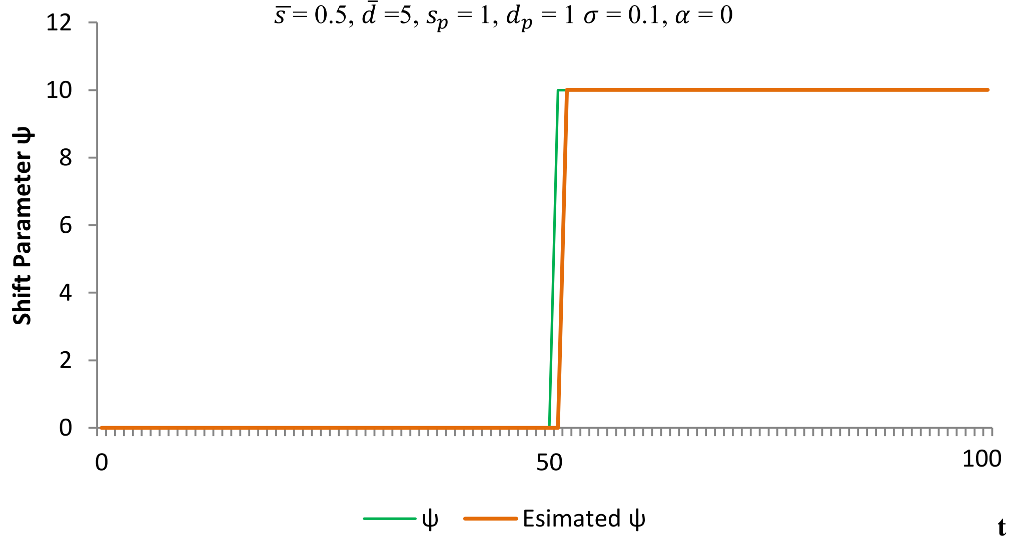


Figure A6: True and Estimate Value of ψ

$\bar{s} = 0.5$, $\bar{d} = 5$, $s_p = 1$, $d_p = 1$ $\sigma = 0.1$, $\alpha = 0$



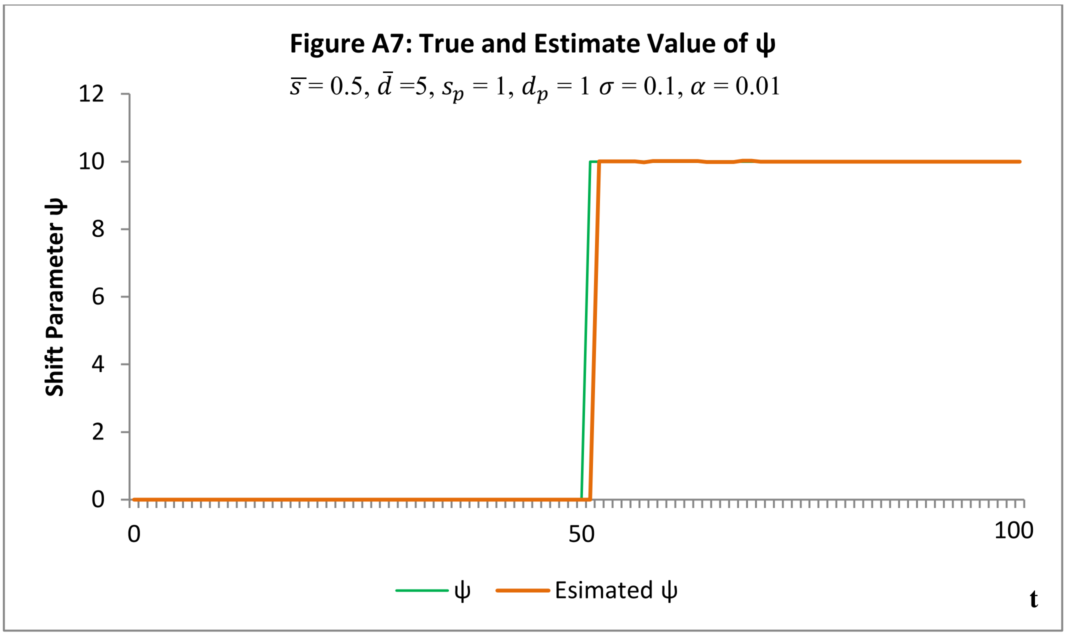


Figure A7: True and Estimate Value of ψ
$\bar{s}$ = 0.5, $\bar{d}$ =5, s_p = 1, d_p = 1 σ = 0.1, α = 0.01
Shift Parameter ψ
12
10
8
6
4
2
0
0
50
100
ψ
Esimated ψ
t

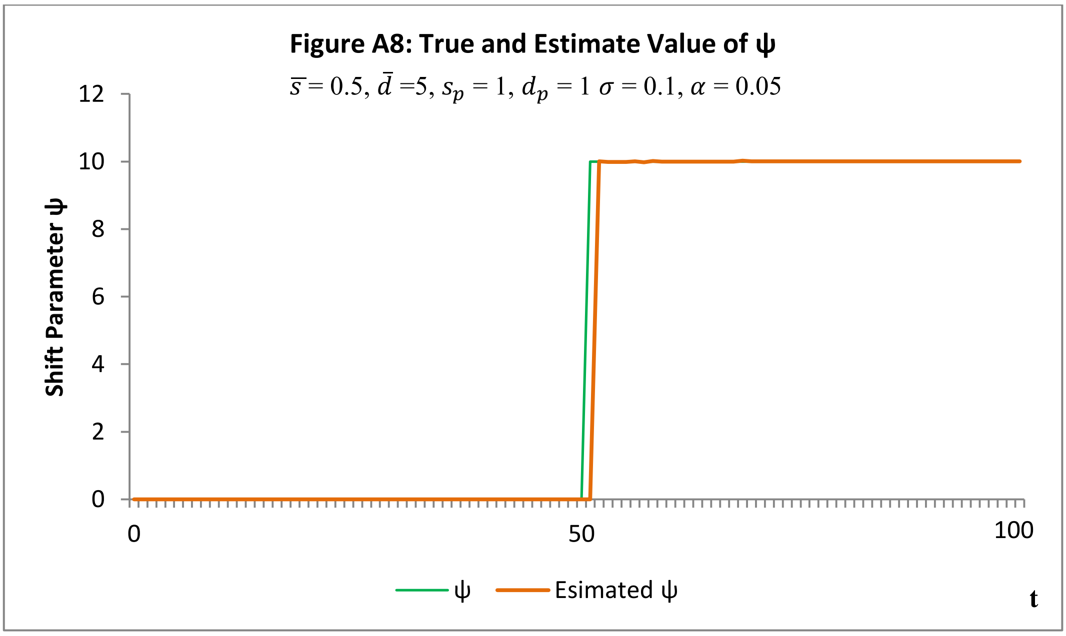


Figure A8: True and Estimate Value of ψ
$\bar{s}$ = 0.5, $\bar{d}$ =5, s_p = 1, d_p = 1 σ = 0.1, α = 0.05
Shift Parameter ψ
12
10
8
6
4
2
0
0
50
100
ψ
Esimated ψ
t

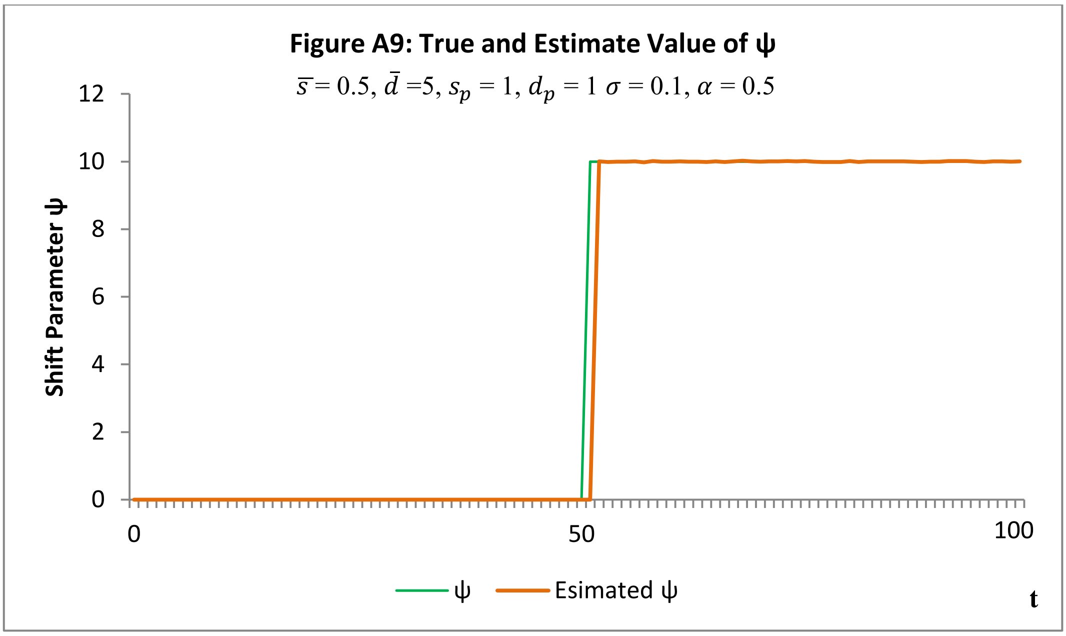


Figure A9: True and Estimate Value of ψ
$\bar{s}$= 0.5, $\bar{d}$ =5, s_p = 1, d_p = 1 σ = 0.1, α = 0.5
Shift Parameter ψ
12
10
8
6
4
2
0
0
50
100
ψ
Esimated ψ
t

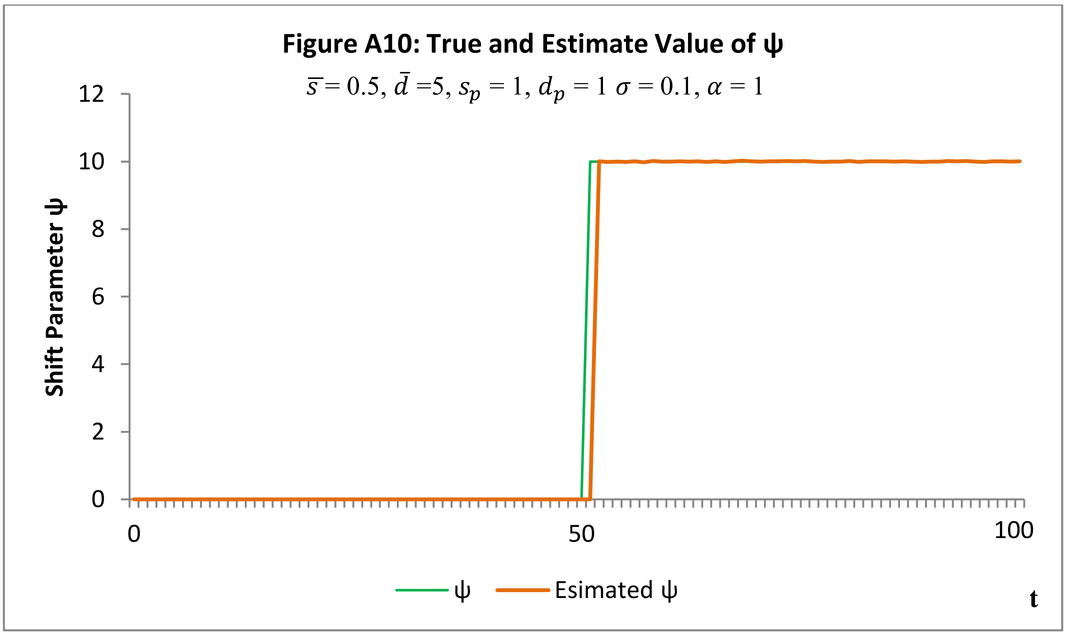


Figure A10: True and Estimate Value of ψ
$\bar{s}$= 0.5, $\bar{d}$ =5, s_p = 1, d_p = 1 σ = 0.1, α = 1
Shift Parameter ψ
12
10
8
6
4
2
0
0
50
100
ψ
Esimated ψ
t

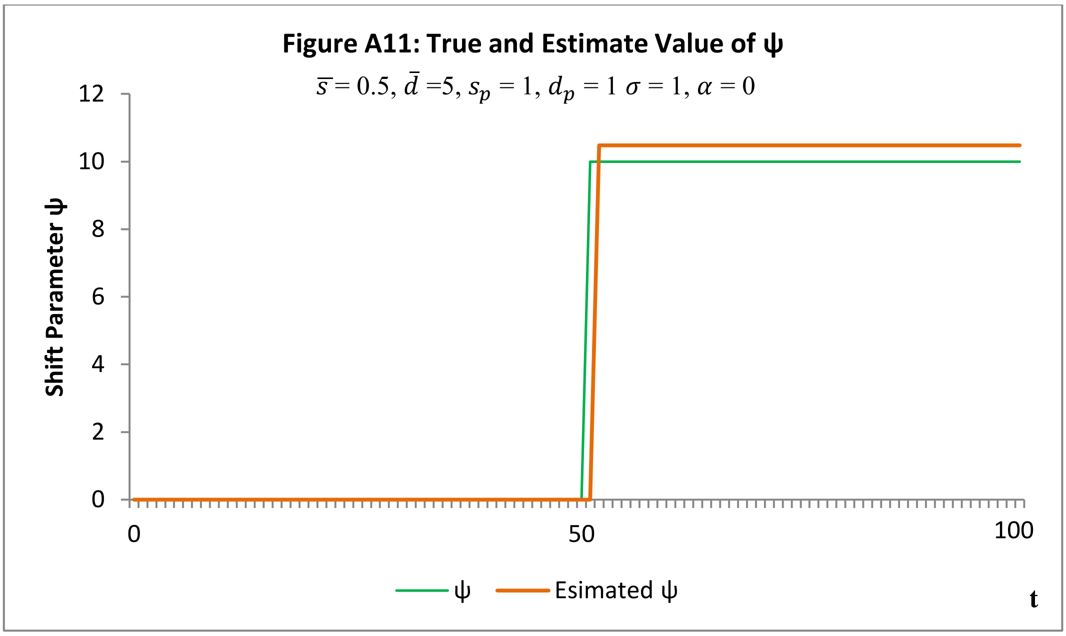


Figure A11: True and Estimate Value of ψ
$\bar{s}$ = 0.5, $\bar{d}$ =5, s_p = 1, d_p = 1 σ = 1, α = 0
Shift Parameter ψ
12
10
8
6
4
2
0
0
50
100
ψ
Esimated ψ
t

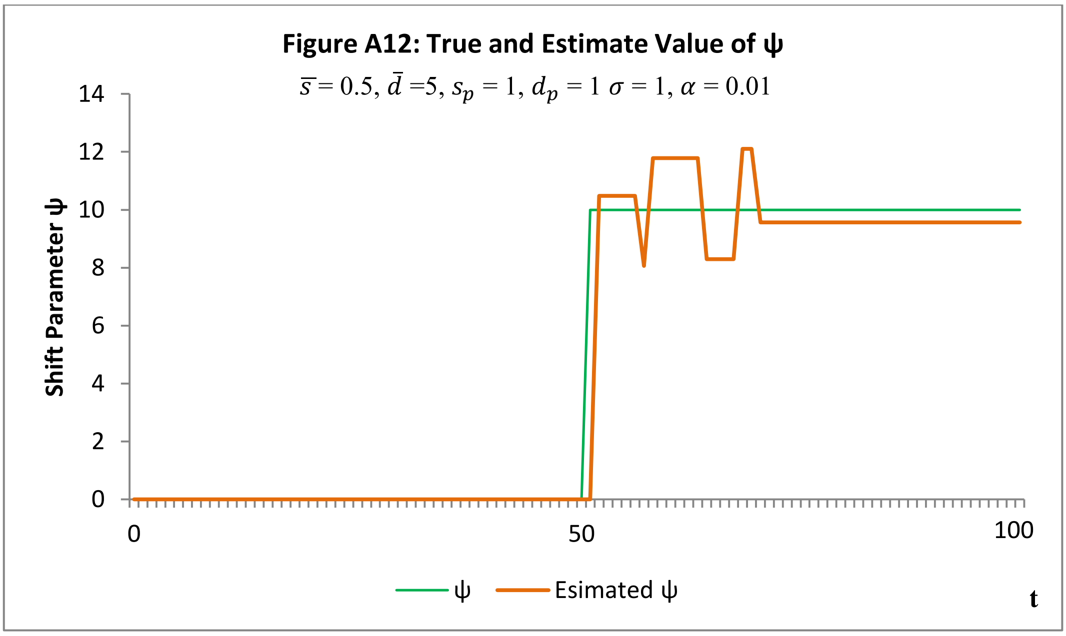


Figure A12: True and Estimate Value of ψ
$\bar{s}$ = 0.5, $\bar{d}$ =5, s_p = 1, d_p = 1 σ = 1, α = 0.01
Shift Parameter ψ
14
12
10
8
6
4
2
0
0
50
100
ψ
Esimated ψ
t

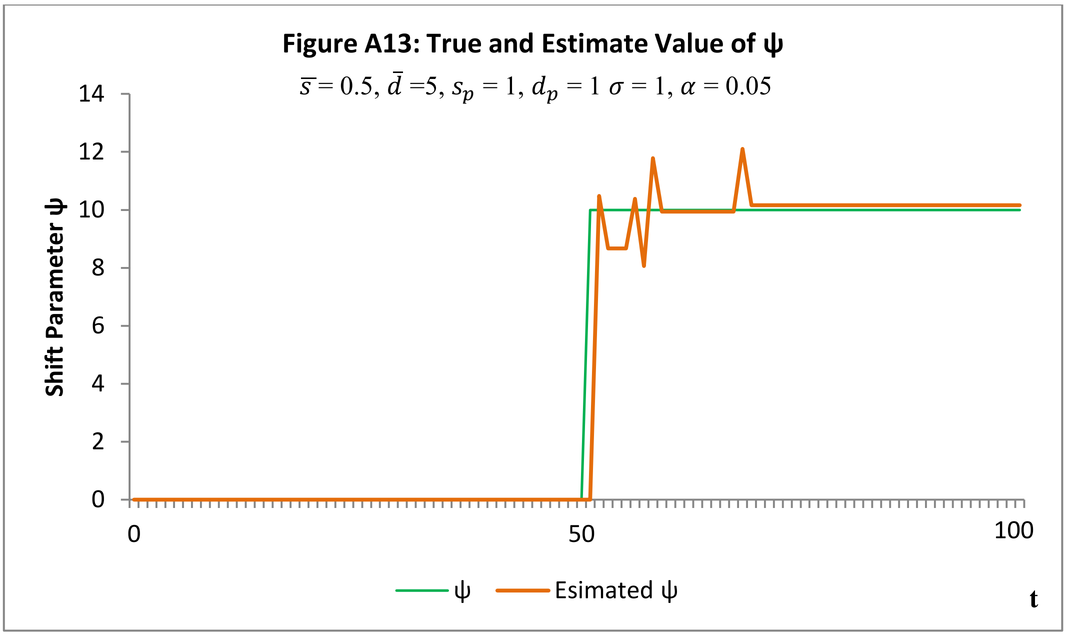


Figure A13: True and Estimate Value of ψ
$\bar{s}$ = 0.5, $\bar{d}$ =5, s_p = 1, d_p = 1 σ = 1, α = 0.05
Shift Parameter ψ
14
12
10
8
6
4
2
0
0
50
100
ψ
Esimated ψ
t

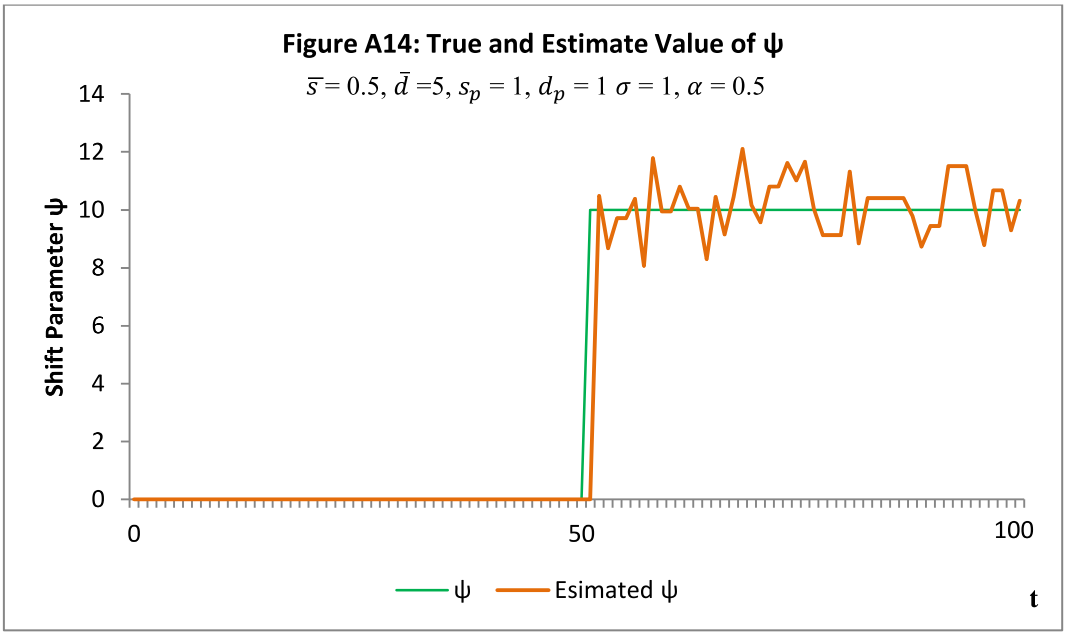


Figure A14: True and Estimate Value of ψ
$\bar{s}$ = 0.5, $\bar{d}$ =5, s_p = 1, d_p = 1 σ = 1, α = 0.5
Shift Parameter ψ
14
12
10
8
6
4
2
0
0
50
100
ψ
Esimated ψ
t

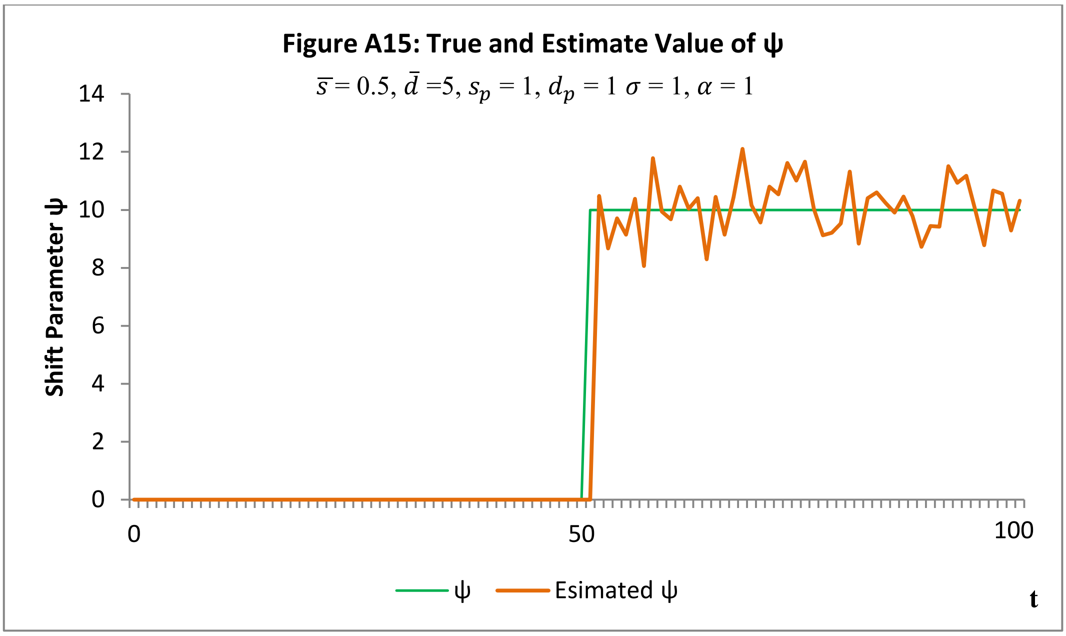


Figure A15: True and Estimate Value of ψ
$\bar{s}$= 0.5, $\bar{d}$ =5, s_p = 1, d_p = 1 σ = 1, α = 1
Shift Parameter ψ
14
12
10
8
6
4
2
0
0
50
100
ψ
Esimated ψ
t

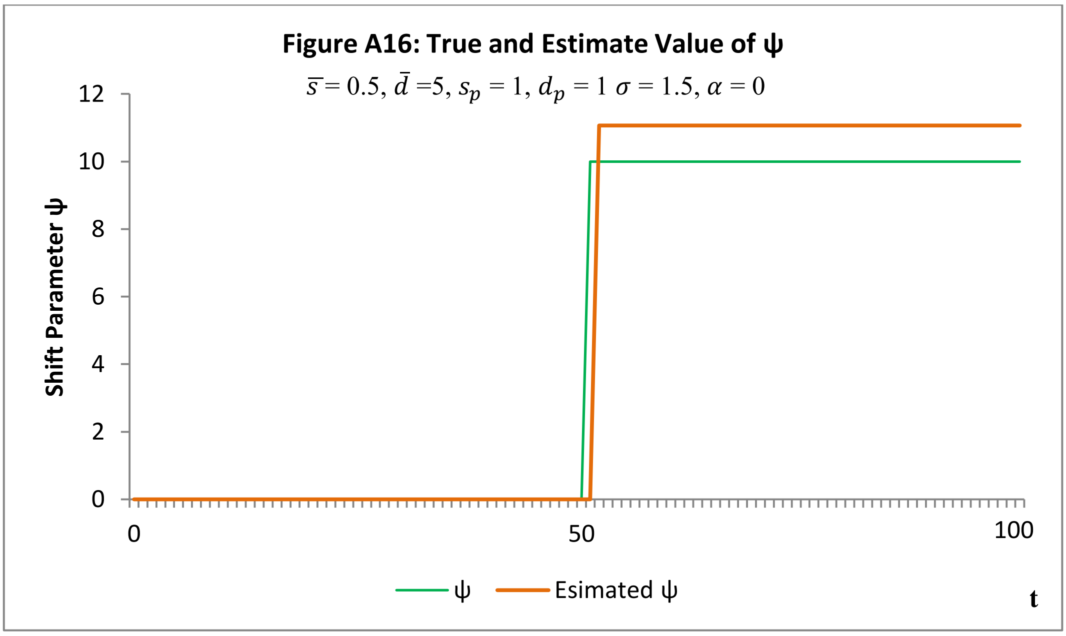


Figure A16: True and Estimate Value of ψ
$\bar{s}$= 0.5, $\bar{d}$ =5, s_p = 1, d_p = 1 σ = 1.5, α = 0
Shift Parameter ψ
12
10
8
6
4
2
0
0
50
100
ψ
Esimated ψ
t

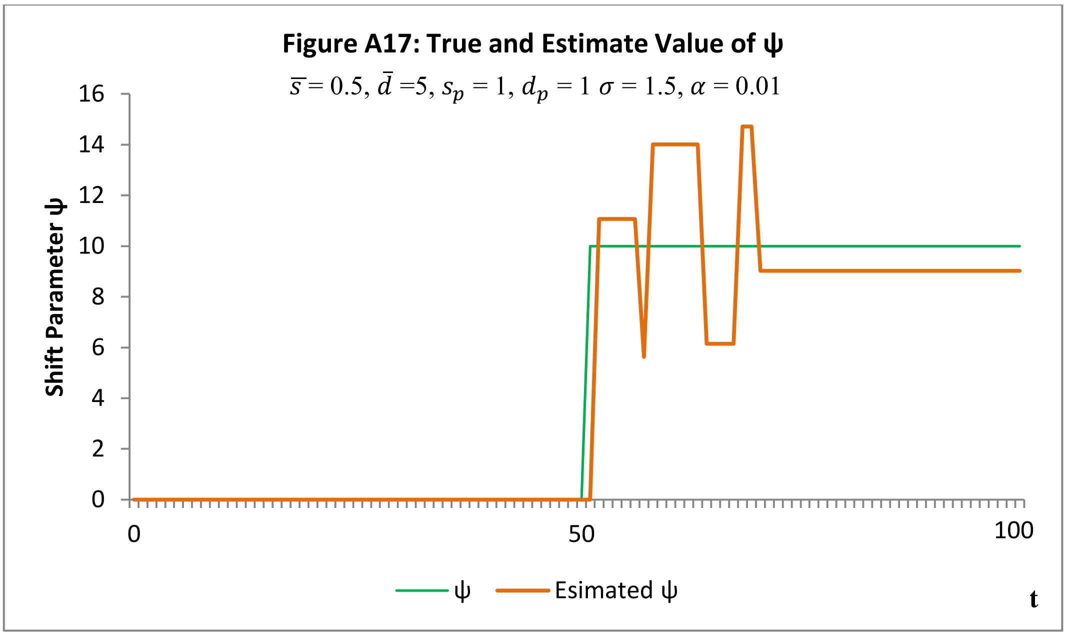


Figure A17: True and Estimate Value of ψ
$\bar{s}$ = 0.5, $\bar{d}$ =5, s_p = 1, d_p = 1 σ = 1.5, α = 0.01
Shift Parameter ψ
16
14
12
10
8
6
4
2
0
0
50
100
ψ
Esimated ψ
t

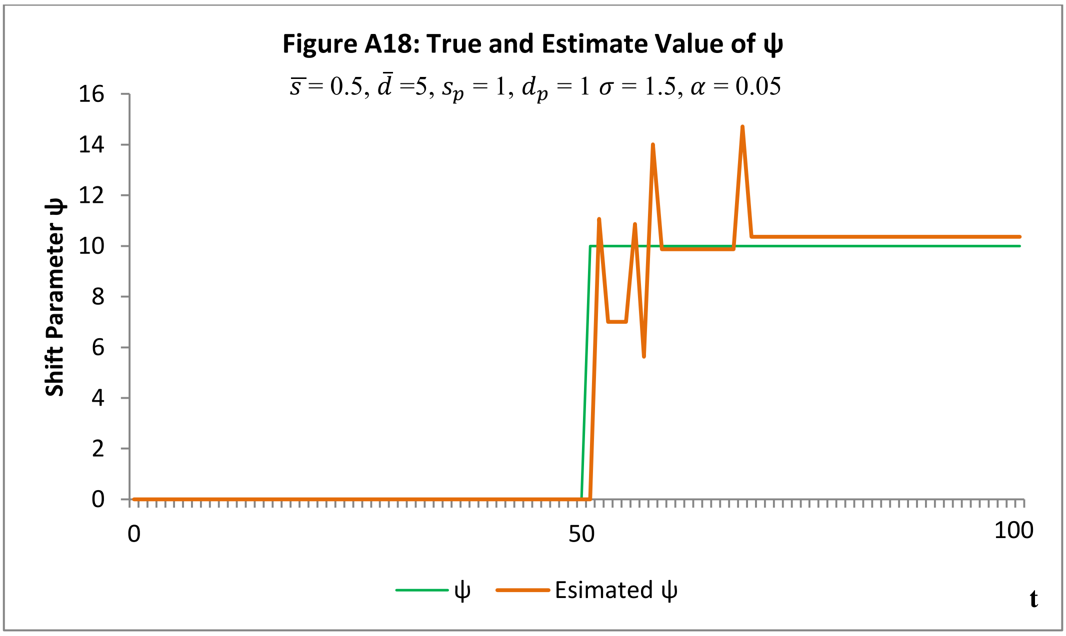


Figure A18: True and Estimate Value of ψ
$\bar{s}$ = 0.5, $\bar{d}$ =5, s_p = 1, d_p = 1 σ = 1.5, α = 0.05
Shift Parameter ψ
16
14
12
10
8
6
4
2
0
0
50
100
ψ
Esimated ψ
t

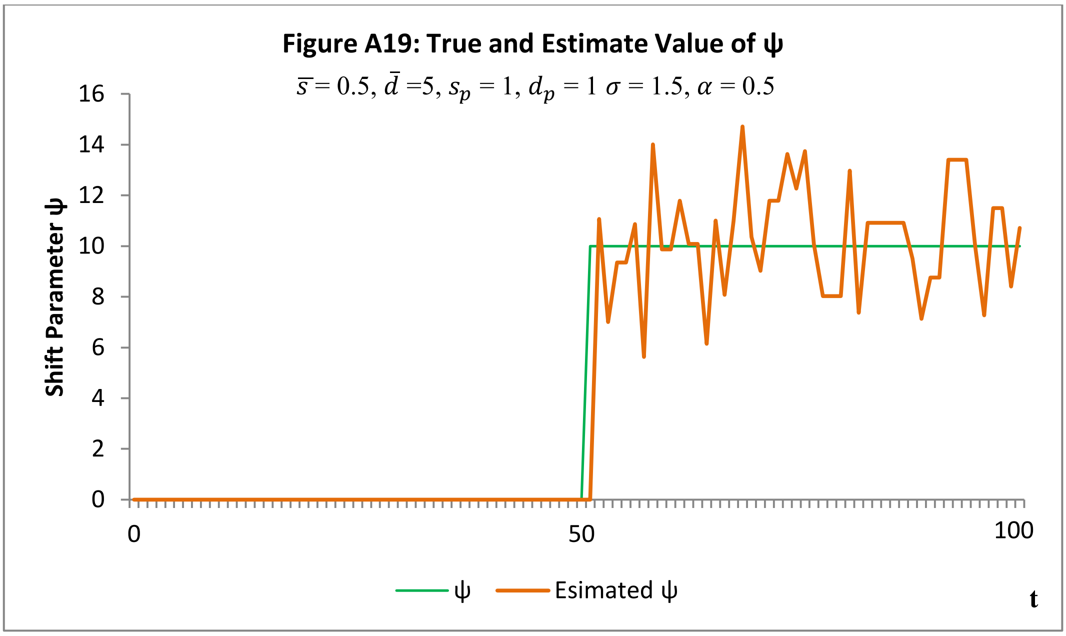


Figure A19: True and Estimate Value of ψ
$\bar{s}$ = 0.5, $\bar{d}$ =5, s_p = 1, d_p = 1 σ = 1.5, α = 0.5
Shift Parameter ψ
16
14
12
10
8
6
4
2
0
0
50
100
ψ
Esimated ψ
t

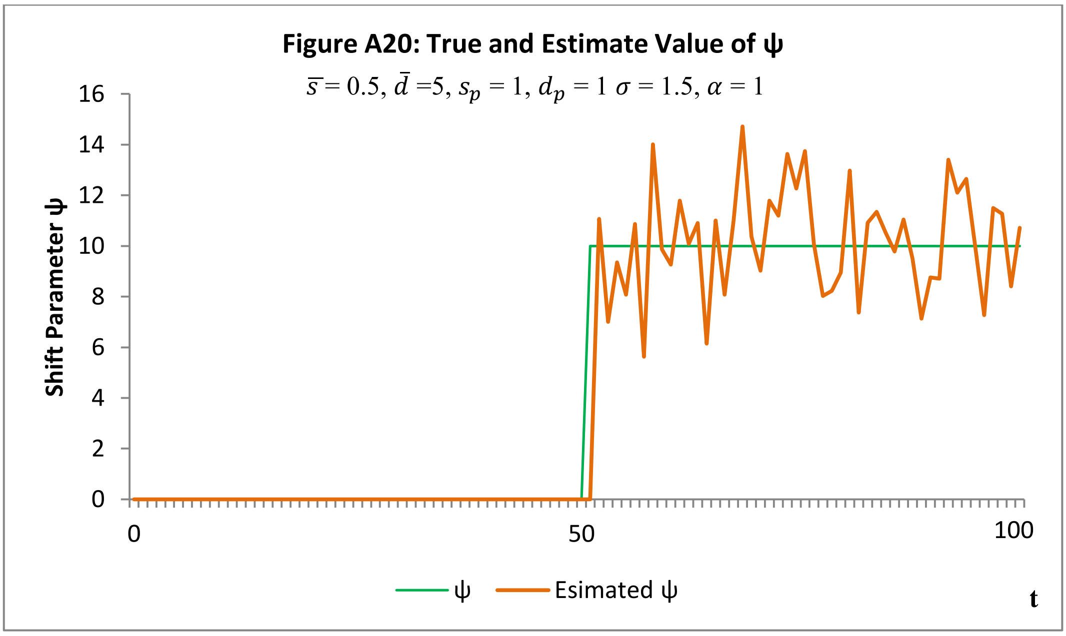


Figure A20: True and Estimate Value of ψ
$\bar{s}$ = 0.5, $\bar{d}$ =5, s_p = 1, d_p = 1 σ = 1.5, α = 1
Shift Parameter ψ
16
14
12
10
8
6
4
2
0
0
50
100
ψ
Esimated ψ
t

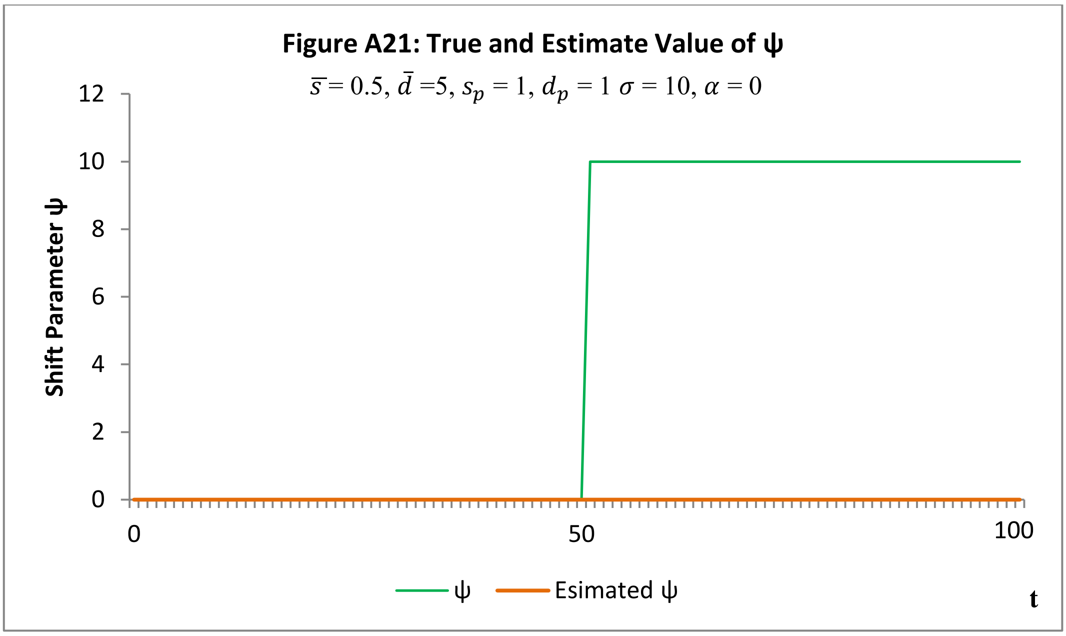


Figure A21: True and Estimate Value of ψ
$\bar{s}$= 0.5, $\bar{d}$ =5, s_p = 1, d_p = 1 σ = 10, α = 0
Shift Parameter ψ
12
10
8
6
4
2
0
0
50
100
ψ
Esimated ψ
t

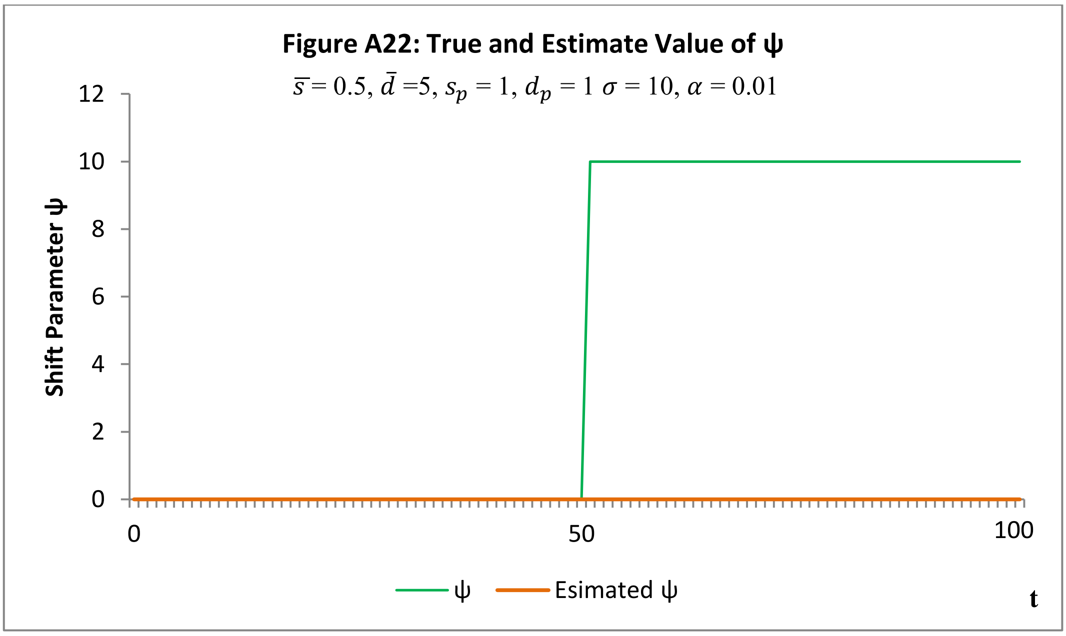


Figure A22: True and Estimate Value of ψ
$\bar{s}$= 0.5, $\bar{d}$ =5, s_p = 1, d_p = 1 σ = 10, α = 0.01
Shift Parameter ψ
12
10
8
6
4
2
0
0
50
100
ψ
Esimated ψ
t

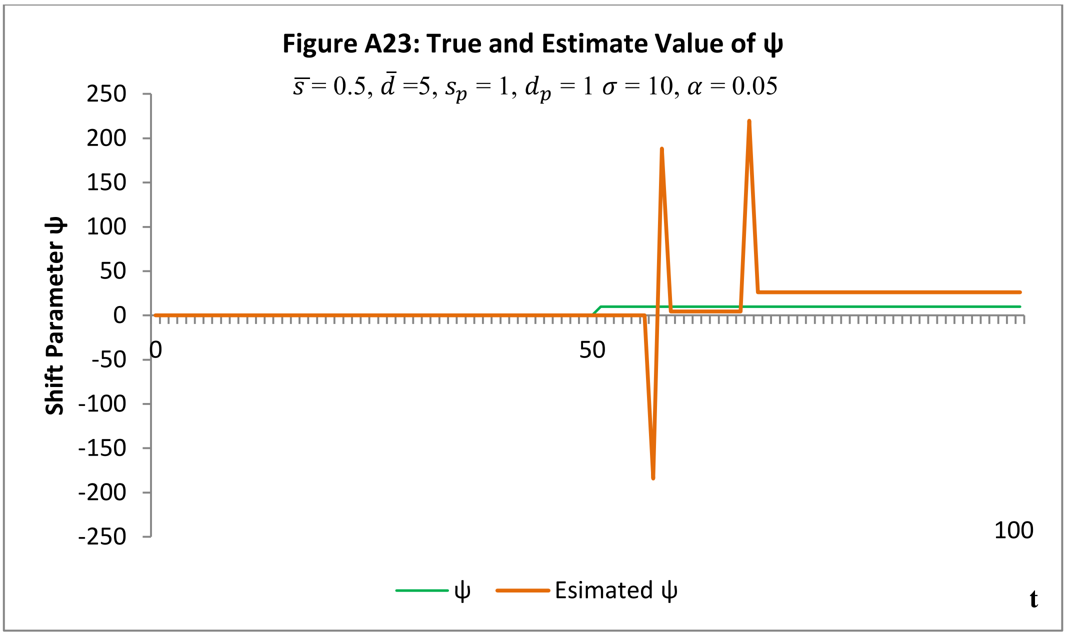


Figure A23: True and Estimate Value of ψ
$\bar{s}$ = 0.5, $\bar{d}$ =5, s_p = 1, d_p = 1 σ = 10, α = 0.05
250
200
150
100
50
0
-50
-100
-150
-200
-250
Shift Parameter ψ
0
50
100
ψ
Esimated ψ
t

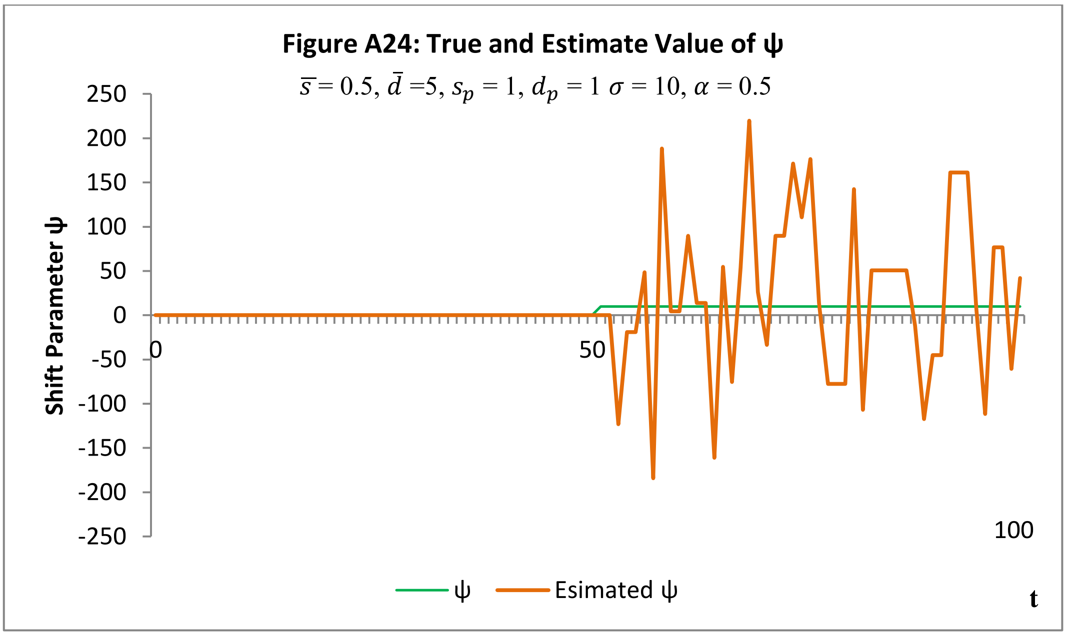


Figure A24: True and Estimate Value of ψ
$\bar{s}$ = 0.5, $\bar{d}$ =5, s_p = 1, d_p = 1 σ = 10, α = 0.5
250
200
150
100
50
0
-50
-100
-150
-200
-250
Shift Parameter ψ
0
50
100
ψ
Esimated ψ
t

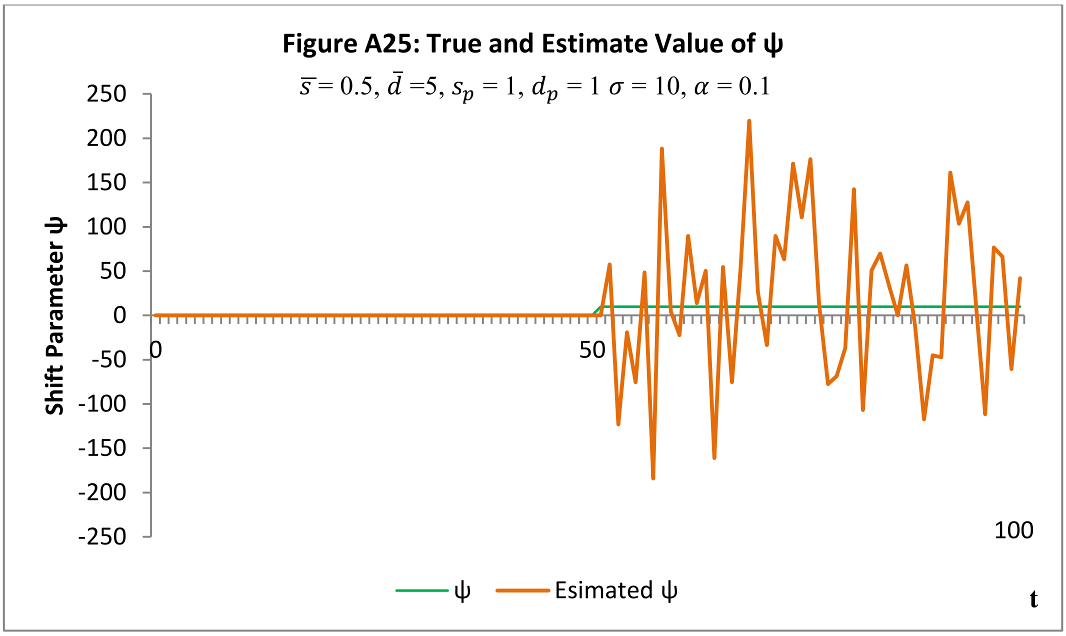


Figure A25: True and Estimate Value of ψ
$\bar{s}$= 0.5, $\bar{d}$ =5, s_p = 1, d_p = 1 σ = 10, α = 0.1
Shift Parameter ψ
250
200
150
100
50
0
-50
-100
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