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Simple analytical Green's functions for ray perturbations in layered media

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SUMMARY
The total Green's function for two-point boundary-value problems can be related to the propagator for initial-value problems. A very simple expression for the Green's function is obtained when the unperturbed medium may be described by material with a constant gradient in quadratic slowness. The derivation requires a correct understanding of assumptions made in the propagator solution. Expressions are also obtained for Green's function in multilayered media.

Key words: Green's functions, layered media, perturbation methods, ray theory.

1 INTRODUCTION

In recent years, ray-perturbation theory has found application in many areas of seismology. Farra & Madariaga (1987) have used ray-perturbation theory to show that essentially the same equations describe both the first-order corrections to ray geometry resulting from small fluctuations in the velocity field and the corrections to ray geometry resulting from small changes to the initial conditions when there are no fluctuations in the velocity field (paraxial ray tracing). Farra (1992) has shown that the ray-bending method (Julian & Gubbins 1977) is also a particular application of first-order perturbation theory and is not intrinsically more complicated than the paraxial shooting method. In that paper expressions are given for the solution to the problem in terms of the propagators appropriate to media without interfaces and transformation matrices which ensure that the boundary conditions are satisfied at the various interfaces crossed by the ray. These are combined to form a composite propagator for the problem. Snieder & Sambridge (1992) have also studied the use of ray-perturbation theory in two-point ray tracing. Moore (1993) has shown that the total Green's function for the two-point boundary-value problems studied in Snieder & Sambridge (1992) can be related to the propagator for initial-value problems. In Moore (1993), analytical expressions were obtained for both propagator and total Green's function for the special case in which the ray parameter is taken to be arc length on the unperturbed ray. In this case, the constraints on the perturbation have a simple geometric interpretation. However, the principles used in that derivation do not depend on the choice of ray parameter and so may be applied to other parametrizations of the ray. In this paper, we study the implications for another choice of the ray parameter—specifically a parameter which allows the propagators to be expressed very simply although imposing constraints on the ray perturbation that are geometrically far less intuitive.

2 CHOICE OF RAY PARAMETERS

Moore (1993) like Snieder & Sambridge (1992) incorporated the corrections for differences in arclength in perturbed and unperturbed media at every instant along the ray path and used s_0, the arclength on the unperturbed ray, as the ray parameter. This means that the equation for the first-order perturbation to the ray must be solved subject to the constraint

$$\mathbf{x}^\epsilon \cdot \frac{d\mathbf{x}^0}{ds_0} = 0 \tag{1}$$

or equivalently,

$$\dot{\mathbf{x}}^\epsilon \cdot \dot{\mathbf{x}}^0 = -\frac{1}{u_0}(\mathbf{x}^\epsilon \cdot \nabla u_0)$$

where $u_0 = u_0(\mathbf{x})$ is the unperturbed slowness field of the material, $\epsilon\mathbf{x}^\epsilon$ denotes the first-order perturbation to the ray trajectory $\mathbf{x}^0$ and $\dot{}$ denotes differentiation with respect to s_0.

Moore (1993) gave an analytical solution to the problem when weak lateral inhomogeneities are superimposed on an unperturbed velocity field which varies with depth only. Both plane and spherical geometries were studied. This choice of parametrization means that the ray perturbation is required to lie in the surface s_0 = constant which has the simple geometric interpretation that it is the plane perpendicular to the ray at the point of evaluation. (This is clear from the constraint given in eq. 1a above.)

On the other hand, Moore (1991) also studied the problem for weak lateral inhomogeneities superimposed on an unperturbed velocity field which varies with depth only, but solved the equation for the first-order perturbation to the ray subject to the constraint

$$\dot{\mathbf{x}}^{\epsilon} \cdot \dot{\mathbf{x}}^{0} = 0. \tag{2}$$

This constraint arises from a requirement that $\partial s / \partial s_0$ equals unity to first order in small quantities.

In other words, the constraint was imposed to ensure that the one parameter would measure arclength on both perturbed (s) and unperturbed (s_0) rays (to first order). It follows that the ray perturbation found in the analytical solution of Moore (1991) was required to lie in the surface s = constant, which is the surface composed of all points lying a fixed distance from the source when distance is measured along the (curved) ray path joining the source to these points. Such surfaces can have quite complex shapes.

Another analytical solution to the problem is obtained when the parameter $dw = ds/u$ is used (ds denotes the element of arclength on the ray and u denotes the local slowness of the material). It is well known that a very simple form of the solution occurs in this case when the unperturbed medium may be described by material with a constant gradient in quadratic slowness. However, implicit in the derivation of this result is the assumption that $\partial w / \partial w_0$ equals unity to first order in small quantities, that is, that the parameters on perturbed (w) and unperturbed (w_0) rays have the same scale length (to first order). This means that we choose to use the same parameter on both perturbed and unperturbed rays. Consequently, the perturbation to the ray $\mathbf{x}^{\epsilon}$ describes the correction to the endpoint of the ray **within the surface** w = const. The corresponding constraint on the solution in this case is

$$\dot{\mathbf{x}}^{\epsilon} \cdot \dot{\mathbf{x}}^{0} = \frac{1}{u_0}(\mathbf{x}^{\epsilon} \cdot \nabla u_0) + \frac{\Delta u_0^2}{2u_0^2} \tag{3}$$

where $\dot{}$ denotes differentiation with respect to arclengths s_0, as before, and Δu_0^2 denotes the local perturbation in quadratic slowness.

(In the notation of eqs 10 and 33 of Farra & Madariaga 1989, this constraint is equivalent to the requirement that $\delta H + \Delta H = 0$; that is, that the sum of perturbations to the Hamiltonian resulting from the variations in the ray trajectory (δH) and local variations in slowness (ΔH) must vanish.) The surface w = constant is the surface composed of all points lying a fixed w length from the source as 'measured' along the (curved) ray paths joining the source to these points. Such surfaces again can have quite complex shapes. Nevertheless, the extreme simplicity of the propagators in this case may well outweigh the drawbacks associated with the geometric complexities of the constraint on this solution, and so we investigate the relationship between the propagators and the total Green's function for the two-point boundary-value problem in this case.

3 THEORY

Červený (1987) has noted that the simplest analytical solution for any type of inhomogeneous medium at all is probably the polynomial solution obtained for the case of constant gradient of quadratic slowness:

$$u^2(\mathbf{x}) = u_0^2 + \boldsymbol{\Gamma} \cdot \mathbf{x} \tag{4}$$

where $\boldsymbol{\Gamma}$ defines the gradient. In such a medium, the rays are given by

$$\mathbf{p}(w) = \mathbf{p}(w_s) + \tfrac{1}{2}\boldsymbol{\Gamma}(w - w_s) \tag{5}$$

$$\mathbf{x}(w) = \mathbf{x}(w_s) + \mathbf{p}(w_s)(w - w_s) + \tfrac{1}{4}\boldsymbol{\Gamma}(w - w_s)^2$$

where

$\mathbf{p}(w_s)$ and $\mathbf{x}(w_s)$ are the initial conditions and

$$w = w_s + \int_0^{s_0} \frac{ds''}{u_0(s'')}.$$

The expression for the ray perturbation in the surface w = constant is simply:

$$\begin{pmatrix} \mathbf{x}^{\epsilon} \\ \mathbf{p}^{\epsilon} \end{pmatrix} = \int_{w_s}^{w} \mathscr{P}(w, w') \begin{pmatrix} \mathbf{0} \\ \nabla \frac{1}{2}\Delta u^2 \end{pmatrix} dw' + \mathscr{P}(w, w_s) \begin{bmatrix} \mathbf{x}^{\epsilon}(w_s) \\ \mathbf{p}^{\epsilon}(w_s) \end{bmatrix} \tag{6}$$

where

$$\mathscr{P}(w, w_s) = \begin{bmatrix} \mathbf{I} & (w - w_s)\mathbf{I} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}$$

and Δu^2 is the perturbation in quadratic slowness.

Using $\mathbf{p}^{\epsilon}$ rather than $d\mathbf{x}^{\epsilon}/ds_0$ appears to introduce some complexities into the application of these results. For example, Farra *et al.* (1989) point out at eq. (33b), that an additional correction needs to be applied to $\mathbf{p}$ to account for local fluctuation in u_0^2. However, such corrections are automatically included in the results which follow.

4 GREEN'S FUNCTION FOR CONTINUOUSLY VARYING MEDIA

The method outlined in Moore (1993) for constructing a Green's function for the two-point problem may also be used in this problem.

The Green's function solution to the two-point boundary-value problem above may be written

$$\mathbf{x}^{\epsilon}(w) = \int_{w_s}^{w_r} \mathbf{G}(w, w') \, \nabla(\tfrac{1}{2}\Delta u^2) \, dw' \tag{7}$$

where w_r is the total w length of the unperturbed ray and w is the w length along the unperturbed ray to the point of interest.

This should be compared with the solution given in eq. (6) when $\mathbf{x}^{\epsilon}(w_s) = \mathbf{0}$ and $\mathbf{p}^{\epsilon}(w_s) = \eta$, a constant vector chosen to ensure that $\mathbf{x}^{\epsilon}(w_r) = \mathbf{0}$, that is

$$\eta = \frac{-1}{w_r - w_s} \int_{w_s}^{w_r} (w_r - w') \, \nabla(\tfrac{1}{2}\Delta u^2) \, dw'. \tag{8}$$

It follows that

$$\mathbf{G}(w, w') = G(w_s, w', w, w_r)\mathbf{I},$$

where

$$G(w_s, w', w, w_r) = (w - w')H(w - w') - \frac{(w - w_s)(w_r - w')}{w_r - w_s}$$

$$= \begin{cases} -\dfrac{(w' - w_s)(w_r - w)}{w_r - w_s}, & w \geq w' \\ -\dfrac{(w - w_s)(w_r - w')}{w_r - w_s}, & w < w' \end{cases} \quad (9)$$

w_s is the value of the ray parameter at the source;
w_r is the value of the ray parameter at the receiver;
w is the value of the ray parameter at the point of interest, and
w' is the value of the ray parameter at some intermediate point along the path of integration.

H is the Heaviside function

$$H(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases}.$$

It is interesting to compare the results obtained by this method with exact ray-tracing results. Suppose, for example that the unperturbed medium has a linear distribution of the square of slowness:

$$u_0^2(\mathbf{x}) = \alpha_0^2 + \gamma \cdot \mathbf{x}$$

and that the perturbed slowness distribution may be written

$$u^2(\mathbf{x}) = u_0^2(\mathbf{x}) + \Delta u^2(\mathbf{x}) \quad (10)$$

where

$$\Delta u^2(\mathbf{x}) = \Delta\alpha^2 + \Delta\gamma \cdot \mathbf{x}.$$

Such a perturbation permits an exact solution using eq. (5) as well as an 'approximate' solution using eq. (6) with $\mathbf{x}^\epsilon(w_s) = \mathbf{0}$ and $\mathbf{p}^\epsilon(w_s)$ given by eq. (8). It is readily shown that the two solutions are identical in this case.

In particular, it is found that

$$\eta = -\tfrac{1}{4}(w_r - w_s)\,\Delta\gamma.$$

However, $\begin{pmatrix}\mathbf{0}\\ \eta\end{pmatrix}$ is **not** a combination of initial conditions permitted by eq. (3). Effectively, what we have done is to choose a combination of initial conditions that **changes** the Hamiltonian of the ray to match the Hamiltonian required for the different initial direction. However, all integrations use the ray specified by the **original** Hamiltonian in the unperturbed medium, and the 'correct' solution is obtained despite the fact that the two rays have different Hamiltonians.

5 GREEN'S FUNCTIONS FOR MULTILAYERED MEDIA

Farra *et al.* (1989) and Farra (1992) outline a method for propagating perturbed and paraxial rays across interfaces with the aid of transformation matrices which first extrapolate the perturbed ray onto the interface and then introduce appropriate changes to the direction of propagation as required by Snell's law. In this way, the formal solution is again given by eqs (6), but this time with a composite propagator;

$$\mathscr{P}(w, w_0) = \mathscr{P}(w, w_N) \prod_{i=1}^{N} [\mathscr{T}_i \pi_i \mathscr{P}(w_i, w_{i-1})] \quad (12)$$

where N denotes the number of interfaces crossed by the ray.

If the interfaces are defined by the relation $f(\mathbf{x}) = 0$ and the reference medium in each layer has constant gradient in quadratic slowness, then the transformation matrices for a single interface may be written

$$\pi_i = \begin{pmatrix} \pi_1 & \mathbf{0} \\ \pi_2 & \mathbf{I} \end{pmatrix} \quad (13)$$

where

$$\pi_1 = \mathbf{I} - \frac{|p^0\rangle\langle\nabla f|}{\langle\mathbf{p}^0 \mid \nabla f\rangle}, \qquad \pi_2 = -\frac{1}{2}\frac{|\Gamma\rangle\langle\nabla f|}{\langle\mathbf{p}^0 \mid \nabla f\rangle}$$

and

$$\mathscr{T} = \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{T}_1 & \mathbf{T}_2 \end{pmatrix} \quad (14)$$

where

$$\mathbf{T}_1 = -\frac{1}{2}\frac{|\nabla f\rangle\langle\hat{\Gamma} - \Gamma|}{\langle\hat{\mathbf{p}}^0 \mid \nabla f\rangle} - \frac{\langle\nabla f \mid \mathbf{p}_0 - \hat{\mathbf{p}}_0\rangle}{\langle\nabla f \mid \nabla f\rangle} \times \left(\mathbf{I} - \frac{|\nabla f\rangle\langle\hat{\mathbf{p}}_0|}{\langle\hat{\mathbf{p}}_0 \mid \nabla f\rangle}\right)\nabla\nabla f$$

$$\mathbf{T}_2 = \mathbf{I} - \frac{|\nabla f\rangle\langle\hat{\mathbf{p}}_0 - \mathbf{p}_0|}{\langle\hat{\mathbf{p}}^0 \mid \nabla f\rangle}$$

and $\hat{}$ denotes properties on the 'transmitting' side of the interface.

The notation $\langle\mathbf{a} \mid \mathbf{b}\rangle$ represents the scalar product of the vectors $\langle\mathbf{a}|$ and $\langle\mathbf{b}|$ while the notation $|\mathbf{a}\rangle\langle\mathbf{b}|$ represents a matrix obtained by the tensor product of the vectors $\langle\mathbf{a}|$ and $\langle\mathbf{b}|$. The elements of this matrix are given by

$$[|\mathbf{a}\rangle\langle\mathbf{b}|]_{ij} = \mathbf{a}_i\mathbf{b}_j.$$

More general expressions for π_i and $\mathscr{T}_i$ are given in Farra (1992). In fact, the generalized propagator given at eq. (12) is not a true propagator as the rank of the submatrix π_1 is two rather than three. This means that the extrapolation procedure at interfaces introduces some ambiguity or that many ray perturbations may produce the same extrapolated deviations at the interface. However, this does not limit the usefulness of the generalized propagator.

We now wish to find a value for $\eta = \mathbf{p}^\epsilon(w_s)$, the change required in the initial direction of the ray to ensure that the perturbed ray reaches the required endpoint at the **same** value of the sampling parameter w as on the unperturbed ray. To facilitate calculations, we partition the propagator

$$\mathscr{P}(w, w_0) = \begin{pmatrix} \mathbf{P}_{11} & \mathbf{P}_{12} \\ \mathbf{P}_{21} & \mathbf{P}_{22} \end{pmatrix}. \quad (15)$$

We then require η to satisfy the equation

$$\mathbf{0} = \int_{w_s}^{w_r} \mathbf{P}_{12}(w_r, w')\,\nabla(\tfrac{1}{2}\Delta u^2)\,dw' + \mathbf{P}_{12}(w_r, w_s)\eta. \quad (16)$$

If $\mathbf{P}_{12}(w_r, w_s)$ is non-singular, we may solve for η to obtain

$$\eta = -\mathbf{P}_{12}^{-1}(w_r, w_s)\int_{w_s}^{w_r} \mathbf{P}_{12}(w_r, w')\,\nabla(\tfrac{1}{2}\Delta u^2)\,dw'. \quad (17)$$

Of course, the presence of interfaces causes the expressions for the various components of the propagator to be far more complicated than before. We consider the case of a reference medium with homogeneous layers and suppose that $\nabla f = \mathbf{k}$. In this case,

$$\pi_1 = \begin{pmatrix} 1 & 0 & -\dfrac{p}{p_3^0} \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\pi_2 = \mathbf{0}, \quad \mathbf{T}_1 = \mathbf{0}, \mathbf{T}_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \dfrac{p_3^0}{\hat{p}_3^0} \end{pmatrix} \tag{18}$$

where we have written $\mathbf{p}_0 = p\mathbf{i} + p_3^0\mathbf{k}$.

For a single interface crossed at ray parameter $w = w_1$,

$$\mathbf{P}_{12}(w_r, w_s) = (w_1 - w_s)\pi_1 + (w_r - w_1)\mathbf{T}_2$$
$$= \begin{bmatrix} w_r - w_s & 0 & -\dfrac{p}{p_3^0(w_1 - w_s)} \\ 0 & w_r - w_s & 0 \\ 0 & 0 & \dfrac{p_3^0}{\hat{p}_3^0(w_r - w_1)} \end{bmatrix} \tag{19}$$

with

$$\mathbf{P}_{12}^{-1}(w_r, w_s) = \frac{1}{w_r - w_s}\begin{bmatrix} 1 & 0 & \dfrac{p\hat{p}_3^0(w_1 - w_s)}{[p_3^0]^2(w_r - w_1)} \\ 0 & 1 & 0 \\ 0 & 0 & \dfrac{\hat{p}_3^0(w_r - w_s)}{p_3^0(w_r - w_1)} \end{bmatrix}. \tag{20}$$

The Green's function solution to the problem is again given by eq. (7).

In this case,

$$\mathbf{G}(w_s, w', w, w_r) = \mathbf{P}_{12}(w, w')H(w - w') - \mathbf{P}_{12}(w, w_s)\mathbf{P}_{12}^{-1}(w_r, w_s)\mathbf{P}_{12}(w_r, w') \tag{21}$$

where $\mathbf{P}_{12}(w, w')$ is given by eq. (19) if the two arguments correspond to points on different sides of the interface but is $(w - w')\mathbf{I}$ otherwise.

When the reference medium contains N homogeneous layers crossed at ray parameters $w = w_i$, $i = 1, N$ and each interface has normal $\nabla f = \mathbf{k}$,

$$\mathbf{P}_{12}(w_r, w_s) = \begin{pmatrix} w_r - w_s & 0 & F_1 \\ 0 & w_r - w_s & 0 \\ 0 & 0 & F_2 \end{pmatrix} \tag{22}$$

where

$$F_1 = -ph_1 \sum_{i=1}^{N} \frac{(w_i - w_{i-1})}{h_i^2}, \qquad w_0 = w_s$$

$$F_2 = \frac{h_1}{h_{N+1}}(w_r - w_N)$$

and h_i denotes the value of p_3^0 in layer i.

In this case, the inverse is simply

$$\mathbf{P}_{12}^{-1}(w_r, w_s) = \frac{1}{w_r - w_s}\begin{pmatrix} 1 & 0 & -\dfrac{F_1}{F_2} \\ 0 & 1 & 0 \\ 0 & 0 & \dfrac{w_r - w_s}{F_2} \end{pmatrix}, \tag{23}$$

and the corresponding Green's function for the problem has a form similar to that given at eq. (21), although the formula used for the various factors of $\mathbf{P}_{12}$ will depend on how many interfaces lies between the two points on the unperturbed ray which correspond to the arguments of $\mathbf{P}_{12}$.

Nevertheless, it is only the $j = 3$ components of G_{ij} which differ from the Green's function obtained at Eq. (9). Now

$$G_{33}(w_s, w', w, w_r) = \begin{cases} -\dfrac{p_3^0(w')}{p_3^0(w)}(w - w_i), & w < w' \\ 0, & \text{otherwise} \end{cases} \tag{24}$$

where w_i denotes the ray parameter at which the unperturbed ray crossed the last interface prior to reaching the point $\mathbf{x}^0(w)$,

$$G_{13}(w_s, w', w, w_r) = F_1(w, w')H(w - w') - \frac{w - w_s}{w_r - w_s}F_1(w_r, w') - \frac{p_3^0(w')}{p_3^0(w_s)}F_1(w, w_s)\left[1 - \frac{w - w_s}{w_r - w_s}\frac{F_1(w_r, w_s)}{F_1(w, w_s)}\right] \tag{25}$$

6 COMPARISON WITH EXACT RAY-TRACING RESULTS

It is instructive to consider the problem of ray propagation across an artificial interface in the material. That is, we may either use the results of Section 4 (no interface), or the results of Section 5, with an interface separating two layers of identical material. We consider the case given by eq. (10) with $\gamma = \mathbf{0}$ and $\Delta\alpha^2 = 0$. Then an analytical solution to the problem is also available.

First of all, we look at the differences in the propagator for the problem as given by eqs (6) and (12). Again we suppose that the interface has normal $\nabla f = \mathbf{k}$. In this case, the propagator given by eq. (12) may be written

$$\mathcal{P} = \begin{bmatrix} \mathbf{I} - \mathbf{J} & (w_r - w_s)\mathbf{I} - (w_1 - w_0)\mathbf{J} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \tag{26}$$

where

$$\mathbf{J} = \frac{|\mathbf{p}^0\rangle\langle\mathbf{k}|}{p_3^0} = \begin{pmatrix} 0 & 0 & \dfrac{p}{p_3^0} \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Because $\mathbf{J}^N = \mathbf{J}$ for all integers N, we find that even if we had introduced N artificial interfaces with normal $\nabla f = \mathbf{k}$, the propagator would have essentially the same form as in eq. (26) namely,

$$\mathcal{P}^N = \begin{bmatrix} \mathbf{I} - \mathbf{J} & (w_r - w_s)\mathbf{I} - (w_N - w_s)\mathbf{J} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}. \tag{27}$$

We see that $\mathcal{P}$ contains no information about any interface except the last ($w = w_N$). Furthermore, when we ignore velocity perturbations in the problem and simply compare how the propagators given by eq. (6) and (26) propagate changes in the initial conditions along the ray, we find that when there are no interfaces,

$$\begin{aligned} \mathbf{x}^\epsilon(w_r) &= \mathbf{x}^\epsilon(w_s) + (w_r - w_s)\mathbf{p}^\epsilon(w_s) \\ \mathbf{p}^\epsilon(w_r) &= \mathbf{p}^\epsilon(w_s) \end{aligned} \tag{28}$$

whilst when there is one interface

$$\mathbf{x}^{\epsilon}(w_r) = [\mathbf{x}^{\epsilon}(w_s) + (w_r - w_s)\mathbf{p}^{\epsilon}(w_s)] - \mathbf{J}\mathbf{x}^{\epsilon}(w_1)$$
$$\mathbf{p}^{\epsilon}(w_r) = \mathbf{p}^{\epsilon}(w_s). \tag{29}$$

The expressions for $\mathbf{x}^{\epsilon}(w_r)$ differ by $-\mathbf{J}\mathbf{x}^{\epsilon}(w_1)$. However, eq. (32b) of Farra (1992) overcomes this problem by introducing an artificial interface at the receiver. This additional application of the projection matrix π has the effect of removing the contribution to the ray perturbation from the previous interface because $(\mathbf{I}-\mathbf{J})\mathbf{J}=\mathbf{0}$. Thus eqs (28) and (29) give the same value for $(\mathbf{I}-\mathbf{J})\mathbf{x}^{\epsilon}(w_r)$ but differ in the component directed along the unperturbed ray. This gives insight into the projection matrix π. This matrix has the effect of modifying the length of the path of integration to the interface without appearing to do so. However, this extrapolation procedure only causes variations in the component of $\mathbf{x}^{\epsilon}(w_i)$ **along** the unperturbed ray, and not transverse to it, with the result that corrections at earlier interfaces become irrelevant once a new interface has been crossed.

In Section 5, we have not included a projection matrix at the endpoint of the ray as required in eq. (33c) of Farra *et al.* (1989): instead we chose to change the initial conditions of the ray to ensure that the perturbed ray arrives at the receiver with the same value of the sampling parameter w as on the unperturbed ray. Clearly, if such a choice of η is possible, then eq. (33c) of Farra *et al.* (1989) will automatically be satisfied. For the particular problem of this section, we find that when there are no interfaces

$$\mathbf{p}^{\epsilon}(w_s) = \eta = -\tfrac{1}{4}(w_r - w_s)\,\Delta\gamma \tag{30}$$

whilst when there is one interface

$$\mathbf{p}^{\epsilon}(w_s) = \eta = -\tfrac{1}{4}[(w_r - w_s)\mathbf{I} + (w_1 - w_s)\mathbf{J}]\,\Delta\gamma. \tag{31}$$

These expressions differ by an amount $-\tfrac{1}{4}(w_1 - w_s)\mathbf{J}\,\Delta\gamma$ which represents a component in the direction of $\mathbf{p}^0$ introduced to compensate for variations in the component along the unperturbed ray that will result from the use of the projection matrix π at each interface.

When there are no interfaces, the perturbation to the ray trajectory is simply

$$\mathbf{x}^{\epsilon}(w) = \tfrac{1}{4}(w - w_s)^2\,\Delta\gamma + (w - w_s)\eta$$
$$= (w - w_s)(w - w_r)\frac{\Delta\gamma}{4}. \tag{32}$$

However, when there is one interface, the perturbation to the ray trajectory is

$$\mathbf{x}^{\epsilon}(w) = \begin{cases} (w - w_s)[(w - w_r)\mathbf{I} - (w_1 - w_s)\mathbf{J}]\dfrac{\Delta\gamma}{4}, & w < w_1 \\[2ex] (w - w_r)[(w - w_s)\mathbf{I} - (w_1 - w_s)\mathbf{J}]\dfrac{\Delta\gamma}{4}, & w > w_1. \end{cases} \tag{33}$$

Again we see the eqs (32) and (33) give the same value for $(\mathbf{I}-\mathbf{J})\mathbf{x}^{\epsilon}(w_1)$, but differ in the component directed along the unperturbed ray.

Each formulation of the problem gives a perturbation to the slowness vector $\mathbf{p}^0$ which may be written

$$\mathbf{p}^{\epsilon}(w) = \mathbf{p}^{\epsilon}(w_s) + \tfrac{1}{2}(w - w_s)\,\Delta\gamma \tag{34}$$

although the expressions for $\mathbf{p}^{\epsilon}(w_s) = \eta$ differ by an amount $-\tfrac{1}{4}(w_1 - w_s)\mathbf{J}\,\Delta\gamma$. It follows that at every point along the ray path, the expressions for $\mathbf{p}^{\epsilon}$ in the two formulations of the problem, differ in the same way that the initial values η vary, as discussed above.

The exact solution for the ray trajectory in the presence of a small gradient in quadratic slowness is the same as that obtained by the perturbation method without interfaces (see Section 4 above). It follows that the use of projection matrices to handle interfaces introduces some ambiguity in the component of the perturbation directed along the unperturbed ray, but maintains the transverse components of the perturbation.

7 LIMITATIONS OF RESULTS

The use of a constant gradient in quadratic slowness to describe the velocity distribution in the unperturbed medium does impose some limitations on the generality of the results obtained. For example, if we consider a simple vertically inhomogeneous half-space $z \geq 0$ with quadratic slowness given by the relation $u^2(z) = u_0^2 - bz$, $b > 0$, we must limit the model to depths $0 \leq z \leq u_0^2/b$ because the velocity becomes infinite at $z = z_b = u_0^2/b$. Furthermore, if we take the point source at $x = z = 0$ and locate receivers along the x-axis with $x \geq 0$, we find that in the 2-D problem there will be a shadow zone along the x-axis for $x > x_b = 2z_b = 2u_0^2/b$ whilst two rays will arrive at any receiver situated in the region $0 < x \leq x_b$ (unless we further limit the model to depths $0 \leq z \leq \tfrac{1}{2}z_b$ and restrict take-off angles to lie within 45° of horizontal). Similar regions of shadows and multipathing will exist for a generally oriented gradient of the quadratic slowness in a 3-D structure. In such circumstances one may need to introduce additional layers into the model to permit a more appropriate description of the unperturbed model. Alternatively, one may prefer to adapt the results of Moore (1993) to multilayered media so that a more general model for the unperturbed medium may be used.

8 CONCLUSIONS

Simple analytical expressions have been found for the total Green's function for the two-point boundary value problem in the context of ray-perturbation theory. These correspond to the special cases in which the unperturbed medium may be described either by a single layer of material with a constant gradient in quadratic slowness, or by several layers of homogeneous material separated by plane parallel interfaces. As outlined in Moore (1993), such formulations of the Green's function are particularly useful in tomographic reconstruction problems. Furthermore, even in the case of more complex reference media, the same strategy may be used to determine the change required in the initial direction of the ray to ensure that the perturbed ray reaches the required endpoint at the same value of the sampling parameter w as on the unperturbed ray (see eq. 21). This involves the calculation of only the $\mathbf{P}_{12}$ component of the generalized propagator $\mathscr{P}$ along the unperturbed ray path joining source and receiver, and finding the inverse of a single rank three matrix. Given these quantities, the solution for the first-order correction to the ray trajectory is completely determined.

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