Pricing Barrier Options under Scalar Diffusions using the Eigenfunction Expansion Approach

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DECLARATION:

I certify that this thesis has not already been submitted for any degree and is not being submitted as part of the candidature for any other degree.

I also certify that the thesis has been written by me and that any help that I have received in preparing this thesis, and all sources used, have been acknowledged in this thesis.

Signature,

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6th December, 2010

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Abstract

In this thesis, we will present some methods used to price barrier options. We first price barrier options under the Black-Scholes model. Then we will discuss some of the shortcomings of the Black-Scholes model. Next we derive prices for barrier options under different classes of scalar diffusions. In particular, we will use eigenfunction expansions to price barrier options under the CEV model of price dynamics.

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1 Introduction

1.1 Evolution of Share Prices

Investors are concerned with how asset prices evolve over time. Anyone who invests in the share market implicitly tries to predict the future behaviour of asset prices by some means. Some investors use emotional reactions, intuitive feelings, the analysis of financial statements and the seeking of trends/patterns in financial time series (i.e charting historical prices). However, it was realised early in the development of the modern theory of quantitative finance that since asset prices evolve randomly over time, the best description of price behaviour would be a probabilistic one. This means that to properly study pricing dynamics we have to use methods from stochastic analysis. Stochastic analysis methodology involves important concepts developed from a variety of disciplines such as theoretical physics, electrical engineering and pure mathematics.

In the modern theory of quantitative finance, share prices are assumed to evolve according to a particular stochastic process S(t), being the price of the share at time t, which is typically an Itó process. Therefore, to model price behaviour, we require the mathematical apparatus of real analysis, measure theory, probability theory, elements of stochastic modelling, stochastic processes and differential equations.

Definition 1.1 *A continuous stochastic process can be thought of a collection of random variables*$\{X(t)|t\epsilon T\}$*, where the time parameter t belongs to an Index set* $T = \{t|0 \leq t < \infty\}$ *and X(t) is the state of the process at time t.*

The most important stochastic process is the Wiener Process, or Brownian motion. In 1900, Bachelier used Brownian motion to model share prices in his PhD thesis *'The Theory of Speculation'*. Independently, a mathematical description of Brownian motion was given by Albert Einstein in 1905 in his paper '*Investigations on the theory of Brownian Movement'*. Einstein derived what we now call the transition probability density for Brownian motion by solving the heat equation. Then in 1923 Norbert Wiener produced a rigourous theory of Brownian motion. In 1951 Kyosi Itô introduced more general stochastic processes, now called Itô processes. He also developed a type of calculus for studying these processes. The ordinary rules of calculus do not generally hold in a stochastic environment. The stochastic processes for prices are usually taken to be Itô processes.

A set of values arising from a stochastic process is a sample path. We can picture each random sample path (random variable) as representing every possible path a share price might move within a certain period. In other words the simulation of share prices can be thought as a possible realisation of the market price over the period (0, T). Thus to price a security we require the probability distribution for the Itô process.

Definition 1.2 *An Itô process, or scalar diffusion, is a stochastic process with the form*

$$X(t) = X(0) + \int_0^t \mu(X(s), s)ds + \int_0^t \sigma(X(s), s)dB(s)$$

where X(0) is a F_0 *adapted process,* $\int_0^t \mu(X(s), s)ds < \infty$ *and* $\int_0^t \sigma^2(X(s), s)ds < \infty$ *are* F_t *adapted process for* $0 \le t \le T$*. The stochastic differential equation for the Itô process may be written*

$$dX(t) = \mu(X(t), t)dt + \sigma(X(t), t)dB(t).$$

The most important result is Itô's formula. Itô Lemma tells us, if we have a random walk model for a share price S and an option on that share, we can find the change of the option price with changes in the share price.

Theorem 2.2 *(Itô's Lemma) Suppose that B(t) is a Brownian motion,f is a* $C^{2,1}$ *function and that X={X(t):t≥0} is an Itô process satisfying the SDE* $dX(t) = \mu(X(t), t)dt + \sigma(X(t), t)dB(t)$*, X(0)=x. That is,*

$$X(t) = x + \int_0^t \mu(X(s), s)ds + \int_0^t \sigma(X(s), s)dB(s)$$

Then,

$$f(X(t), t) = f(X(0), 0) + \int_0^t f_s(X(s), s)ds + \int_0^t f_x(X(s), s)dX(s) + \frac{1}{2}\int_0^t f_{xx}(X(s), s)dX^2(s)$$

$$= f(X(0), 0) + \int_0^t f_s(X(s), s)ds + \int_0^t f_x(X(s), s)dX(s) + \frac{1}{2}\int_0^t \sigma^2(X(s), s)f_{xx}(X(s), s)ds$$

In differential form this is:

$$df(X(t),t) = f_s(X(t),t)dt + f_x(X(t),t)dX(t) + \frac{1}{2} f_{xx}(X(t),t)dX^2(t)$$
$$= f_s(X(t),t)dt + f_x(X(t),t)dX(t) + \frac{1}{2} \sigma^2(X(t),t) f_{xx}(X(t),t)dt$$

See [30].

The probability distribution for an Itô process can be obtained by solving a partial differential equation (PDE). These equations can either be solved by separation of variables, integral transforms and numerical methods. We will discuss this further when we approach the Black-Scholes model.

An option on a share is a contract that gives the purchaser or seller the right (not the obligation) to buy or sell the share for a predetermined (initiated at time 0) price at some time in the future time t. Entering an option contract minimises the uncertainty associated with asset prices. Options lets investors hedge (reduce risk), speculate, and find arbitrage opportunities to make riskless profits.

Definition 1.3 *An European call option gives the investor the right, but not the obligation, to buy the underlying asset for an agreed price (strike price K) at maturity t=T of the option contract (which was initiated at t=0). A European put option gives the seller the right, but not the obligation, to sell the underlying asset for an agreed price (strike price K) at maturity (t=T) of the option contract (which was initiated at t=0).*

In order to do this, the investor has to determine the value of the option at some time prior to maturity T, say time 0. The valuation of the option is basically the discounted expected cash flow calculation of the possible payoffs of the option back to time 0. The two things to work out in the valuation of the option, are the appropriate discount rate and the precise distribution to be used in calculating the expected value of the payoff. In the case of a European call option, we are interested in the probability that the share price at T (maturity) finishes above the strike price K, to value the option. It should be clear now that the valuation of the option requires some theory about how share prices move stochastically and the calculation of the expectations of payoffs with respect to their distributions.

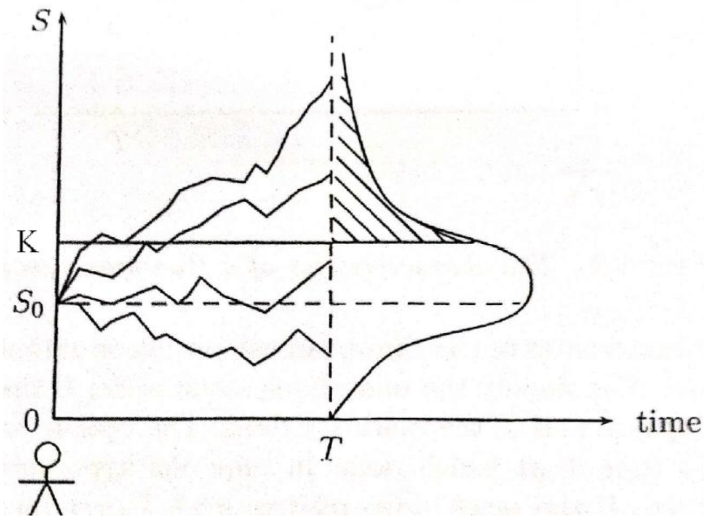

Figure 1.1 Visualising the distribution of S_T conditional on S_0.
Taken from Carl Chiarella (2004), "An Introduction to Derivative Security Pricing".

Throughout the development of the modern theory of finance there have been several models and methods to find the value of the option. We will demonstrate a few approaches in this thesis.

1.2 Exotic Options

Exotic options are a category of options which include complicated features and complex payoffs. The option payoff or other key values often depend on outside factors which vary over time. Because of their complexity, exotic options are traded over the counter (OTC), rather through an exchange and they have become becoming increasingly popular since the 1980s, because of their favourable payoffs in certain market expectations to some investors.

Path dependent options are a type of exotic option whose payoff at exercise or expiry date depends on the past history of the underlying asset price as well as its spot price at exercise or expiry.

A barrier option is a path-dependent exotic option, where the exercise price depends on the underlying asset crossing or reaching a given 'barrier level'. Barrier options are usually cheaper than options without the barrier because the barrier creates an upper limit on the potential profit. Therefore the premium (price of the option writer sets) needs to be reduced compared to the plain-vanilla option to compensate for the limited profit the investor holds. Barrier options were created to provide the insurance value of an option without charging as much premium. There are a several complex methods to value a barrier option i.e. Law of maximum and minimum distributions, Monte Carlo option model, PDE approach etc.

The aim of this thesis is to demonstrate some of the approaches of valuing Barrier Options. In particular, we will be investigating the eigenfunction expansion approach.

Barrier options are weakly path-dependent and slot very easily into the Black-Scholes framework. The Black-Scholes equation is a partial differential equation for valuing an option as a function of the underlying asset and time. The partial differential equations (PDE) satisfied by a barrier option is the same one satisfied by a vanilla option under Black & Scholes assumptions, but with extra boundary conditions demanding that the option becomes worthless (if it is a knock-out barrier option) or conversely only has value (if it is a knock-in barrier option) when the underlying touches the barrier. We will discuss the types of Barrier options further in section 3.

We also will see later why valuing the Barrier option in particular under the Black-Scholes model is not appropriate. There are other several approaches to value the Barrier option. We will focus on the Eigenfunction Expansion approach to value path dependent options, such as the Barrier option.

2 Black-Scholes Model

2.1 History

Fischer Black, Myron Scholes and Robert Merton's original work was published in two separate papers in 1973. First paper "*The pricing of options and corporate liabilities*" was written by Black & Scholes and the second paper "*Theory of rational option pricing*" was written by Merton. They derived the Black-Scholes equation for options. Black and Scholes (1973) showed that almost all corporate liabilities can be viewed as combinations of options and the equation can be used to derive the discount that should be applied to a corporate bond because of the possibility of default. Merton (1973) expanded the mathematical understanding of the options pricing model and invented the term the Black-Scholes options pricing model.

Fischer Black passed away in August 1995. In October 1997 Myron Scholes and Robert Merton were awarded the Nobel Prize for Economics for their work. *The New York Times* of Wednesday, 15th October 1997 wrote: 'They won the Nobel Memorial Prize in Economic Science for work that enables investors to price accurately their bets on the future, a breakthrough that has helped power the explosive growth in financial markets since the 1970's and plays a profound role in the economics of everyday life.'

The most important principle of financial derivative pricing is the no-arbitrage principle. A portfolio should be self-financing and there should be no opportunities for arbitrage.

Definition 2.1 *No-arbitrage principle is equivalent to the impossibility to invest zero today and receive a nonnegative amount tomorrow with positive probability. In other words, two portfolios having the same payoff at a given future date must have the same price today (law of one price).*

In the derivation of the Black-Scholes equation for valuing an option in terms of the price of the share V(S,T), we must state assumptions in the market for the share and for the option:

1. There are no arbitrage opportunities.

2. Risk-free rate μ and volatility of the share σ are known over the lifetime of the option.
3. The distribution of possible share prices at the end of any finite interval is lognormal.
4. Short selling of share is possible at all times, at the short-term risk free rate.
5. No transaction costs or taxes in buying or selling the share or option
6. All securities are perfectly divisible. Trading can take place continuously.
7. The option is European (can only be exercised at maturity).
8. The underlying share pays no dividends or other distributions.

2.2 Geometric Brownian Motion

Geometric Brownian motion is the most commonly used model for asset prices that follow a lognormal distibution. The Black-Scholes model is based on geometric Brownian motion.

Definition 2.2 *Geometric Brownian motion (GBM) S={S(t):t≥0} is the unique stochastic process satisfying the stochastic differential equation (SDE)*

$$dS(t) = \mu S(t)dt + \sigma S(t)dB(t),$$

where μ and σ>0 are constants and B(t) is a Brownian motion.

Let us use Itô's formula to solve the equation for GBM. We take f(S)=ln S. Then we find the derivatives of f. We need the derivatives

$$f_s = \frac{1}{S} \quad f_{ss} = -\frac{1}{S^2} \quad f_t = 0.$$

Then Itó's formula gives:

$$\begin{aligned}
df(S) &= d\ln S(t) \\
&= f_t(S(t),t) + f_s(S(t),t)dS(t) + \frac{1}{2}\sigma^2(S(t),t)\, f_{ss}(S(t),t)dt \\
&= 0 + \frac{1}{S(t)}dS(t) + \frac{1}{2}\sigma^2\left(-\frac{1}{S(t)^2}\right)S(t)^2 dt \\
&= \frac{1}{S(t)}\big(\mu S(t)dt + \sigma S(t)dB(t)\big) + \left(-\frac{1}{2}\right)\sigma^2 dt \\
&= \mu dt + \sigma dB(t) - \frac{1}{2}\sigma^2 dt
\end{aligned}$$

Therefore

$$d \ln S(t) = \left(\mu - \frac{1}{2}\sigma^2\right) dt \ + \sigma dB(t)$$

$$\begin{aligned}
\ln S(t) &= lnS(0) + \int_0^t \frac{1}{S(u)} dS(u) + \frac{1}{2}\int_0^t -\frac{1}{S^2(u)}\sigma^2 S^2(u) ds \\
&= lnS(0) + \int_0^t \frac{1}{S(u)}\big(\mu S(u)du + \sigma S(u)dB(u)\big) + \frac{1}{2}\int_0^t -\sigma^2 ds \\
&= lnS(0) + \int_0^t \mu \, du + \int_0^t \sigma \, dB(u) - \frac{1}{2}\sigma^2 t \\
&= lnS(0) + \mu t + \sigma B(t) - \tfrac{1}{2}\sigma^2 t
\end{aligned}$$

$$lnS(t) = lnS(0) + \left(\mu - \tfrac{1}{2}\sigma^2\right) t + \sigma B(t).$$

This gives the solution of the Black-Scholes SDE as

$$S(t) = S(0)e^{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma B(t)}.$$

S(t) is a geometric Brownian motion with constant drift and volatility where μ is the rate of return and B(t) is a Brownian motion. It follows from this that the returns follow a log-normal distribution, which is one of the assumptions on how stock prices evolve under the Black-Scholes framework.

Geometric Brownian motion has become the most widely used process for modelling asset prices in modern finance. However, real world asset prices do not follow geometic Brownian motion. We will discuss this further later.

2.3 Deriving the Black-Scholes PDE

The derivation of the famous Black-Scholes partial differential equation uses Itó's lemma and a simple hedging argument. The resulting equation is readily generalisable to allow incorporation of dividends, other payoffs, stochastic volatility, jumping processes etc. to price other options with different underlyings and other exotic contracts. Assuming that the movement of share price follows the geometric Brownian motion given by

$$dS(t) = \mu S(t)dt + \sigma S(t)dX(t).$$

The value V of an option is a function of share price 'S' and time 't'. Applying Itô's Lemma, we get:

$$dV = \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}dS + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}dt.$$

Itô's Lemma finds the change of the option price with changes in the share price. From this follows the idea of hedging. Now consider a portfolio of value Π constructed by longing one option and shorting Δ amount of shares: $\Pi = \mathrm{V(S,t)} - \Delta S$

Differentiating this gives the change in portfolio as

$$\begin{aligned} d\Pi &= d\mathrm{V} - \Delta dS \\ &= \left(\frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}dS + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}dt\right) - \Delta \mathrm{dS}, \end{aligned}$$

to eliminate the randomness associated with the portfolio (the random terms are those with the dS), let $\Delta = \frac{\partial V}{\partial S}$ (the value Δ should be chosen such that the portfolio will earn the same rate of return as other risk-free securities). So change is completely riskless:

$$d\Pi = \frac{\partial V}{\partial t}dt + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}dt$$

Since the change in portfolio is riskless, then it must have the same as the amount if we invest the same amount of outlay in a risk-free interest-bearing account:

let $d\Pi_{\mathrm{riskfree}} = \mathrm{r}\Pi dt$ and $d\Pi = d\Pi_{\mathrm{riskfree}}$

This follows from the no-arbitrage principle.

So $(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2})dt = r\Pi dt$ where $\Pi = \mathrm{V} - \Delta \mathrm{S}$

$$(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2})d\mathrm{t} = \mathrm{r(V - \Delta S)}d\mathrm{t}$$

$$\frac{\partial V}{\partial t}dt + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\mathrm{dt} = \mathrm{rV}d\mathrm{t} - \mathrm{rS}\frac{\partial V}{\partial S}d\mathrm{t}.$$

Therefore the Black-Scholes partial differential equation (PDE) is:

$$\frac{\partial V}{\partial t} + \mathrm{rS}\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - \mathrm{rV} = 0$$

It is a PDE because it has more than one independent variable, the PDE is in two dimensions, S and t. It is a linear second-order parabolic equation because it has a second derivative with respect to S, and a first derivative with respect to the other variable t.

The equation requires two parameters to be known, the risk-free interest rate 'r' and the asset volatility. The volatility is rather harder to forecast accurately compared to the risk-free interest rate 'σ'. The first term of the equation represents how much the option value changes by if the share price doesn't change, the second term represents how much a hedged position makes on average from share moves, third term represents the drift term for the growth in the share at the risk-free rate and the last term is the term discounting the payoff, since the option is being valued at t=0.

In deriving this equation, recall we assumed that:

1. There are no arbitrage opportunities.
2. Risk-free rate μ and volatility of the share σ are known over the lifetime of the option.
3. The distribution of possible share prices at the end of any finite interval is lognormal.
4. Short selling of share is possible at all times, at the short-term risk free rate.
5. No transaction costs or taxes in buying or selling the share or option
6. All securities are perfectly divisible. Trading can take place continuously.
7. The option is European (can only be exercised at maturity).
8. The underlying share pays no dividends or other distributions.

2.4 Transformation into heat equation

It is useful to transform the basic Black-Scholes equation into a constant coefficient diffusion equation. We transform our Black-Scholes partial differential equation into the heat equation by a change of variables. The problem we transform is:

$\frac{\partial V}{\partial t} + rs\frac{\partial V}{\partial s} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - \mathrm{rV} = 0$ with boundary conditions: (2.1)

$V(0,t) = 0 \; for \; all \; t$

$V(0,t) \rightarrow S$ as $S \rightarrow \infty$

$V(S,T)=\max(S_T - K,0)$.

The PDE is clearly in backward form, with final data given at t=T. Let us now change the variable such that:

$$V(S,t) = U(S,T-t) = U(S,\tau)$$

$$\frac{\partial V}{\partial t} = \frac{\partial U}{\partial \tau}\frac{\partial \tau}{\partial t} = -\frac{\partial U}{\partial \tau}$$

Changing the variables will make the PDE dimensionless (have constant coefficients) and turn it into a forward equation. Substitute $\frac{dV}{dt}$ into the PDE (2.1):

$$-\frac{\partial U}{\partial \tau}+rS\frac{\partial U}{\partial S}+\frac{1}{2}\sigma^2S^2\frac{\partial^2 U}{\partial S^2}-\mathrm{rU}=0,$$

hence,

$$\frac{\partial U}{\partial \tau}=rS\frac{\partial U}{\partial S}+\frac{1}{2}\sigma^2S^2\frac{\partial^2 U}{\partial S^2}-\mathrm{rU}. \qquad (2.2)$$

Next we let $x=\ln \mathrm{S}$ So $\mathrm{S}=\mathrm{e}^{\mathrm{x}}$.

$U(S,\tau)=U(e^x,T-t)=\bar{U}(lnS,\tau)$.

Then find the derivatives;

$$\frac{\partial U}{\partial S}=\frac{\partial \bar{U}}{\partial x}\frac{dx}{dS}=\frac{1}{S}\frac{\partial \bar{U}}{\partial x},$$

$$\frac{\partial^2 U}{\partial S^2}=\frac{1}{S}\frac{\partial}{\partial S}\left(\frac{\partial \bar{U}}{\partial x}\right)-\frac{1}{S^2}\frac{\partial \bar{U}}{\partial x}=\frac{1}{S^2}\frac{\partial^2 \bar{U}}{\partial x^2}-\frac{1}{S^2}\frac{\partial \bar{U}}{\partial x}=\frac{1}{S^2}(\bar{U}_{xx}-\bar{U}_x).$$

Therefore substitute the derivatives back in the PDE (2.2):

$$\frac{\partial \bar{U}}{\partial \tau}=rS\frac{1}{S}\frac{\partial \bar{U}}{\partial x}+\frac{1}{2}\sigma^2S^2\left(\frac{1}{S^2}(\bar{U}_{xx}-\bar{U}_x)\right)-\mathrm{r}\bar{U}$$

or

$$\bar{U}_\tau=\frac{1}{2}\sigma^2\bar{U}_{xx}+(r-\frac{1}{2}\sigma^2)\bar{U}_x-\mathrm{r}\bar{U} \qquad (2.3)$$

The change of variables also lets the final condition be seen as an initial condition:

$$\mathrm{V(S,T)}=U(e^x,T-T)=\bar{U}(lnS,0)=\bar{U}(x,0)=\max(e^x-K,0)$$

Next let $u(x,\tau)=e^{\phi(x)+b\tau}w(x,\tau)$ which is a common transformation for parabolic equations.

Given any ODE $\alpha(x)u''+\beta(x)u'+\gamma(x)u=0$ we can always eliminate the u' term.

To do this we use:

$$\begin{aligned}\alpha(x)u''+\beta(x)u'+\gamma(x)u \\ &=\alpha\left(\phi''e^{\phi(x)}w+(\phi')^2e^{\phi(x)}w+2\phi'e^{\phi(x)}w'+e^{\phi(x)}w''\right)\\ &+\beta\left(\phi'e^{\phi(x)}w+e^{\phi(x)}w'\right)+\gamma\left(e^{\phi(x)}w\right)\\ &=e^{\phi(x)}(\alpha w''+(2\alpha\phi'+\beta)w'+(\alpha\phi''+\alpha(\phi')^2+\beta\phi'+\gamma)w)=0 \ (2.4)\end{aligned}$$

Set $\phi'=-\frac{\beta}{2\alpha}=-\frac{\left(r-\frac{1}{2}\sigma^2\right)}{\sigma^2}=\frac{1}{2}-\frac{r}{\sigma^2}$ as this eliminates w'.

So $\phi''=0 \ and \ \phi=\left(\frac{1}{2}-\frac{r}{\sigma^2}\right)x$

And we know from (2.3) $\alpha = \frac{1}{2}\sigma^2$ $\beta = \left(r - \frac{1}{2}\sigma^2\right)$ $\gamma = -$r.

Substitute in (2.4) to get $w_\tau = \frac{1}{2}\sigma^2 w_{xx} + (\frac{1}{2}\sigma^2\left(\frac{r-\frac{1}{2}\sigma^2}{\sigma^2}\right)^2 + \left(r - \frac{1}{2}\sigma^2\right)\left(\frac{r-\frac{1}{2}\sigma^2}{\sigma^2}\right) - r)w$

i.e. $w_\tau = \frac{1}{2}\sigma^2 w_{xx} + cw$ where 'c' is constant

Let $w = e^{c\tau}W$

By the product rule $w_\tau = ce^{c\tau}W + e^{c\tau}W_\tau$, so that

$$ce^{c\tau}W + e^{c\tau}W_\tau = \frac{1}{2}\sigma^2 e^{c\tau}W_{xx} + ce^{c\tau}W.$$

So therefore $W_\tau = \frac{1}{2}\sigma^2 W_{xx}$.

Now we rescale time. Put $\tau = zt$, so that

$W(x,\tau) = u\left(x, \frac{\tau}{z}\right)$. Then

$$W_\tau = u_t \frac{\partial s}{\partial \tau} = \frac{1}{z}u_t$$

So $\frac{1}{z}u_t = \frac{1}{2}\sigma^2 u_{xx}$

or

$$u_t = \frac{z}{2}\sigma^2 u_{xx}.$$

So taking $z = \frac{2}{\sigma^2}$ gives the heat equation:

$$u_t = u_{xx}$$

The initial condition is found by using:

$$u(x,\tau) = e^{\phi(x)+b\tau}w(x,\tau), \text{ so that}$$

$$u(x,0) = e^{\phi(x)}w(x,0).$$

or

$$u(x,0) = e^{\phi(x)}w(x,0) = e^{\left(\frac{1}{2}-\frac{r}{\sigma^2}\right)x}(e^x - K)^+$$

If we let $k = \frac{r}{\frac{1}{2}\sigma^2}$, then

$$u(x,0) = \max\left(e^{\frac{1}{2}(k+1)x} - Ke^{\frac{1}{2}(k+1)x}, 0\right)$$

The simpler constant coefficient diffusion equation is easier to handle than the Black-Scholes equation, when seeking closed-form solutions.

2.5 Solving the heat equation via Fourier Transform

We now have the heat equation. Such equations have been used to model all sorts of physical phenomena. The equation goes back to the beginning of the 19th century. Grindrod (1991) includes many problems successfully modelled by the diffusion equation.

Solving the linear diffusion equation

$$u_t = u_{xx},$$

with the initial condition

$$u(x,0) = \mathrm{f(x)} = \max\left(e^{\frac{1}{2}(k+1)x} - Ke^{\frac{1}{2}(k+1)x}, 0\right) \tag{2.5}$$

will give us the Black-Scholes call option formula. Boundary conditions tell us how the solution must behave for all times at certain values of the asset. We will solve the heat equation by Fourier transform

Definition 2.3 *The Fourier transform of u is:*

$$\hat{u}(y,\tau) = \int_{-\infty}^{\infty} u(x,\tau)e^{-ixy}dx$$

The inverse Fourier transform of $\hat{u}$ is:

$$u(x,\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{u}(y,\tau)e^{iyx}dy$$

Take the Fourier transform of u_{xx} and u_s:

$$\hat{u}(y,\tau) = \int_{-\infty}^{\infty} u(x,\tau)e^{-ixy}dx$$

$\hat{u}_{xx}(y) = \int_{-\infty}^{\infty} u_{xx}(x,\tau)e^{-ixy}dx$ using integration by parts we get

$$= u_x e^{-ixy}|_{-\infty}^{\infty} + iy\int_{-\infty}^{\infty} u_x(x,\tau)e^{-ixy}dx$$

$$= iy\int_{-\infty}^{\infty} u_x(x,\tau)e^{-ixy}dx$$

$$= iyu(x,\tau)e^{-ixy}|_{-\infty}^{\infty} + (iy)^2\int_{-\infty}^{\infty} u(x,\tau)e^{-ixy}dx$$

$$= -y^2\hat{u}(y,\tau).$$

Also

$$\hat{u}_t(y) = \int_{-\infty}^{\infty} u_t(x,\tau)e^{-ixy}dx$$
$$= \frac{\partial}{\partial t}\int_{-\infty}^{\infty} u(x,\tau)e^{-ixy}dx$$

So taking the Fourier transforms of both sides of the heat equation gives

$\hat{u}_t(y) = -y^2\hat{u}(y,\tau)$, and

$$\hat{u}(y,0) = \int_{-\infty}^{\infty} u(x,0)e^{-ixy}dx = \hat{f}(y).$$

Since

$\hat{u}_t(y) = -y^2\hat{u}(y,\tau)$, we find

$$\hat{u}(y,0) = \hat{u}(y,0)e^{-\frac{1}{2}y^2\tau}$$

But because

$\hat{u}(y,0) = \hat{f}(y)$ then

$$\hat{u}(y,\tau) = \hat{f}(y)e^{-y^2\tau} = \hat{u}(y,0)e^{-y^2\tau} = \int_{-\infty}^{\infty} u(z,0)e^{-iyz-y^2\tau}dz.$$

Now by Fourier inversion

$$u(x,\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{u}(y,\tau)e^{iyx}dy$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(y)e^{-y^2\tau}e^{iyx}dy$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u(z,0)e^{-iyz-y^2\tau}\, e^{iyx}dzdy$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u(z,0)e^{-y(iz+y\tau-ix)}\, dzdy$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u(z,0)e^{iy(x-z)-y^2\tau}\, dzdy$$

We reverse the order of integration and complete the square to obtain

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u(z,0)e^{-\tau\left(y-\frac{i}{\tau}(x-z)^2\right)-\frac{(x-z)^2}{4\tau}}\, dydz$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} u(z,0)\sqrt{\frac{\pi}{\tau}}\,e^{-\frac{(x-z)^2}{4\tau}}dz$$
$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi\tau}}u_0(z)e^{-\frac{(x-z)^2}{4\tau}}dz$$

$$= \int_{-\infty}^{\infty} \frac{1}{2\sqrt{\pi\tau}} u_0(z) e^{-\frac{(x-y)^2}{4\tau}} dz$$

The function $K(x,t) = \frac{1}{2\sqrt{\pi\tau}} u_0(z) e^{-\frac{(x-z)^2}{4\tau}}$ is the fundamental solution of the heat equation.

2.6 Pricing a European Call Option

The PDE is readily generalisable to allow incorporation of dividends, other payoffs, stochastic volatility, jumping processes etc. to price other options with different underlying and other exotic contracts. The problem of evaluating the option price has been reduced to computing the standard integral. We can now work out the price of the European Call option with:

$$u(x,\tau) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi\tau}} u_0(z) e^{-\frac{(x-z)^2}{4\tau}} dz$$

where $u_0(z) = \max\,(e^{\frac{1}{2}(k+1)z} - Ke^{\frac{1}{2}(k+1)z}, 0)$. Working out the integration then replacing the original variables will give the price of an European call option under the Black-Scholes model:

$C(S,t) = SN(d_1) - e^{-r(T-t)}KN(d_2)$

where $d_1 = \frac{\ln\left(\frac{S_0}{K}\right)+\left(r+\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$ and $d_2 = \frac{\ln\left(\frac{S_0}{K}\right)+\left(r-\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} = d_1 - \sigma\sqrt{T-t}$

The full details of the integration can be found on page 78 of Paul Wilmott’s book “*The Mathematics of Financial Derivatives*”, 1995.

3 Barrier Options

3.1 Introduction

Definition 2.3 *A barrier option is a path-dependent exotic option, where the exercise price depends on the underlying asset crossing or reaching a given 'barrier level'.*

The most basic type of barrier option is the single barrier option. The main types of single barrier options are:

"Knock-In" options are initially worthless and become active in the event of a predetermined knock-in barrier level is breached (right to exercise is active).

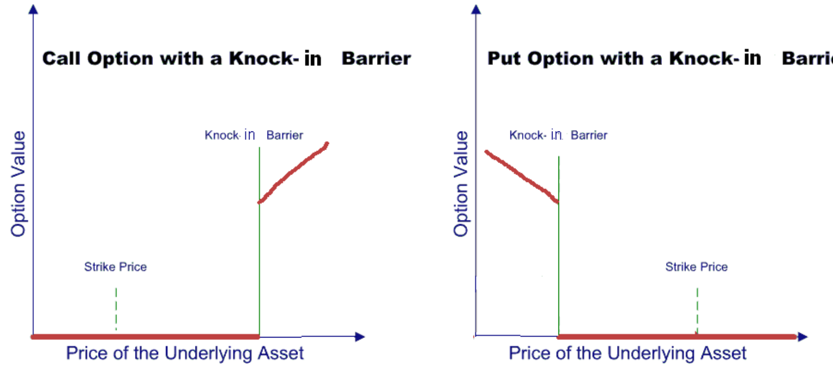

Figure 3.2 Visualising the payoff of a Knock-In barrier.

Taken from http://thismatter.com/money/options/exotic_options.htm

"Knock-Out" options are initially active and become terminated in the event that a predetermined knock-out barrier is violated. Once it's out, it's out for good (right to exercise is lost). The option ceases to exist.

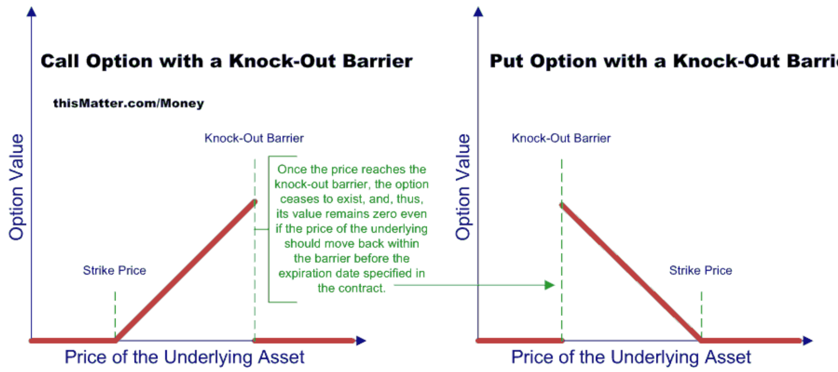

Figure 3.2 Visualising the payoff of a Knock-Out barrier.

Taken from http://thismatter.com/money/options/exotic_options.htm

Barrier events occur when the underlying hits the barrier level. We further characterise the option by the position of the barrier level relative to the initial asset value. The main types of single barrier options are:

*Up & Knock-Out – the spot price is below the barrier level and has to move up for option to be knocked out.

*Down & Knock-Out - the spot price is above the barrier level and has to move down for option to be knocked out.

*Up & Knock-In – the spot price is below barrier level and has to move up for option to become activated.

*Down & Knock-In – the spot price is above barrier level and has to move down for option to become activated.

Another type of barrier option is the double barrier option. The double barrier option has both an upper and lower barrier relative to the current asset price. In a double-out barrier option, contract ceases to exist if either of the barriers is reached. In a double-in barrier option, barriers must be reached before expiry in order to activate the contract.

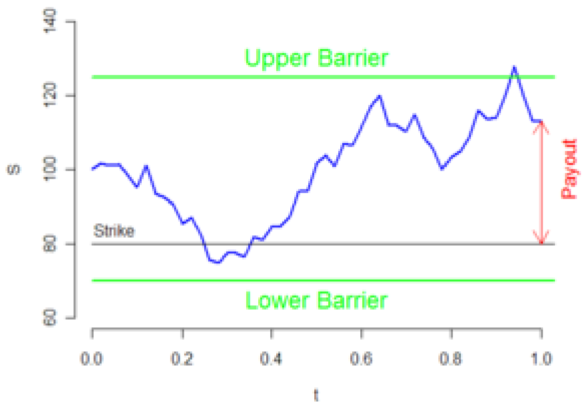

Figure 3.3 Visualising the characteristics of a double knock-out barrier option. Taken from http://formulapages.com/live/OptionBarrierDouble.html

Barriers can take either American or European forms. They have been traded on the OTC since the late 60's and have been used extensively to manage risks related to commodities, FX and interest rate exposures. Barrier options have become popular in the past decade for a number of reasons. An investor who would buy a barrier option, would have very precise views about the direction of the market and will take advantage of the cheaper option, relative to the plain-vanilla option, which doesn't pay for all the upside potential.

Barrier options are sometimes accompanied by a rebate, which is a payoff to the option holder in case of a barrier event. The rebate can either be paid at the time of the event or at expiration. Barrier options with rebates increases the value of the barrier option but are not traded as much as barriers without the rebate.

There are numerous types of other sorts of barrier options. Just to name a few, a discrete barrier is one for which the barrier event is considered at discrete times, rather than the normal continuous barrier case and a Parisian option is a barrier option where the barrier condition applies only once the price of the underlying instrument has spent at least a given period of time on the wrong side of the barrier.

A barrier option is usually cheaper (premium is low) than a standard plain-vanilla European call option because of the limited profit the barrier option holder may have. Whereas the standard European call option may have unlimited payoff profit and the writer holds a higher risk potential compared to the barrier option. For example, in order to compensate for the limited profit the holder of a down-and-out call option, the writer of the down-and-out call option may reduce the premium (cost of the option). Barrier options were created to provide the insurance value of an option without charging as much premium.

The buyer of the down-and-out call option might believe the stock price of a company will go up this year and is willing to bet it won't go below a certain level (barrier level). Whereas the writer of the down-and-out option believes that the stock price of a company will fall below a certain level (barrier level) and will make money from the premium. Remember that once a "out" barrier option is knocked out, it's out for good (if the underlying goes back above the barrier level, option is still void, the right to exercise is already lost once it's knocked out).

Merton (1973), see [27], provided the first analytical formula for a down-and-out option which was followed by a more detailed paper by Reiner & Rubinstein (1991), see [32], which provides the formulas for all single barriers.

There are fundamentally two different ways to price barrier options. They are the expectations approach of Rubenstein and Reiner (1991), see [32], and the PDE approach explained by Wilmott, see [39]. In this section we will tend to focus more on the latter approach. The expectations approach requires the calculation of the risk-neutral densities (often difficult to calculate) as the barrier level is breached, which involves the reflection principle of the Brownian motion. See Konstandatos [25]. The PDE approach involves solving the Black-Scholes PDE, similar to section 2.4, but with extra boundary conditions to account for the barrier events.

3.2 Barrier Option PDE

The barrier option is weakly path dependent. We only have to know whether or not and when the barrier has been reached. The value of the barrier option still satisfies the Black-Scholes equation for the plain-vanilla option

$$\frac{\partial V}{\partial t} + rs\frac{\partial V}{\partial s} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - \mathrm{rV} = 0$$

but has extra boundary conditions that account for the barrier events and in different domains.

So the boundary conditions for an up and out barrier call option are:

$V(S_u, T) = 0$ for when the barrier is triggered or $V(S_u, T) = R$ if a rebate is paid when the barrier is triggered and

$V(S, T) = \max(S - K, 0)$ for when the barrier is not triggered.

The boundary conditions for a down and out barrier call option are:

$V(S_d, T) = 0$ for when the barrier is triggered or $V(S_d, T) = R$ if a rebate is paid when the barrier is triggered and

$V(S, T) = \max(S - K, 0)$ for when the barrier is not triggered.

The boundary conditions for an up and in barrier call option are:

$V(S_u, T) = \max(S - K, 0)$ for when the barrier is triggered and

$V(S, T) = 0$ for when the barrier is not triggered.

The boundary conditions for a down and in barrier call option are:

$V(S_d, T) = \max(S - K, 0)$ for when the barrier is triggered and

$V(S, T) = 0$ for when the barrier is not triggered.

The boundary conditions for a double barrier out call option are:

$V(S_u, T) = 0$ or $V(S_d, T) = 0$ for when either barrier is triggered and

$V(S, T) = \max(S - K, 0)$ for when the barrier is not triggered.

The boundary conditions for a double barrier in call option are:

$V(S_u, T) = \max(S - K, 0)$ or $V(S_d, T) = \max(S - K, 0)$ for when either barrier is triggered and

$V(S, T) = 0$ for when the barrier is not triggered.

S_u is the upper barrier and S_d is the lower barrier.

3.3 Method of Images

The theory of the method of images can be used to price a barrier type option. Otto Konstandatos, see [25], prices all eight of the classic barrier options using the method of images in his book. The amount of calculations can be minimised by noting symmetries inbuilt in the structure of the barrier problems.

The traditional approach to solving the Black-Scholes PDE, illustrated in section 2.4, is to transform to the heat equation, and to solve it by Fourier transform to find the fundamental solution or to use the well known Green's function to solve the problem in the transformed variables. Either way, we'll end up with the same solution:

$$u(x,\tau) = \int_{-\infty}^{\infty} G(x,z,t) f(z) dz$$

where

$$G(x,z,\tau) = \frac{1}{2\sqrt{\pi k\tau}} e^{-\frac{(x-z)^2}{4k\tau}}$$

is the given Green's function.

We then seek a solution of the initial value problem for the Heat Equation in the truncated domain $x > a$ for $u(x,\tau)$.

Theorem 3.1 *(Method of Images for the Heat Equation)*

To solve the problem:

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} \ \ for \ \ x > a \ \ and \ \ t > 0$$

$$u(x,0) = f(x)$$

$$u(a,\tau) = 0 \ \ for \ \ t > 0$$

we first solve the related full-range problem with the introduction of an indicator function 1 in the initial condition:

$$\frac{\partial u_a}{\partial \tau} = \frac{\partial^2 u_a}{\partial x^2} \ \ for \ \ x > a \ \ and \ \ t > 0$$

$$u_a(x,0) = f(x) 1_{\{x>a\}}$$

The solution for u(x,t) is then given by:

$$u(x,\tau) = u_a(x,t) - u_a(2a-x,t)$$

See Konstandatos [25].

The solution to the original problem is obtained by back-transforming the solution for the transformed heat problem form space back to Black-Scholes space.

3.4 Pricing Formulas of Single Barrier Options

Paul Wilmott's (2006), see [38], Quantitative Finance book summarises all the pricing formulas of single barrier options in his Barrier Options chapter. To summarise here:

Use N(*) to denote the culmulative distribution function for a standardised Normal variable, q is the dividend yield, S_b is the barrier level,

$$a = \left(\frac{S_b}{S}\right)^{-1+\left(\frac{2(r-q)}{\sigma^2}\right)}, \quad b = \left(\frac{S_b}{S}\right)^{1+\left(\frac{2(r-q)}{\sigma^2}\right)},$$

$$d_1 = \frac{\log\left(\frac{S}{K}\right) + \left(r - q + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2 = \frac{\log\left(\frac{S}{K}\right) + \left(r - q - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_3 = \frac{\log\left(\frac{S}{S_b}\right) + \left(r - q + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_4 = \frac{\log\left(\frac{S}{S_b}\right) + \left(r - q - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_5 = \frac{\log\left(\frac{S}{S_b}\right) - \left(r - q - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_6 = \frac{\log\left(\frac{S}{S_b}\right) - \left(r - q + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_7 = \frac{\log\left(\frac{SK}{S_b^2}\right) - \left(r - q - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_8 = \frac{\log\left(\frac{SK}{S_b^2}\right) - \left(r - q + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}$$

- **Up-and-Out Call**

$$Se^{-q(T-t)}\left(N(d_1)-N(d_3)-b\big(N(d_6)-N(d_8)\big)\right)-Ke^{-r(T-t)}\left(N(d_2)-N(d_4)-a\big(N(d_5)-N(d_7)\big)\right)$$

- **Up-and-In Call**

$$Se^{-q(T-t)}\left(N(d_3)+b\big(N(d_6)-N(d_8)\big)\right)-Ke^{-r(T-t)}\left(N(d_4)+a\big(N(d_5)-N(d_7)\big)\right)$$

- **Down-and-Out Call**

  If K> S_b

$$Se^{-q(T-t)}\left(N(d_1)-b\big(1-N(d_8)\big)\right)-Ke^{-r(T-t)}\left(N(d_2)-a\big(1-N(d_7)\big)\right)$$

  If K< S_b

$$Se^{-q(T-t)}\left(N(d_3)-b\big(1-N(d_6)\big)\right)-Ke^{-r(T-t)}\left(N(d_4)-a\big(1-N(d_7)\big)\right)$$

- **Down-and-In Call**

  If K> S_b

$$Se^{-q(T-t)}\left(b\big(1-N(d_8)\big)\right)-Ke^{-r(T-t)}\left(a\big(1-N(d_7)\big)\right)$$

  If K< S_b

$$Se^{-q(T-t)}\left(N(d_1)-N(d_3)+b\big(1-N(d_6)\big)\right)-Ke^{-r(T-t)}\left(N(d_2)-N(d_4)+a\big(1-N(d_5)\big)\right)$$

- **Up-and-Out Put**

  If K> S_b

$$Ke^{-r(T-t)}\big(1-N(d_4)-aN(d_5)\big)-Se^{-q(T-t)}\big(1-N(d_3)-bN(d_6)\big)$$

  If K< S_b

$$Ke^{-r(T-t)}\big(1-N(d_2)-aN(d_7)\big)-Se^{-q(T-t)}\big(1-N(d_1)-bN(d_8)\big)$$

- **Up-and-In Put**

  If K> S_b

$$Ke^{-r(T-t)}\big(N(d_4)-N(d_2)+aN(d_5)\big)-Se^{-q(T-t)}\big(N(d_3)-N(d_1)+bN(d_6)\big)$$

  If K< S_b

$$Ke^{-r(T-t)}\big(aN(d_7)\big)-Se^{-q(T-t)}\big(bN(d_8)\big)$$

- **Down-and-Out Put**

$$Ke^{-r(T-t)}\left(N(d_4) - N(d_2) - a\big(N(d_7) - N(d_5)\big)\right) - Se^{-q(T-t)}\left(N(d_3) - N(d_1) - b\big(N(d_8) - N(d_6)\big)\right)$$

- **Down-and-In Put**

$$Ke^{-r(T-t)}\left(1 - N(d_4) + a\big(N(d_7) - N(d_5)\big)\right) - Se^{-q(T-t)}\left(1 - N(d_3) + b\big(N(d_8) - N(d_6)\big)\right)$$

4 Pricing Barrier Options on Scalar Diffusions

4.1 An Eigenfunction Expansion Approach

Linetsky and Davydov (2003) worked out how to price a whole range of options under any process. Linetsky and Davydov stated in an earlier paper (2001), that the prices of options which depend on extrema, such as the path-dependent barrier options, can be much more sensitive to the specification of the underlying price process than standard plain-vanilla options and shows that a financial institution that uses the standard geometric Brownian motion assumption is exposed to significant pricing and hedging errors when dealing in path-dependent options.

Linetsky and Davydov (2003), see [18], developed an option pricing methodology based on unbundling all contingent claims into portfolios of primitive securities called eigensecurities (eigenfunctions). They solved the pricing PDE by separation of variables obtaining eigenfunctions and the eigenvalues of the pricing operator. The pricing is then immediate by the linearity of the pricing operator and the eigenvector property of eigensecurities. We will apply this to double-barrier options under the geometric Brownian process and constant elasticity of variance (CEV) process.

4.1.1 General Set-Up of Pricing a Double-Barrier Option

Under an equivalent martingale measure Q, the underlying asset price follows a scalar diffusion process (Itó Process)

$$dX_t = b(X_t)dt + a(X_t)dB_t,$$

with generator:

$$(Gf)(x) = \frac{1}{2}a^2(x)f''(x) + b(x)f'(x),$$

where $\{B_t, t \geq 0\}$ is a standard Brownian motion.

Let f be a square-integrable function on I that claim pays off an amount $f(X_T)$ at expiration T>0 if the process X does not leave the interval I=(L,U), such that the claim pays prior to expiration, and zero otherwise. Then the price of this double-barrier claim at time t=0 is given by the risk-neutral expectation of the discounted payoff:

$$V(x,T) = E_x[e^{-\int_0^T r(X_t)dt} f(X_T) 1_{\{F_{(L,U)}>T\}}].$$

Where E_x denotes expectation assuming process $X_0 = x$ and $F_{(L,U)}=\inf\{t\geq 0: X_t \notin [L,U]\}$ is the first exit time from (L,U) that occurs after T and $1_{\{A\}}$ is the indicator function of the event A.

The following proposition (from the Linetsky Davydov (2003 paper)) summarises the eigenfuntion expansion method:

Proposition 4.1

To price an option using the eigenfunction expansion method:

1. *Find the scale function of the particular scalar diffusion process*

$$s(x) = e^{-\int^{x}\frac{2b(y)}{a^2(y)}dy} \qquad (4.1)$$

2. *Find the speed density of the particular scalar diffusion process*

$$m(x) = \frac{2}{a^2(x)s(x)}$$

3. *Find the eigenfunctions (eigenvectors) of the pricing operator*

   Let H=L^2([L,U],m) be the Hilbert Space of functions on (L,U) square integrable with the m(x) endowed with the inner product $< f, g \geq= \int_L^U f(x)g(x)m(x)dx$.

   Then:

   i) *H admits a complete orthonormal basis $\{\psi_n(x)\}_{n=1,}^{\infty}$*

   $$< \psi_n, \psi_m \geq= \int_L^U \psi_n\psi_m m(x)dx = 1 \; if \; n = m$$

   $$< \psi_n, \psi_m \geq= \int_L^U \psi_n\psi_m m(x)dx = 0 \; if \; n \neq m,$$

   Such that $\psi_n$are eigenfunctions of the pricing operator (eigenvector property)

   $$V(x,T) = E_x\left[e^{-\int_0^T r(X_t)dt}1_{\{F_{(L,U)}>T\}}\psi_n(X_T)\right] = e^{-\lambda_n T}\psi_n(x)$$

   $with\; \lambda_n \to \infty \; as \; n \to \infty.$

   If $\psi_n$has this eigenvector property for some λ, then its price time 0 is $V(x,T) = e^{-\lambda T}\psi(x)$

   Any payoff f∉H is in the span of eigenpayoffs ψ_n:

   $$f = \sum_{n=1}^{\infty} c_n\psi_n \quad c_n =< f, \psi_n >$$

   and convergence of the eigenfunction expansion f is in the norm of the Hilbert space.

ii) *Let A be the second order differential operator (the negative of the infinitesimal generator of the pricing semi-group)*

$$(Af)(x) = -\frac{1}{2}a^2(x)f''(x) - b(x)f'(x) + r(x)f(x)$$

where r(x) is the risk-free rate. This can be reduced in terms of the scale and speed density function of the scalar diffusion:

$$(Af)(x) = -\frac{1}{m(x)}\left(\frac{f'(x)}{s(x)}\right)' + r(x)f(x)$$

This can be seen as a second-order ODE of the regular Sturm-Liouville type $(Au)(x) = \lambda u(x)$ *with the two Dirichlet boundary conditions u(L)=0 and u(U)=0.*

4. *Find the eigenvalue corresponding to the eigenfunction.*
5. *Compute the coefficient* c_n *to satisfy the initial condition by integrating the payoff against the eigenfunction. Coefficients* c_n *in* $f = \sum_{n=1}^{\infty} c_n\psi_n$ *are determined by calculating the inner products of the payoff function with the eigenpayoffs*

$$c_n = < f, \psi_n > = \int_L^U f(x)\psi_n(x)m(x)dx.$$

6. *Value of the option is found – this is the price of the option. Finally the pricing formula* $V(x,T) = \sum_{n=1}^{\infty} c_n e^{-\lambda_n T}\psi_n(x)$ *follows from the eigenfunction expansion of the payoff* $f = \sum_{n=1}^{\infty} c_n\psi_n$ *, the linearity of the pricing operator, and the eigenvector property of*

$$\psi_n => E_x\left[e^{-\int_0^T r(X_t)dt}\psi_n(X_T)1_{\{F_{(L,U)}>T\}}\right] = e^{-\lambda_n T}\psi_n(x).$$

Note: (4.1) Linetsky defines the scale function as the derivative of the original scale function

4.1.2 Separation of Variables

As stated earlier, partial differential equations can be solved to predict phenomena. Typically, in describing natural phenomena the dependent variable will depend on one or more space variables and time t. The goal is to find the function (i.e the solution) that satisfies the partial differential equation and the initial and boundary conditions.

Separation of Variables is one of the most important methods for solving partial differential equations with boundary conditions. This technique reduces a partial differential equation in n independent variables to n ordinary differential equations. We use the separation of variables in the eigenfunction expansion approach.

The first step is to find product solutions which are separated by a separable constant eigenvalue. Next is to find the fundamental solutions by satisfying the boundary conditions. We will find the eigenvalues of the problem, and the corresponding solutions are called the eigenfunctions of the problem. These fundamental solutions which satisfies the PDE and the boundary equations are the building blocks for the final solution. The last step is the summing of the fundamental solutions, which also satisfies the coefficients.

4.1.3 Sturm-Liouville Problem

Proposition 4.1 unbundles any European-style, double-barrier contingent claim with the payoff in H into a portfolio of eigensecurities with eigenpayoffs ψ_n. The pricing is then automatic by the linearity of the pricing operator and the eigenvector property of the eigenpayoffs $e^{-\lambda_n T}\psi_n(x)$. This is accomplished by solving the regular Sturm-Liouville boundary value problem:

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + q(x)y = \lambda r(x)y \tag{4.2}$$

with boundary conditions:

$$\alpha_1 y(a) - \alpha_2 y'(a) = 0$$

$$\beta_1 y(b) - \beta_2 y'(b) = 0$$

$(a < x < b)$ with neither $\alpha_1$and α_2 both zero nor $\beta_1$and β_2 are both zero.

The Swiss mathematician Jacques Sturm and the French mathematician Joseph Liouville studied the solutions of this type of second-order ordinary differential equations under appropriate boundary conditions. These solutions are complete orthogonal sets of functions in L^2. The differential equations considered here arise directly as mathematical models of motion according to Newton's law, but more often as a result of using the method of separation of variables to solve the classical partial differential equations of physics.

Theorem 4.1 *All eigenvalues of problem (4.2) are real. See [2].*

Theorem 4.2 *The eigenfunctions of problem (4.2) form an orthogonal basis for* $L^2([a,b], r(x))$*. See [2].*

4.2 Pricing a double-barrier option under the Geometric Brownian Motion

In this section we will price a double-barrier option under geometric Brownian motion. Assume that under the risk-neutral measure Q the underlying asset price follows a geometric Brownian motion

$$S(t) = S(0)e^{\left(r-q-\frac{1}{2}\sigma^2\right)t+\sigma B(t)} = S(0)e^{\nu\sigma t+\sigma B(t)} = S(0)e^{\sigma(\nu t+B(t))}$$

$t \geq 0$ *where* $\{B(t), t \geq 0\}$ is a S.B.M, σ>0 is the constant volatility, S(0)>0 is the initial asset price at t=0, r is the risk-free rate, q is the dividends and $\nu = \frac{1}{\sigma}(r - q - \frac{1}{2}\sigma^2)$.

This scalar diffusion process solves the SDE

$$dS(t) = (r-q)S(t)dt + \sigma S(t)dB(t) \qquad S(0) = S$$

We will price a double-barrier knock-out call option that has strike K, expiration T and two knock-out barriers 0≤L<K<U, with payoff

$$1_{\{F_{(L,U)}>T\}}(S(T)-K)^+ where \;\; F_{(L,U)}=\inf\{t\geq 0:S(t)\notin[L,U]\} \text{ and } x^+\equiv\max\{x,0\}.$$

Then the double-barrier call price at t=0 is given by the risk-neutral expectation of the discounted payoff $C(S,T) = e^{-rT}E_S\left[1_{\{F_{(L,U)}>T\}}(S(T)-K)^+\right]$

In order to price the option we proceed as follows:

1. The scale function of the Geometric Brownian Motion

$$s(x) = e^{-\int^x \frac{2b(y)}{a^2(y)}dy}$$

$$= \exp\{-\int^x \frac{2(r-q)y}{\sigma^2 y^2} dy\}$$

$$= \exp\{-\int^x \frac{2(r-q)}{\sigma^2 y} dy\}$$

$$= exp\left\{-\frac{2(r-q)}{\sigma^2}\ lnx\right\}$$

$$= x^{\frac{2(q-r)}{\sigma^2}}$$

Linetsky defines $v = \frac{1}{\sigma}\left(r - q - \frac{1}{2}\sigma^2\right)$, hence:

$$s(x) = x^{\frac{-2\left(r-q-\frac{1}{2}\sigma^2\right)}{\sigma^2}-1} = x^{\frac{-2v}{\sigma}-1}$$

2. The speed density of the Geometric Brownian Motion is

$$m(x) = \frac{2}{a^2(x)s(x)}$$

$$= \frac{2}{\sigma^2 x^2 x^{\frac{-2v}{\sigma}-1}}$$

$$= \frac{2x^{-1+\frac{2v}{\sigma}}}{\sigma^2}.$$

3. We find the eigenfunctions (eigenvectors) of the pricing operator and the corresponding eigenvalue.

Let H=L^2([L,U],m) be the Hilbert Space of functions on (L,U) square integrable with the m(x) endowed with the inner product

$$< f, g \geq= \int_L^U f(x)g(x)m(x)dx$$

Let A be given by:

$$(Af)(x) = -\frac{1}{2}a^2(x)f''(x) - b(x)f'(x) + r(x)f(x)$$

$$= -\frac{1}{m(x)}\left(\frac{f'(x)}{s(x)}\right)' + rf(x)$$

$$= \left(\frac{f'(x)}{s(x)}\right)' - rm(x)f(x) = -\lambda m(x)f(x).$$

This is a second-order ODE of the regular Sturm-Liouville type

$(Af)(x) = \lambda u(x)$ with $r(x) = r$ is constant and the two Dirichlet boundary conditions f(L)=0 and f(U)=0. We can reduce this ODE to:

$$\left(\frac{f'(x)}{s(x)}\right)' + (\lambda - r)m(x)f(x) = 0$$

which is

$$\left[x^{\frac{2v}{\sigma}+1} f'(x)\right]' + (\lambda - r)\frac{2x^{-1+\frac{2v}{\sigma}}}{\sigma^2} f(x) = 0$$

$$\left[x^{\frac{2v}{\sigma}+1} f''(x) + \left(\frac{2v}{\sigma}+1\right)x^{\frac{2v}{\sigma}} f'(x)\right] + (\lambda - r)\frac{2x^{-1+\frac{2v}{\sigma}}}{\sigma^2} f(x) = 0$$

or

$$x^2 f''(x) + \left(\frac{2v}{\sigma}+1\right)x f'(x) + (\lambda - r)\frac{2}{\sigma^2} f(x) = 0.$$

To solve this let x=e^{t}, this converts the ODE into a constant conversion equation. We want the conditions:

$f(x) = F(\ln x)$ $t = \ln x$

$f(L) = f(U) = 0$ transform to

$F(\ln L) = F(\ln U) = 0$

Then,

$$\frac{df}{dx} = \frac{dF}{dt}\frac{dt}{dx} = \frac{1}{x}\frac{dF}{dt} \ and \ \frac{d^2f}{dx^2} = \frac{1}{x^2}F'' - \frac{1}{x^2}F'$$

Thus,

$$x^2 f''(x) + \left(\frac{2v}{\sigma}+1\right)x f'(x) + (\lambda - r)\frac{2}{\sigma^2} f(x)$$

$$= F'' - F' + \left(\frac{2v}{\sigma}+1\right)F' + (\lambda - r)\frac{2}{\sigma^2}F = 0$$

i.e $F'' + \left(\frac{2v}{\sigma}\right)F' + (\lambda - r)\frac{2}{\sigma^2}F = 0$, and

$F(\ln L) = F(\ln U) = 0$

For solutions to

$$F(t) = e^{at}, F'(t) = ae^{at}, F''(t) = a^2e^{at}.$$

Hence

$$a^2 + \left(\frac{2v}{\sigma}\right)a + (\lambda - r)\frac{2}{\sigma^2} = 0.$$

The quadratic formula gives:

$$a = \frac{\frac{-2v}{\sigma} \pm \sqrt{\frac{4v^2}{\sigma^2} - \frac{8}{\sigma^2}(\lambda - r)}}{2}.$$

Thus

$$F(t) = Ae^{\frac{1}{2}(\frac{-2v}{\sigma} + \sqrt{\frac{4v^2}{\sigma^2} - \frac{8}{\sigma^2}(\lambda - r)})t} + Be^{\frac{1}{2}(\frac{-2v}{\sigma} - \sqrt{\frac{4v^2}{\sigma^2} - \frac{8}{\sigma^2}(\lambda - r)})t}$$

which is equivalent to

$$F(t) = e^{\frac{-vt}{\sigma}}(Ae^{\frac{1}{2}\sqrt{\frac{4v^2}{\sigma^2} - \frac{8}{\sigma^2}(\lambda - r)}t} + Be^{-\frac{1}{2}\sqrt{\frac{4v^2}{\sigma^2} - \frac{8}{\sigma^2}(\lambda - r)}t}).$$

If $\frac{4v^2}{\sigma^2} - \frac{8}{\sigma^2}(\lambda - r) > 0$ or $\frac{4v^2}{\sigma^2} - \frac{8}{\sigma^2}(\lambda - r) = 0$,

these give A=B=0..

If $\frac{4v^2}{\sigma^2} - \frac{8}{\sigma^2}(\lambda - r) < 0 \;\; A \neq B \neq 0$

So put $\frac{4v^2}{\sigma^2} - \frac{8}{\sigma^2}(\lambda - r) = -k^2$

This gives us the form of the eigenfunctions:

$$F(t) = e^{\frac{-vt}{\sigma}}\left(Acos\left(\frac{kt}{2}\right) + Bsin\left(\frac{kt}{2}\right)\right)$$

The eigenvalues are chosen by requiring $F(\ln L) = F(\ln U) = 0$;

$$F(\ln U) = e^{\frac{-vlnU}{\sigma}}\left(Acos\left(\frac{klnU}{2}\right) + Bsin\left(\frac{klnU}{2}\right)\right) = 0$$

and

$$F(\ln L) = e^{\frac{-vlnL}{\sigma}}\left(Acos\left(\frac{klnL}{2}\right) + Bsin\left(\frac{klnL}{2}\right)\right) = 0.$$

We let $A = -\frac{BSin(\frac{klnU}{2})}{Cos(\frac{klnU}{2})}$, hence

$$Acos\left(\frac{klnL}{2}\right) + Bsin\left(\frac{klnL}{2}\right)$$

$$= \left(-\frac{BSin\left(\frac{klnU}{2}\right)}{Cos\left(\frac{klnU}{2}\right)}Cos\left(\frac{klnL}{2}\right) + Bsin\left(\frac{klnL}{2}\right)\right)$$

$$= -BSin\left(\frac{klnU}{2}\right)Cos\left(\frac{klnL}{2}\right) + BSin\left(\frac{klnL}{2}\right)Cos\left(\frac{klnU}{2}\right)$$

$$=B(Sin\left(\frac{klnU}{2}\right)Cos\left(\frac{klnL}{2}\right) + Sin\left(\frac{klnL}{2}\right)Cos\left(\frac{klnU}{2}\right)) = 0$$

and we want $B \neq 0$.

So $Sin\left(\frac{klnU}{2}\right)Cos\left(\frac{klnL}{2}\right) + Sin\left(\frac{klnL}{2}\right)Cos\left(\frac{klnU}{2}\right) = 0$

Using the trig identity SinxCosy+SinyCosx=Sin(x-y) we can reduce this to

$$Sin\left(\frac{klnU}{2}-\frac{klnL}{2}\right)=0$$

$$Sin(\frac{k}{2}(\ln\ U-\ln L))=0$$

$$\frac{k}{2}(\ln\ U-\ln L)=n\pi$$

Therefore $k=\frac{2n\pi}{\ln\left(\frac{U}{L}\right)}$.

Recall that $\frac{4v^2}{\sigma^2}-\frac{8}{\sigma^2}(r-\lambda)=-k^2$.

Thus

$$\frac{4v^2}{\sigma^2}-\frac{8}{\sigma^2}(\lambda-r)=-\left(\frac{2n\pi}{\ln\left(\frac{U}{L}\right)}\right)^2.$$

Solving gives the eigenvalues

$$\lambda=\frac{\sigma^2}{8}\left[\left(\frac{2n\pi}{\ln\left(\frac{U}{L}\right)}\right)^2+\frac{4v^2}{\sigma^2}\right]+r,$$

or

$$\lambda=r+\frac{v^2}{2}+\frac{\sigma^2n^2\pi^2}{2ln^2(\frac{U}{L})}\quad .$$

To find the corresponding eigenfunction, substitute $A=-\frac{BSin(\frac{klnU}{2})}{Cos(\frac{klnU}{2})}$ in F(t):

$$F(t)=e^{\frac{-vt}{\sigma}}\left(\frac{BSin\left(\frac{klnU}{2}\right)}{Cos\left(\frac{klnU}{2}\right)}Cos\left(\frac{kt}{2}\right)+Bsin\left(\frac{kt}{2}\right)\right)$$

$$=e^{\frac{-vt}{\sigma}}\left(\frac{Sin\left(\frac{klnU}{2}\right)Cos\left(\frac{kt}{2}\right)+sin\left(\frac{kt}{2}\right)Cos\left(\frac{klnU}{2}\right)}{Cos\left(\frac{klnU}{2}\right)}\right)$$

$$=e^{\frac{-vt}{\sigma}}\left(\frac{Sin\left(\frac{klnU}{2}-\frac{kt}{2}\right)}{Cos\left(\frac{klnU}{2}\right)}\right)=\ e^{\frac{-vt}{\sigma}}\left(\frac{Sin\left(\frac{k}{2}(\ln U-t)\right)}{Cos\left(\frac{klnU}{2}\right)}\right).$$

Recall that $x=e^t$ and $k=\frac{2n\pi}{\ln\left(\frac{U}{L}\right)}$ so

$$F(t)=\ e^{\frac{-vt}{\sigma}}\left(\frac{Sin\left(\frac{k}{2}(\ln U-t)\right)}{Cos\left(\frac{klnU}{2}\right)}\right)$$

$$= x^{\frac{-v}{\sigma}}\left(\frac{Sin\left(\frac{n\pi}{\ln\left(\frac{\mathrm{U}}{\mathrm{L}}\right)}(\ln U - \ln x)\right)}{Cos\left(\frac{n\pi \ln U}{\ln\left(\frac{\mathrm{U}}{\mathrm{L}}\right)}\right)}\right)$$

$$= x^{\frac{-v}{\sigma}}\left(\frac{Sin\left(\frac{n\pi}{\ln\left(\frac{\mathrm{U}}{\mathrm{L}}\right)}(-\frac{\ln x}{\ln U})\right)}{Cos\left(\frac{n\pi \ln U}{\ln\left(\frac{\mathrm{U}}{\mathrm{L}}\right)}\right)}\right)$$

$$= -x^{\frac{-v}{\sigma}}\left(\frac{Sin\left(\frac{n\pi}{\ln\left(\frac{\mathrm{U}}{\mathrm{L}}\right)}(\frac{\ln x}{\ln U})\right)}{Cos\left(\frac{n\pi \ln U}{\ln\left(\frac{\mathrm{U}}{\mathrm{L}}\right)}\right)}\right)$$

$$= x^{\frac{-v}{\sigma}}\left(Sin\left(\frac{n\pi}{\ln\left(\frac{\mathrm{U}}{\mathrm{L}}\right)}(\frac{\ln x}{\ln U})\right)\right).$$

Therefore $\psi(x) = \frac{\sigma}{\sqrt{\ln\left(\frac{U}{L}\right)}}\, x^{\frac{-v}{\sigma}} Sin\left(\frac{n\pi \ln\left(\frac{x}{U}\right)}{\ln\left(\frac{U}{L}\right)}\right)$ are the eigenfunctions.

Multiplying by $\frac{\sigma}{\sqrt{\ln\left(\frac{U}{L}\right)}}$ normalises the eigenfunction.

4. Next we compute the coefficient c_n to satisfy the initial condition by integrating the payoff against the eigenfunction
   The Call Payoff is $f(x)=(x-K)^+$ on [L,U]. Then

$$c_n = < f, \psi_n > = \int_L^U f(x)\psi_n(x)m(x)dx$$

$$= \int_L^U (x-K)^+ . \frac{\sigma}{\sqrt{\ln\left(\frac{U}{L}\right)}}\, x^{\frac{-v}{\sigma}} Sin\left(\frac{n\pi \ln\left(\frac{x}{U}\right)}{\ln\left(\frac{U}{L}\right)}\right) . \frac{2x^{-1+\frac{2v}{\sigma}}}{\sigma^2} dx$$

If $x<K$ i.e. $L \le x \le K$ then $(x-K)^+=0$ So

$$c_n = \frac{2}{\sigma\sqrt{\sigma u}} \int_K^U (x-K)^+ .\ x^{\frac{v}{\sigma}-1} Sin\left(\frac{n\pi \ln\left(\frac{x}{U}\right)}{\ln\left(\frac{U}{L}\right)}\right) dx$$

5. Value of the option is found – this is the price of the option

$$V(x,T) = \sum_{n=1}^{\infty} c_n e^{-\lambda_n T} \psi_n(x)$$

4.3 Pricing a double-barrier option under the Constant Elasticity Process

As stated earlier, a financial institution that prices a path dependent option under the standard geometric Brownian motion assumption is exposed to significant pricing and hedging errors. Barrier options, can be much more sensitive to the specification of the underlying price process than standard plain-vanilla options. Therefore we will price the double-barrier under the constant elasticity of variance (CEV) process of Cox (1975). The constant elasticity of variance (CEV) process is a diffusion model where the volatility is a function of the underlying asset price. Geometric Brownian motion is a special case of the CEV process; take $\beta = 0$.

Assume that under the risk-neutral measure Q the underlying asset price follows a CEV process.

This scalar diffusion process solves the SDE

$$\begin{aligned} dS_t &= b(S_t)dt + a(S_t)dB_t \\ &= \mu S_t dt + \delta S_t^{\beta+1} dB_t\,, \qquad\qquad S(0) = S \end{aligned}$$

where μ risk-neutral drift rate, $r \geq 0$ constant risk-free rate and $q \geq 0$ is the dividend yield and β is the elasticity of the local volatility function.

$\beta < 0$ means the local volatility $\sigma(S) = \delta S^{\beta}$ is a decreasing function of the asset price.

$\beta > 0$ means the local volatility $\sigma(S) = \delta S^{\beta}$ is an increasing function of the asset price.

The parameters β and δ can be interpreted as the elasticity of the local volatility function.

We will price a double-barrier knock-out call option that has strike K, expiration T and two knock-out barriers $0 \leq L < K < U$, with payoff

$1_{\{F_{(L,U)} > T\}}(S(T) - K)^{+}$ *where* $F_{(L,U)} = \inf\{t \geq 0 : S(t) \notin [L,U]\}$ and $x^{+} \equiv \max\{x,0\}$.

Then the double-barrier call price at t=0 is given by the risk-neutral expectation of the discounted payoff $C(S,T) = e^{-rT}E_S\left[1_{\{F_{(L,U)}>T\}}(S(T)-K)^+\right]$.

In order to price the option we proceed as follows:

1. The scale function of the CEV process:

$$s(S) = e^{-\int^S \frac{2b(y)}{a^2(y)}dy}$$

$$= \exp\{-\int^S \frac{2\mu y}{\delta^2 y^{2\beta+2}}dy\}$$

$$= \exp\left\{-\frac{\mu S^{-2\beta}}{\delta^2|\beta|}\right\}.$$

2. The speed density of the CEV process

$$m(S) = \frac{2}{a^2(S)s(S)}$$

$$= \frac{2}{\delta^2 S^{2\beta+2}\exp\left\{-\frac{\mu S^{-2\beta}}{\delta^2|\beta|}\right\}}$$

$$= \frac{2\exp\{\frac{\mu S^{-2\beta}}{\delta^2|\beta|}\}}{\delta^2 S^{2\beta+2}}.$$

3. We find the eigenfunctions (eigenvectors) of the pricing operator and the corresponding eigenvalues.

Let H=L^2([L,U],m) be the Hilbert Space of functions on (L,U) square integrable with the m(x) endowed with the inner product

$$< f,g >= \int_L^U f(x)g(x)m(x)dx$$

Let A be the second order differential operator (the negative of the infinitesimal generator of the pricing semi-group). To find the eigenfunction and corresponding eigenvalues, we solve the equation coming from a CEV Process:

$$(Af)(x) = -\frac{1}{2}a^2(x)f''(x) - b(x)f'(x) + r(x)f(x)$$

$$= -\frac{1}{m(x)}\left(\frac{f'(x)}{s(x)}\right)' + rf(x)$$

$$= \left(\frac{f'(x)}{s(x)}\right)' - rm(x)f(x)$$

$$= -\lambda m(x)f(x)$$

This can be seen as a second-order ODE of the regular Sturm-Liouville type $(Af)(x) = \lambda u(x)$ with $r(x) = r$ a constant and the two Dirichlet boundary conditions f(L)=0 and f(U)=0. We can reduce this ODE to:

$$\left(\frac{f'(x)}{s(x)}\right)' + (\lambda - r)m(x)f(x) = 0$$

which becomes

$$\tfrac{1}{2}S^{2+2\beta}\delta^2 f''(S) + S\mu f'(S) + \lambda f(S) = 0.$$

To solve this let the change of variables be

$$f(S) = e^{-\frac{S^{-2\beta}}{2|\beta|\sigma^2}}\mathrm{S}^{\frac{1}{2}+\beta}\mathrm{w}\left(\frac{S^{-2\beta}\mu}{|\beta|\sigma^2}\right), \tag{4.3}$$

this converts the ODE into a simpler equation.

This gives

$$w''(x) - 2w'(x) + \frac{1}{16\mu^2x^2\beta^2}(\mathrm{A} + 12x^2\beta^2 + Bx) = 0$$

with $A = 4\beta^2 - \mu,\ B = 4\beta + 8\beta^2 + 8\beta\lambda.$

Then taking another transformation $w = e^x v$ gives the simpler equation

$$v'' + \left(\frac{1+2\beta+2\lambda}{4x\beta\mu^2} + \frac{4\beta^2-\mu}{16\mu^2x^2\beta^2} + \left(-\frac{4\mu^2-3}{4\mu^2}\right)\right)v = 0 \tag{4.4}$$

The solution is

$$v(x) = C_1 M_{\varepsilon,\xi}\left(\frac{\sqrt{4\mu^2-3}}{\mu}x\right) + C_2 W_{\varepsilon,\xi}\left(\frac{\sqrt{4\mu^2-3}}{\mu}x\right) \tag{4.5}$$

where $\varepsilon = \frac{1+2\beta+2\lambda}{4\beta\mu\sqrt{4\mu^2-3}}$, $\xi = \frac{\sqrt{4\mu^2\beta^2+\mu-4\beta^2}}{4\beta\mu}$ and $\mu^2 \neq \frac{3}{4}$.

$v(x)$ gives the solution in terms of Whittaker functions where $M_{\varepsilon,\xi}$ and $W_{\varepsilon,\xi}$ are Whittaker functions.

Note: Linetsky and Davydov's, see [18], use the change of variables given here as (4.3) but obtain a different form of Whittaker's equation, (4.4) was verified in Mathematica.

Definition 4.1 *Whittaker's equation is an equation in the form*

$$\frac{d^2w}{dz^2}+\left[-\frac{1}{4}+\frac{\kappa}{z}+\frac{\left(\frac{1}{4}-\mu^2\right)}{z^2}\right]w=0$$

and the solutions to these type of equations are Whittaker's functions:

$$M_{\kappa,\mu}(z)=e^{-\frac{1}{2}z}z^{\frac{1}{2}+\mu}M(\frac{1}{2}+\mu-\kappa,1+2\mu,z)$$

$$W_{\kappa,\mu}(z)=e^{-\frac{1}{2}z}z^{\frac{1}{2}+\mu}U(\frac{1}{2}+\mu-\kappa,1+2\mu,z)$$

$$W_{\kappa,\mu}(z)=\frac{\Gamma(-2\mu)}{\Gamma\left(\frac{1}{2}-\mu-\kappa\right)}M_{\kappa,\mu}(z)+\frac{\Gamma(2\mu)}{\Gamma\left(\frac{1}{2}+\mu-\kappa\right)}M_{\kappa,-\mu}(z).$$

M is Kummer's confluent hypergeometric function

$$M(a,b,z)=\frac{\Gamma(b)}{\Gamma(b-a)\Gamma(a)}\int_0^1 e^{zt}t^{a-1}(1-t)^{b-a-1}dt$$

and U is Tricomi's confluent hypergeometric function

$$U(a,b,z)=\frac{1}{\Gamma(a)}\int_0^\infty e^{-zt}t^{a-1}(1+t)^{b-a-1}dt$$

where

$$\Gamma(z)=\int_0^\infty t^{z-1}e^{-t}\,dt.$$

So the eigenfunctions are $w=e^x(\upsilon(x))$

$$f(S)=e^{-\frac{S^{-2\beta}}{2|\beta|\sigma^2}}\mathrm{S}^{\frac{1}{2}+\beta}\mathrm{w}(\frac{S^{-2\beta}\mu}{|\beta|\sigma^2})$$

Then $f(L)=f(U)=0$

Now let $=\frac{\sqrt{4\mu^2-3}\mu}{\mu|\beta|\sigma^2}=\frac{\sqrt{4\mu^2-3}}{|\beta|\sigma^2}$.

We know that f(L)=0.

Cancelling the factors gives

$$C_1M_{\varepsilon,\xi}\left(\rho L^{-2\beta}\right)+C_2W_{\varepsilon,\xi}\left(\rho L^{-2\beta}\right)=0.$$

Making C_1 the subject, we have:

$$C_1=-\frac{C_2W_{\varepsilon,\xi}\left(\rho L^{-2\beta}\right)}{M_{\varepsilon,\xi}\left(\rho L^{-2\beta}\right)}.$$

Then the eigenvalues are the zeroes of :

$$M_{\varepsilon(n),\xi}\left(\rho L^{-2\beta}\right)M_{\varepsilon(n),\xi}\left(\rho U^{-2\beta}\right)-W_{\varepsilon(n),\xi}\left(\rho L^{-2\beta}\right)W_{\varepsilon(n),\xi}\left(\rho U^{-2\beta}\right)=0$$

as a function of λ. They can be found numerically. If the nth eigenvalue is λ_n, let $\varepsilon(n)=\frac{1+2\beta+2\lambda_n}{4\beta\mu\sqrt{4\mu^2-3}}$.

So the eigenfunctions are

$$\psi_n(S)=e^{-\frac{S^{-2\beta}}{2|\beta|\sigma^2}}S^{\frac{1}{2}+\beta}[M_{\varepsilon(n),\xi}\left(\rho L^{-2\beta}\right)M_{\varepsilon(n),\xi}\left(\rho S^{-2\beta}\right)-W_{\varepsilon(n),\xi}\left(\rho L^{-2\beta}\right)W_{\varepsilon(n),\xi}\left(\rho S^{-2\beta}\right)].$$

4. Next we compute the coefficient c_n to satisfy the initial condition by integrating the payoff against the eigenfunction

The Call Payoff is $f(S)=(S-K)^+$ on [L,U]. Then

$$c_n=<f,\psi_n>=\int_L^U f(S)\psi_n(S)m(S)dx$$

5. Value of the option is found – this is the price of the option

$$V(S,T)=\sum_{n=1}^{\infty}c_n e^{-\lambda_n T}\psi_n(S)$$

5 Conclusion

This thesis shows some of the different methods of pricing the increasingly popular barrier options. It prices barrier options under all different scalar diffusion processes. The eigenfunction expansion method is a very powerful computational tool for derivatives pricing. The eigensecurities (eigenfunctions of the negative infinitesimal generator of the pricing semigroup) are solutions to the static pricing equation (the second-order ordinary differential equation of the Sturm-Liouville type) without the derivative term. The eigenfunction expansion method is readily generalisable to allow incorporation of dividends, other payoffs, stochastic volatility, jumping processes etc. to price other options with different underlyings and other exotic contracts. Further applications of the eigenfunction expansion method in financial engineering will be certainly be investigated by practitioners in the industry and academia.

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