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# ON MOMENTS OF PITMAN ESTIMATORS: THE CASE OF FRACTIONAL BROWNIAN MOTION*

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(*Translated and revised by the authors*)

**Abstract.** In some nonregular statistical estimation problems, the limiting likelihood processes are functionals of fractional Brownian motion (fBm) with Hurst's parameter $H$, $0 < H \leqq 1$. In this paper we present several analytical and numerical results on the moments of Pitman estimators represented in the form of integral functionals of fBm. We also provide Monte Carlo simulation results for variances of Pitman and asymptotic maximum likelihood estimators.

**Key words.** Pitman estimators, fractional Brownian motion, integral functionals, Riemann-zeta function

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**1. Introduction.** Pitman estimators [19], also known as Bayesian estimators with a constant a priori distribution on the real line [3], for parameters of stochastic processes are optimal under various continuous- and discrete-time settings [10], [12]. For example, one may consider the estimation problem of parameter $\theta$ by observing the diffusion process $X = \{X_t,\ 0 \leqq t \leqq T\}$ that is a solution of the stochastic differential equation

$$dX_t = s(X_t, t, \theta)\, dt + \sigma(X_t)\, dW_t, \qquad 0 \leqq t \leqq T,$$

where the drift $s(x,t,\theta)$ is a nonregular function, e.g., $s(x,t,\theta) = |x-\theta|^p$, $p < \frac{1}{2}$, or $s(x,t,\theta) = I\{\theta > t\}$. For such nonregular statistical estimation problems it is a typical situation when the respective limit likelihood process $Z_t$ is generated by a fractional Brownian motion (fBm) $W_t^H$ with Hurst's parameter $H \in (0,1]$, namely

$$Z_t := e^{W_t^H - |t|^{2H}/2}, \qquad t \in \mathbf{R} := (-\infty, \infty);$$

see [15], [7]. Note that the case $H = \frac{1}{2}$ appears in a study of a change point problem for a Brownian motion (Bm) in [10], [12] and processes with a time delay in [9]. The case $H \neq \frac{1}{2}$ appears in various continuous-time settings (see [15, Chap. 3] and [7]) and discrete-time frameworks (see [12], [2]).

Distributional properties of Pitman estimators for large sample sizes have not been studied in much detail. In this paper, in continuation of our results from [18], we study the limit distribution of Pitman estimators, which can be defined as the distribution of a random variable

$$\zeta_H = \int_{-\infty}^{\infty} t q_t\, dt, \tag{1}$$

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where

$$q_t = Z_t\Big(\int_{-\infty}^{\infty} Z_u\,du\Big)^{-1}; \tag{2}$$

$\zeta_H$ represents a conditional expectation with respect to a posteriori density $q_t$. Recall that $W^H = \{W_s^H,\ s \in R\}$ is a Gaussian process with continuous trajectories

$$W_0^H = 0, \quad \mathbf{E}\,W_s^H = 0, \quad \mathbf{E}\,|W_s^H - W_t^H|^2 = |s-t|^{2H}, \qquad s \in \mathbf{R}, \quad t \in \mathbf{R}.$$

This implies that the covariance function of $W_s^H$ is

$$R_H(t,s) := \mathbf{E}\,(W_t^H W_s^H) = \frac{1}{2}\,(|t|^{2H} + |s|^{2H} - |t-s|^{2H}).$$

Note that, even when $H = \frac{1}{2}$, i.e., in the case of standard Bm, neither the distribution nor the moments $\zeta_{\frac{1}{2}}$ (greater than 2) are known in an explicit form. For other cases, except $H = 1$, the essential difficulty in studying functionals of fBm is due to the fact that $W^H$ is not a semimartingale, and therefore standard tools of stochastic calculus (based on the Itô formula) are not applicable.

The case $H = 1$ corresponds to the regular statistical estimation problems where the limit distribution is normal, $\zeta_1 \sim N(0,1)$.

In this paper we obtain several results on the variance and higher moments of $\zeta_H$, $0 < H \leqq 1$, using the measure transformation technique and Gaussian property of fBm. In [18] we showed that, for $H > 0.309\ldots$, the random variable (r.v.) $|\zeta_H|^{2H}$ is *exponentially bounded*; i.e., there exists a constant $\alpha_H > 0$ such that

$$\mathbf{E}\,e^{\delta|\zeta_H|^{2H}} < \infty \quad \text{for } \delta < \alpha_H. \tag{3}$$

This result implies, of course, finiteness of all moments of $\zeta_H$. In section 2 we improved this result (see Theorem 1) by showing that (3) holds for all $H \in (0,1]$. Improvement is achieved due to application of the measure transformation technique; see Lemma 1 in section 2. Note that Lemma 1 also will be used in the proof of Theorem 2 in section 3 which presents a general identity for expectations of functions of $\zeta_H$. Then, using the obtained identity, we derive a useful representation for the variance of $\zeta_H$ when $H \in (0,1]$, (see Corollary 1). Corollary 2 provides a lower bound for the moments $\mathbf{E}\,\zeta_H^k$, $k = 2,4,6,\ldots$.

In section 4, Theorem 3, for the case $H \in [\frac{1}{2}, 1)$ (see Theorem 3) we also derive a new representation for $\operatorname{Var}(\zeta_H)$ in terms of the function

$$g(m) := \mathbf{E}\ln\int_{-\infty}^{\infty} Z_u e^{mu}\,du. \tag{4}$$

This result was formulated in [18] without a proof. Earlier in [18] we derived another expression for $\operatorname{Var}(\zeta_H)$ in terms of the function

$$g(m_1, m_2) = \mathbf{E}\ln\int_0^{\infty} (Z_u e^{-m_1 u} + Z_{-u} e^{-m_2 u})\,du.$$

In [18] it was shown that in the case $H = \frac{1}{2}$ the function $g(m_1,m_2)$ (as well as $g(m) = g(-m,m)$) can be expressed in terms of the PolyGamma function leading to much shorter derivation of the following result from [20],

$$\operatorname{Var}\zeta_{1/2} = 16\,\mathrm{Zeta}\,[3] \approx 19.23, \tag{5}$$

where Zeta$[k]$ is the Riemann-zeta function; see details in [18].

In section 5 we present Monte Carlo simulation results for $\mathrm{Var}\,(\zeta_H)$ and the variance of the asymptotic maximum likelihood estimator which is the *argmax* of $W^H = \{W_s^H, s \in \mathbf{R}\}$.

**2. Exponentially boundedness of $|\zeta_H|^{2H}$.** The following lemma on a measure transformation for Gaussian processes (and hence for fBm $W_t^H$, $t \in \mathbf{R}$, as well) plays a key role in the proof of Theorem 1 below.

We formulate Lemma 1 in terms of a Gaussian system $(\xi, \{X_s\},\ s \in D)$ (see [25]) defined on probability space $(\Omega, F, P)$. Recall that it means that $(\xi, \{X_{t_i}\}, t_i \in D, i = 1, \dots, n)$ is a Gaussian vector for any $n$. We use the upper index, e.g., $Q$, to indicate that expectations are taken with respect to a measure $Q$, so $E^Q(\cdot)$ is used for the expectation with respect to measure $Q$. Lemma 1 gives a new result which belongs to a group of results broadly known as Cameron–Martin–Girsanov–Maruyama-type measure transformations.

LEMMA 1. *Let $(\xi, \{X_s\},\ s \in D)$ be a Gaussian system on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$. Set*

$$\mathbf{E}\,\xi = 0, \quad \mathrm{Var}\,\xi = \sigma^2$$

*and consider the measure transformation*

$$\mathbf{Q}(A) = \mathbf{E}\,I(A)e^{\xi - \sigma^2/2}, \qquad A \in \mathcal{F}.$$

*Then on the probability space $(\Omega, \mathcal{F}, \mathbf{Q})$*

(1) *the system $(\xi, \{X_s\},\ s \in D)$ is Gaussian*;

(2) $\mathbf{E}^{\mathbf{Q}} X_s = \mathbf{E}\,\mathbf{X}_s + \mathrm{Cov}\,(\xi, X_s)$, $\mathrm{Cov}^{\mathbf{Q}}(X_t, X_s) = \mathrm{Cov}\,(X_t, X_s)$.

*Proof.* Property (1) is a consequence of the definition of a Gaussian system and the fact that any linear transformation of a Gaussian vector is a Gaussian vector.

To check the second property one should write out the joint moment generating function of $X_s$ and $X_t$ with respect to measure $\mathbf{Q}$ for $t, s \in \mathbf{R}$

$$\begin{aligned}(6) \qquad &\mathbf{E}^{\mathbf{Q}} e^{z_1 X_t + z_2 X_s} = \mathbf{E}\, e^{z_1 X_t + z_2 X_s} e^{\xi - \sigma^2/2} \\ &\quad = \exp\left\{ z_1\, \mathbf{E}^{\mathbf{P}} X_s + z_2\, \mathbf{E}^{\mathbf{P}} X_t - \frac{\sigma^2}{2} + \frac{1}{2}\,\mathrm{Var}^{\mathbf{P}}(\xi + z_1 X_s + z_2 X_t) \right\}.\end{aligned}$$

Since

$$\begin{aligned}\frac{1}{2}\,\mathrm{Var}^{\mathbf{P}}(\xi + z_1 X_s + z_2 X_t) &= \frac{1}{2}\,(\sigma^2 + z_1^2\,\mathrm{Var}^{\mathbf{P}} X_s + z_2^2\,\mathrm{Var}^{\mathbf{P}} X_t) \\ &\quad + z_1\,\mathrm{Cov}^{\mathbf{P}}(\xi, X_s) + z_2\,\mathrm{Cov}^{\mathbf{P}}(\xi, X_t) + z_1 z_2\,\mathrm{Cov}^{\mathbf{P}}(X_t, X_s),\end{aligned}$$

differentiating in (6) with respect to $z_1$ and $z_2$ we obtain

$$\begin{aligned}\mathbf{E}^{\mathbf{Q}} X_s &= \frac{\partial}{\partial z_1}\,\mathbf{E}^{\mathbf{Q}} e^{z_1 X_s + z_2 X_t}\Big|_{z_1 = z_2 = 0} = \mathbf{E}\,X_s + \mathrm{Cov}\,(\xi, X_s), \\ \mathbf{E}^{\mathbf{Q}} X_t X_s &= \frac{\partial^2}{\partial z_1\,\partial z_2}\,\mathbf{E}^{\mathbf{Q}} e^{z_1 X_t + z_2 X_s}\Big|_{z_1 = z_2 = 0} \\ &= (\mathbf{E}\,X_s + \mathrm{Cov}\,(\xi, X_s))(\mathbf{E}\,X_t + \mathrm{Cov}\,(\xi, X_t)) + \mathrm{Cov}\,(X_t, X_s) \\ &= \mathbf{E}^{\mathbf{Q}} X_t\, \mathbf{E}^{\mathbf{Q}} X_s + \mathrm{Cov}\,(X_t, X_s).\end{aligned}$$

This completes the proof.

While dealing with integrals of a Gaussian process $X_s$ we assumed that there is a progressively measurable modification of $X_s$ such that integrals are well defined.

For proving Theorem 1 we also need the following lemma.

LEMMA 2. *Let $X_s$ be a Gaussian process with $\mathbf{E}\, X_s = 0$. Then for any $t > 0$ and $r > 0$*

$$\mathbf{E}\biggl(\int_0^t e^{X_s}\,ds\biggr)^{-r} \leqq \frac{1}{t^r}\,\exp\biggl\{\frac{r^2}{2t^2}\,\mathrm{Var}\int_0^t X_s\,ds\biggr\}. \tag{7}$$

*Proof.* Applying Jensen's inequality we obtain for any $t > 0$

$$\int_0^t e^{X_s}\,ds \geqq t\exp\biggl\{\frac{1}{t}\int_0^t X_s\,ds\biggr\},$$

and hence for any $t > 0$ and $r > 0$

$$\mathbf{E}\biggl(\int_0^t e^{X_s}\,ds\biggr)^{-r} \leqq \frac{1}{t^r}\,\mathbf{E}\exp\biggl\{-\frac{r}{t}\int_0^t X_s\,ds\biggr\} = \frac{1}{t^r}\,\exp\biggl\{\frac{r^2}{2t^2}\,\mathrm{Var}\int_0^t X_s\,ds\biggr\}.$$

*Remark* 1. Integrals of type $\mathbf{E}\,(\int_0^t e^{X_s}\,ds)^{-1}$ were considered in [14], [17], and [13].

THEOREM 1. *For any $H \in (0,1]$, there exists a positive number $\alpha_H$ such that*

$$\mathbf{E}\, e^{\delta|\zeta_H|^{2H}} < \infty \quad \textit{for } \delta < \alpha_H. \tag{8}$$

*Proof.* If $H = 1$, then $\alpha_1 = \frac{1}{2}$ in (8) because it is well known that in this case $\zeta_1 \sim N(0,1)$, and therefore

$$\mathbf{E}\, e^{\delta\zeta_1^2} < \infty \quad \text{for } \delta < \frac{1}{2}; \qquad \mathbf{E}\, e^{\zeta_1^2/2} = \infty. \tag{9}$$

In the case $H \in (0,1)$ we need to find a proper estimate for the expectation of $\mathbf{E}\, q_t$ which leads to (8).

Note that the function $e^{\delta|x|^{2H}}$ is a convex function for $H \geqq \frac{1}{2}$, and for $H \in (0,\frac{1}{2})$ it is dominated by the convex function $\max(C_{\delta,H}, e^{\delta|x|^{2H}})$, where $C_{\delta,H}$ is a sufficiently large number. The random process $q_t$ represents a density function as it is a normalized nonnegative function of $t$ such that $\int_{-\infty}^{\infty} q_t dt = 1$. Thus by Jensen's inequality, for any $\delta \geqq 0$, from (1) we have

$$\begin{aligned} e^{\delta|\zeta_H|^{2H}} &\leqq C_{\delta,H} + \int_{-\infty}^{\infty} e^{\delta|t|^{2H}} q_t\,dt \\ &\leqq C_{\delta,H} + \int_{-1}^{1} e^{\delta|t|^{2H}} q_t\,dt + \int_{-\infty}^{\infty} I\{|t|>1\} e^{\delta|t|^{2H}} q_t\,dt \\ &\leqq C_{\delta,H} + e^{\delta} + \int_{-\infty}^{\infty} I\{|t|>1\} e^{\delta|t|^{2H}} q_t\,dt, \end{aligned}$$

where the constants $C_{\delta,H} \geqq 0$ and $C_{\delta,H} = 0$ in the case $H \geqq \frac{1}{2}$; above we also used the fact that $\int_{-\infty}^{\infty} q_t\,dt = 1$. In view of the symmetry property of fBm in distributional sense

$$\{W_u^H, u \geqq 0\} \overset{d}{=} \{W_{-u}^H, u \geqq 0\}, \tag{10}$$

we have

$$\mathbf{E}\, e^{\delta|\zeta_H|^{2H}} \leqq C_{\delta,H} + e^{\delta} + 2\int_1^{\infty} e^{\delta t^{2H}}\, \mathbf{E}\, q_t\, dt.$$

For finding a proper upper bound for $\mathbf{E}\, q_t$ for $t \geqq 1$ we use Lemma 1 with

$$\xi = \lambda W_t^H, \quad X_s = W_s^H,$$

where $\lambda$ is a real number. Then

$$\mathbf{E}\, X_s = 0, \quad \sigma^2 = \frac{\lambda^2 t^{2H}}{2}; \quad \mathbf{E}^{\mathbf{Q}} W_t^H = \lambda R_H(t,s), \quad R^{\mathbf{Q}}(t,s) = R_H(t,s).$$

This means that with respect to the measure $Q$ the process $\{W_t^H - \lambda R_H(t,s),\ t \in \mathbf{R}\}$ is a (standard) fBm. Using this fact we obtain

$$\begin{aligned}\mathbf{E}\, q_t &= \mathbf{E}\, \exp\left\{\lambda W_t^H - \frac{\lambda^2}{2}\, t^{2H}\right\} \exp\left\{(1-\lambda)W_t^H - \frac{(1-\lambda^2)}{2}\, t^{2H}\right\} \\ &\quad \times \left(\int_{-\infty}^{\infty} e^{W_s^H - |s|^{2H}/2}\, ds\right)^{-1} \\ &= \mathbf{E}^{\mathbf{Q}} \exp\left\{(1-\lambda)W_t^H - \frac{(1-\lambda^2)}{2}\, t^{2H}\right\}\left(\int_{-\infty}^{\infty} e^{W_s^H - |s|^{2H}/2}\, ds\right)^{-1}.\end{aligned}$$

Applying Lemma 1 we have

$$\begin{aligned}\mathbf{E}\, q_t &= \mathbf{E}\, \exp\left\{(1-\lambda)W_t^H + \lambda(1-\lambda)t^{2H} - \frac{(1-\lambda^2)}{2}\, t^{2H}\right\} \\ &\quad \times \left(\int_{-\infty}^{\infty} e^{W_s^H + \lambda R_H(t,s) - |s|^{2H}/2}\, ds\right)^{-1} \\ &= \exp\left\{-\frac{(1-\lambda)^2}{2}\, t^{2H}\right\} \mathbf{E}\, e^{(1-\lambda)W_t^H}\left(\int_{-\infty}^{\infty} e^{W_s^H + t^{2H} f(s/t)}\, ds\right)^{-1}, \qquad (11)\end{aligned}$$

where $t^{2H} f(\frac{s}{t}) := \lambda R(t,s) - \frac{|s|^{2H}}{2}$. Note that

$$f(u) = \frac{\lambda + (\lambda-1)|u|^{2H} - \lambda|1-u|^{2H}}{2}, \quad f(0) = 0, \quad f(1) = \lambda - \frac{1}{2}. \qquad (12)$$

To simplify the exposition of the proof we choose $\lambda = \frac{1}{2}$ (although it seems that a somewhat better estimator for $\alpha_H$ can be obtained with a proper choice of $\lambda$ depending on $H$). Then assuming $\lambda = \frac{1}{2}$ we can rewrite (11) as

$$\mathbf{E}\, q_t = e^{-t^{2H}/8}\, \mathbf{E}\, e^{W_t^H/2}\left(\int_{-\infty}^{\infty} e^{W_s^H + t^{2H} f(s/t)}\, ds\right)^{-1}, \qquad (13)$$

where

$$f(u) = f(1-u) = \frac{1 - |u|^{2H} - |1-u|^{2H}}{4}, \quad f(0) = 0, \quad f(1) = 0. \qquad (14)$$

It is easy to see that for $H \in (0, 1/2)$ the function $f(u)$, $0 < u < 1$ is negative; for $H \in (1/2, 1)$ the function $f(u)$, $0 < u < 1$ is positive.

Applying the inequality $e^{x/2} \leqq (1+e^x)/2$ for the term $e^{W_t^H/2}$ in (13) and reducing the range of integration to $s \in [0,t]$ instead of $s \in R$ for the integral, we obtain

$$\mathbf{E}\, q_t \leqq \frac{1}{2}\, e^{-t^{2H}/8} \biggl[ \mathbf{E} \biggl( \int_0^t e^{W_s^H + t^{2H} f(s/t)}\, ds \biggr)^{-1} + \mathbf{E} \biggl( \int_0^t e^{W_s^H - W_t^H + t^{2H} f(s/t)}\, ds \biggr)^{-1} \biggr].$$

In view of the following translation-invariance property of fBm

$$\{W_s^H - W_t^H,\ s \in \mathbf{R}\} \stackrel{d}{=} \{W_{t-s}^H,\ s \in \mathbf{R}\}, \tag{15}$$

the expectation $\mathbf{E}\,(\int_0^t e^{W_s^H - W_t^H + t^{2H} f(s/t)}\, ds)^{-1}$ equals $\mathbf{E}\,(\int_0^t e^{W_s^H + t^{2H} f(1-s/t)}\, ds)^{-1}$ after the change of variable $t-s$ to $s$. Thus we get the estimate

$$\mathbf{E}\, q_t \leqq \frac{1}{2}\, e^{-t^{2H}/8} \biggl[ \mathbf{E} \biggl( \int_0^t e^{W_s^H + t^{2H} f(s/t)}\, ds \biggr)^{-1} + \mathbf{E} \biggl( \int_0^t e^{W_s^H + t^{2H} f(1-s/t)}\, ds \biggr)^{-1} \biggr].$$

Since $f(u) = f(1-u)$ the above inequality now can be rewritten as

$$\mathbf{E}\, q_t \leqq e^{-t^{2H}/8}\, \mathbf{E} \biggl( \int_0^t e^{W_s^H + t^{2H} f(s/t)}\, ds \biggr)^{-1}. \tag{16}$$

**The case $H \in [1/2, 1)$.** Then it is easy to see $f(s/t) \geqq 0$, and therefore by Lemma 2 for $t \geqq 1$

$$\mathbf{E}\, q_t \leqq e^{-t^{2H}/8}\, \mathbf{E} \biggl( \int_0^1 e^{W_s^H}\, ds \biggr)^{-1} < \infty.$$

Combining all estimates obtained above for $H \in [\frac{1}{2}, 1)$ and $\delta < \frac{1}{8}$ we have

$$\mathbf{E}\, e^{\delta |\zeta_H|^{2H}} \leqq C_{\delta,H} + e^{\delta} + 2\, \mathbf{E} \biggl( \int_0^1 e^{W_s^H}\, ds \biggr)^{-1} \int_1^\infty e^{-(1/8-\delta) t^{2H}}\, dt < \infty.$$

Since $E(\int_0^1 e^{W_s^H}\, ds)^{-1}$ is finite due to the result of Lemma 2 we have proved Theorem 1 with $\alpha_H \geqq \frac{1}{8}$.

**The case $H \in (0, \frac{1}{2})$.** Obviously, the function $f(u)$ (defined in (12)) is decreasing on the interval $u \in (0, \frac{1}{2})$. This fact implies that $f(s/t) \geqq f(\varepsilon)$ for all $s \in (0, \varepsilon t)$ and any $\varepsilon \in (0, \frac{1}{2})$, where $f(\varepsilon) = -(\varepsilon^{2H}/4)(1+o(1))$ as $\varepsilon \to 0$. Thus, from (16) we have

$$\mathbf{E}\, q_t \leqq e^{-(1/8 + f(\varepsilon)) t^{2H}}\, \mathbf{E} \biggl( \int_0^{t\varepsilon} e^{W_s^H}\, ds \biggr)^{-1}.$$

Again, the fact that $\mathbf{E}\,(\int_0^{t\varepsilon} e^{W_s^H}\, ds)^{-1}$ is a bounded and decreasing function of $t \in [1, \infty)$ easily follows from Lemma 2.

Combining all estimates obtained above for $H \in (0, \frac{1}{2}]$ we have the following estimate:

$$\mathbf{E}\, e^{\delta |\zeta_H|^{2H}} \leqq C_{\delta,H} + e^{\delta} + 2\, \mathbf{E} \biggl( \int_0^{\varepsilon} e^{W_s^H}\, ds \biggr)^{-1} \int_1^\infty e^{-(1/8 + f(\varepsilon) - \delta) t^{2H}}\, dt,$$

where the right-hand side is obviously finite when $\varepsilon$ is small enough and $\delta < \frac{1}{8}$.

This completes the proof of Theorem 1 with $\alpha_H \geqq \frac{1}{8}$ for all $H \in (0,1]$.

*Remark* 2. Using a different approach in [18] we found that

$$\alpha_H \geqq \frac{4H^2+2H-1}{2(2H+2)(2H+1)} \quad \text{for } H > \frac{\sqrt{5}-1}{4} = 0.3090\dots.$$

Comparing this estimate with the one obtained in the proof of Theorem 1 we get the following lower bounds:

$$\alpha_H \geqq \frac{1}{8} \quad \text{for } H \in (0, H_0],$$
$$\alpha_H \geqq \frac{4H^2+2H-1}{2(2H+2)(2H+1)} \quad \text{for } H \in [H_0, 1),$$

where $H_0 = (\sqrt{73}-1)/12 = 0.6287\dots$, $H_0$ is the largest root of the equation

$$\frac{4H^2+2H-1}{2(2H+2)(2H+1)} = \frac{1}{8}.$$

Recall that for $H=1$ the index $\alpha_1 = \frac{1}{2}$; see (9).

CONJECTURE. *There exists an index $\alpha_H$ such that*

$$\mathbf{E}\, e^{\delta|\zeta_H|^{2H}} < \infty \quad \textit{for } \delta < \alpha_H, \qquad \mathbf{E}\, e^{\delta|\zeta_H|^{2H}} = \infty \quad \textit{for } \delta > \alpha_H.$$

*Remark* 3. This conjecture is motivated by the result of Theorem 1 and the following result on the limit distribution of maximum likelihood estimator (MLE) $\xi_H$ for the case $H = 1/2$.

It is well known that the distribution of $\xi_{\frac{1}{2}}$ coincides with the distribution of a location of maximum of two-sided Brownian motion and is

$$\mathbf{P}(|\xi_{1/2}| > t) = (t+5)\Phi\Big(-\frac{\sqrt{t}}{2}\Big) - \sqrt{\frac{2t}{\pi}}\, e^{-t/8} - 3e^t\Phi\Big(-\frac{3\sqrt{t}}{2}\Big), \tag{17}$$

where $\Phi(t)$ is the standard normal distribution. This result can be easily derived from [24] and [23]. Using the well-known formula $\Phi(-x) = \sqrt{\frac{1}{2\pi x^2}}\, e^{-x^2/2}(1+o(1))$, $x \to \infty$, we have

$$\mathbf{P}(|\xi_{1/2}| > t) = \sqrt{\frac{32}{\pi t}}\, e^{-t/8}(1+o(1)), \qquad t \to \infty,$$

and hence

$$\mathbf{E}\, e^{\delta|\xi_{1/2}|} < \infty \quad \text{for } \delta < \frac{1}{8}, \qquad \mathbf{E}\, e^{|\xi_{1/2}|/8} = \infty.$$

Note that from (17) one can directly obtain

$$\operatorname{Var} \xi_{1/2} = 26. \tag{18}$$

This result appeared in [21] for the first time.

*Remark* 4. The reviewer of this paper indicated that the existence of exponential moments for $|\zeta_H|^{2H}$ can be retraced from the general results of Ibragimov–Hasminskii theory (see [12]) in combination with some results from [6].

**3. Identities for expectations of functions of $\zeta_H$.**

THEOREM 2. *Let $G(\zeta_H)$ be a measurable bounded function of $\zeta_H$, $H \in (0,1]$. Then*

$$\mathbf{E}\, G(\zeta_H) = \int_{-\infty}^{\infty} \mathbf{E}\,(G(\zeta_H - t)q_t)\, dt. \tag{19}$$

*Proof.* Using (1) and Lemma 1 with $\xi = W_t^H$, $X_s = W_s^H$, we obtain

$$\int_{-\infty}^{\infty} \mathbf{E}\,(G(\zeta_H - t)q_t)\, dt = \int_{-\infty}^{\infty} \mathbf{E}^{\mathbf{Q}}\biggl(G(\zeta_H - t)\biggl(\int_{-\infty}^{\infty} Z_s\, ds\biggr)^{-1}\biggr)\, dt$$

(using Lemma 1, after simplifications we get)

$$= \int_{-\infty}^{\infty} \mathbf{E}\, G\biggl(\int_{-\infty}^{\infty} s e^{W_s^H - |t-s|^{2H}/2}\, ds \biggl(\int_{-\infty}^{\infty} e^{W_s^H + |t|^{2H}/2 - |t-s|^{2H}/2}\, ds\biggr)^{-1} - t\biggr)$$
$$\times \biggl(\int_{-\infty}^{\infty} e^{W_s^H + |t|^{2H}/2 - |t-s|^{2H}/2}\, ds\biggr)^{-1} dt$$

(next, using (15) we have)

$$= \int_{-\infty}^{\infty} \mathbf{E}\biggl(G\biggl(\int_{-\infty}^{\infty} s e^{W_{s-t}^H - |t-s|^{2H}/2}\, ds \biggl(\int_{-\infty}^{\infty} e^{W_{s-t}^H - |t-s|^{2H}/2}\, ds\biggr)^{-1} - t\biggr)$$
$$\times e^{W_t^H - |t|^{2H}/2} \biggl(\int_{-\infty}^{\infty} e^{W_{s-t}^H - |t-s|^{2H}/2}\, ds\biggr)^{-1}\biggr)\, dt$$

(setting $s - t = u$)

$$= \int_{-\infty}^{\infty} \mathbf{E}\,\biggl(G\biggl(\int_{-\infty}^{\infty} u e^{W_u^H - |u|^{2H}/2}\, du \biggl(\int_{-\infty}^{\infty} e^{W_u^H - |u|^{2H}/2} du\biggr)^{-1}\biggr)$$
$$\times e^{W_t^H - |t|^{2H}/2} \biggl(\int_{-\infty}^{\infty} e^{W_u^H - |u|^{2H}/2}\, du\biggr)^{-1}\biggr)\, dt$$
$$= \int_{-\infty}^{\infty} \mathbf{E}\,(G(\zeta_H)q_t)\, dt = \mathbf{E}\biggl(G(\zeta_H)\int_{-\infty}^{\infty} q_t\, dt\biggr) = \mathbf{E}\, G(\zeta_H).$$

This completes the proof.

*Remark* 5. A discrete-time analogue of (19) for independent identically distributed (i.i.d.) observations can be found in [4, Lemma 2.18.1]; see also [8].

Further we use the notation

$$B_i = \int_{-\infty}^{\infty} t^i Z_t\, dt, \quad i = 0, 1, 2 \ldots, \qquad A_p = \int_{-\infty}^{\infty} |t|^p q_t\, dt, \quad p > 0.$$

Due to the fact that

$$q_t = Z_t \biggl(\int_{-\infty}^{\infty} Z_u\, du\biggr)^{-1} = \frac{Z_t}{B_0}$$

is a density function, and due to the Hölder inequality, we obtain from (1) that for any $p \geqq 1$

$$|\zeta_H|^p \leqq A_p. \tag{20}$$

COROLLARY 1. *For any* $H \in (0,1]$

$$\operatorname{Var}\zeta_H = \mathbf{E}\,\zeta_H^2 = \frac{1}{2}\,\mathbf{E}\,A_2. \tag{21}$$

*Proof.* Let $G(\zeta_H) = \min(|\zeta_H|^2, K)$ with a finite parameter $K > 0$. Then in view of Theorem 1 and passing to the limit as $K \to \infty$ (using the Lebesgue theorem and Fatou's lemma) we obtain

$$\mathbf{E}\,\zeta_H^2 = \int_{-\infty}^{\infty} \mathbf{E}\left((\zeta_H - t)^2 q_t\right) dt.$$

Note that by (20) for any $H \in (0,1]$

$$\mathbf{E}\,\zeta_H^2 = \mathbf{E}\,A_1^2 < \infty, \qquad \int_{-\infty}^{\infty} t^2\mathbf{E}\,q_t\,du < \infty.$$

Expanding $(\zeta_H - t)^2 = \zeta_H^2 - 2\zeta_H t + t^2$ gives

$$\begin{aligned}\mathbf{E}\,\zeta_H^2 &= \int_{-\infty}^{\infty} \mathbf{E}\left((\zeta_H^2 - 2\zeta_H t + t^2)\,\frac{Z_t}{B_0}\right) dt\\ &= \mathbf{E}\,\zeta_H^2 - 2\,\mathbf{E}\,\zeta_H\zeta_H + \mathbf{E}\,A_2\\ &= \mathbf{E}\,\zeta_H^2 - 2\,\mathbf{E}\,\zeta_H^2 + \mathbf{E}\,A_2.\end{aligned}$$

After simplifying we get (21).

*Remark* 6. Originally the identity (21) was proved in [8] for $H > \frac{1}{2}$. The method used in [8] was based on the fact that a similar identity is valid for Pitman estimators of a location parameter for independent identically distributed observations.

Theorem 1 can be used for derivation of various useful properties of the distribution of $\zeta_H$. As another example we present the following result.

COROLLARY 2. *For any* $H \in (0,1]$ *and* $k = 2, 4, 6, \ldots$ *there exist constants* $c_k > 0$ *such that*

$$\mathbf{E}\,\zeta_H^k \geqq c_k^k\,\mathbf{E}\,A_k, \tag{22}$$

*where* $c_k$ *is a unique positive root of the equation*

$$(x+1)^k - (x-1)^k + 2k(x^k - x^{k-1}) = 2. \tag{23}$$

*Proof.* Validity of the result for $k = 2$ can be seen from Corollary 1.

For the case $k \geqq 4$ we apply Theorem 1 with the polynomial $G(x) = x^k$. Then we obtain

$$\mathbf{E}\,\zeta_H^k = \mathbf{E}\,\zeta_H^k - k\,\mathbf{E}\,\zeta_H^{k-1}A_1 + \sum_{i=2}^{k-1}(-1)^i C_k^i\,\mathbf{E}\,\zeta_H^{k-i}A_i + \mathbf{E}\,A_k,$$

where $C_k^i$ are binomial coefficients. This implies

$$k\,\mathbf{E}\,\zeta_H^k = \sum_{i=2}^{k-1}(-1)^i C_k^i\,\mathbf{E}\,\zeta_H^{k-i}A_i + \mathbf{E}\,A_k,$$

and hence

$$k\,\mathbf{E}\,\zeta_H^k \geqq -\sum_{\substack{3\leqq i\leqq k-1\\ i-\text{odd}}} C_k^i\,\mathbf{E}\,|\zeta_H^{k-i}|A_i + \mathbf{E}\,A_k.$$

By the Hölder inequality

$$\mathbf{E}\,|\zeta_H^{k-i}|A_i \leqq (\mathbf{E}\,\zeta_H^k)^{1-i/k}(\mathbf{E}\,A_i^{k/i})^{i/k} \leqq (\mathbf{E}\,\zeta_H^k)^{1-i/k}(\mathbf{E}\,A_k)^{i/k},$$

we have

$$k\,\mathbf{E}\,\zeta_H^k \geqq \mathbf{E}\,A_k - \sum_{\substack{3\leqq i\leqq k-1\\ i-\text{odd}}} C_k^i(\mathbf{E}\,\zeta_H^k)^{1-i/k}(\mathbf{E}\,A_k)^{i/k}.$$

Set $x^k = (\mathbf{E}\,\zeta_H^k)/\mathbf{E}\,A_k$. Then

$$\frac{(\mathbf{E}\,\zeta_H^k)^{1-i/k}}{(\mathbf{E}\,A_k)^{1-i/k}} = \left(\frac{\mathbf{E}\,\zeta_H^k}{\mathbf{E}\,A_k}\right)^{1-i/k} = x^{k-i},$$

and therefore the last inequality is equivalent to

$$kx^k + \sum_{\substack{3\leqq i\leqq k-1\\ i=\text{odd}}} C_k^i x^{k-i} \geqq 1.$$

We can find a short expression for $\sum_{i=\text{odd}\geqq 3}^{k-1} C_k^i x^{k-i}$ using the following elementary identity:

$$2\sum_{\substack{3\leqq i\leqq k-1\\ i=\text{odd}}} C_k^i x^{k-i} = (x+1)^k - (x-1)^k - 2kx^{k-1}.$$

Hence we obtain $(x+1)^k - (x-1)^k + 2k(x^k - x^{k-1}) \geqq 2$. This implies the result.

*Remark* 7. One can easily verify that

$$c_k = \frac{D}{k}\,(1+o(1)), \qquad k\to\infty, \tag{24}$$

where $D = \log(1+\sqrt{2})$ is a unique positive root of the equation

$$\sinh D = 1.$$

The derivation of (24) is elementary and thus is omitted. We restrict ourselves to the illustration of the accuracy of the approximation (24) for $k = 100$, $D/100 \approx 8.813\times 10^{-3}$, and in this case the exact solution of (23) is $c_{100} = 8.8412\times 10^{-3}$.

## 4. Representation for $\mathbf{Var}\,(\boldsymbol{\zeta_H})$.

In this section, for the case $H \in [\frac{1}{2}, 1]$ we derive another representation for $\mathrm{Var}\,(\zeta_H)$ in terms of the function $g(m)$ defined in (4).

Furthermore we use the following parametrized random functions,

$$\alpha(m) = \int_{-\infty}^{\infty} ue^{mu} Z_u\,du, \quad \beta(m) = \int_{-\infty}^{\infty} e^{mu} Z_u\,du,$$

where $m$ is an auxiliary parameter. With this notation we have

$$\int_{-\infty}^{\infty} Z_u\,du = \beta(0), \quad \int_{-\infty}^{\infty} uZ_u\,du = \alpha(0), \quad \alpha(m) = \frac{\partial}{\partial m}\,\beta(m),$$

and

$$\zeta_H = \frac{\alpha(0)}{\beta(0)}.$$

Note that due to the symmetry property of fBm we have $g(m) = g(-m)$, and from inequality $\ln(a+b) \leqq \ln(a+1) + \ln(b+1)$, $a > 0$, $b > 0$, we have

$$g(m) \leqq \mathbf{E} \log\left(\int_{-\infty}^{0} e^{mu} Z_u\, du + 1\right) + \mathbf{E} \log\left(\int_{0}^{\infty} e^{mu} Z_u\, du + 1\right). \tag{25}$$

Let $m > 0$. The finiteness of the first expectation on the right-hand side of (25) is obvious due to the inequality $\log(x+1) \leqq x$ and the equality $\mathbf{E}\, Z_u = 1$.

The finiteness of the second expectation on the right-hand side of (25) for $m > 0$ can be shown as follows.

Note that

$$\begin{aligned}\ln\left(\int_0^\infty e^{mu} Z_u\, du + 1\right) &= \int_0^\infty \left(\int_0^s e^{mu} Z_u\, du + 1\right)^{-1} d\int_0^s e^{mu} Z_u\, du \\ &\leqq \int_1^\infty e^{ms} Z_s \left(\int_0^s Z_u\, du\right)^{-1} ds + e^m \int_0^1 Z_s \left(\int_0^s Z_u\, du\right)^{-1} ds \\ &\leqq \int_1^\infty e^{ms} Z_s \left(\int_0^s Z_u\, du\right)^{-1} ds + e^m.\end{aligned}$$

Since $\mathbf{E}\, Z_s(\int_0^s Z_u\, du)^{-1} = \mathbf{E}\, q_s \leqq Ce^{-\delta s^{2H}}$ for $s \geqq 1$ and $\delta < \frac{1}{8}$ (see the proof of Theorem 1 and (3); from now on $C$ is a generic constant) we can claim that $g(m), 0 < m < \frac{1}{8}$ is finite (recall that we assumed $H \in [\frac{1}{2}, 1]$). Obviously, $g(m)$ is a continuous function.

THEOREM 3. *Let $H \in [\frac{1}{2}, 1]$. Then the function $g(m)$ is twice continuously differentiable on the interval $m \in (-1/8, 1/8)$ and*

$$\operatorname{Var} \zeta_H = \left.\frac{\partial^2 g(m)}{\partial m^2}\right|_{m=0}.$$

*Proof.* Using the notation introduced above we have

$$\operatorname{Var} \zeta_H = \mathbf{E}\, \frac{\alpha^2(0)}{\beta^2(0)} = \lim_{m \to 0} \mathbf{E}\, \frac{\alpha^2(m)}{\beta^2(m)}.$$

The last equality can be justified by (3) and the estimate

$$\mathbf{E}\, \frac{\alpha^2(m)}{\beta^2(m)} = \mathbf{E}\left(\int_{-\infty}^{\infty} u\, \frac{e^{mu} Z_u}{\beta(m)}\, du\right)^2 \leqq \mathbf{E} \int_{-\infty}^{\infty} u^2\, \frac{e^{mu} Z_u}{\beta(m)}\, du < \infty.$$

By direct calculations we obtain for $m > 0$ that

$$\frac{\partial^2 \log \beta(m)}{\partial m^2} = -\frac{\alpha^2(m)}{\beta^2(m)} + \frac{\int_{-\infty}^{\infty} u^2 e^{mu} Z_u\, du}{\beta(m)}.$$

Applying the expectation to both sides of the last equality and using well-known theorems about differentiability of expectations with respect to a parameter, we obtain

$$\frac{\partial^2 g(m)}{\partial m^2} = \mathbf{E}\, \frac{\partial^2 \ln \beta(m)}{\partial m^2} = -\mathbf{E}\, \frac{\alpha^2(m)}{\beta^2(m)} + \mathbf{E}\, \frac{\int_{-\infty}^{\infty} u^2 e^{mu} Z_u\, du}{\beta(m)}, \tag{26}$$

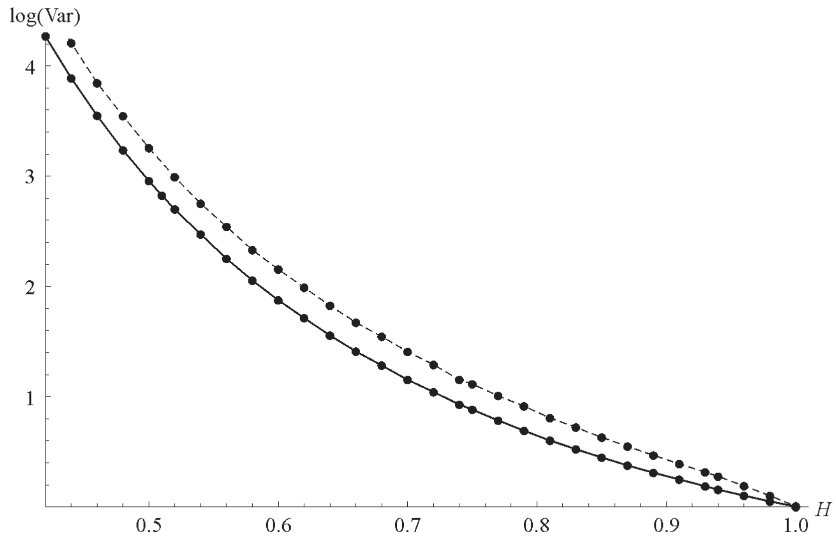


FIG. 1. *Values of* $\mathrm{Var}\,(\zeta_H)$ *(solid line) and* $\mathrm{Var}\,(\xi_H)$ *(dashed line) for* $H \in [0.4, 1]$ *are given on a logarithmic axis.*

where the right-hand side is a continuous function of $m$. This implies that $\frac{\partial^2 g(m)}{\partial m^2}$ is a continuous function for $m \in (0, 1/8)$ and (due to symmetry) also for $m \in (-1/8, 0)$. Passing to the limit in (26) as $m \to 0$, we obtain

$$\left.\frac{\partial^2 g(m)}{\partial m^2}\right|_{m=0} = -\mathbf{E}\,\frac{\alpha^2(0)}{\beta^2(0)} + \mathbf{E}\,\frac{\int_{-\infty}^{\infty} u^2 Z_u\,du}{\beta(0)} = -\mathrm{Var}\,\zeta_H + 2\,\mathrm{Var}\,\zeta_H = \mathrm{Var}\,\zeta_H.$$

This completes the proof.

**5. Modeling results.** To the best of our knowledge the problem of evaluation of integral functionals numerically remains unsolved. The only known explicit result is given by formula (5). These integral functionals can be modeled using the Monte Carlo simulation method. The results of Monte Carlo modeling for variances of Pitman estimator $\zeta_H$ and asymptotic MLE $\xi_H$ for $H \in [0.4, 1)$ are given in Table 1.

TABLE 1. *Monte Carlo estimates for* $\mathrm{Var}\,(\zeta_H)$ *and* $\mathrm{Var}\,(\xi_H)$ *using* $10^6$ *trajectories. Each trajectory is generated based on* $2^{18}$ *equally spaced discretization points on the interval* $(-10^5, 10^5)$.

| $H$ | $\widehat{\mathrm{Var}}\,\zeta_H$ | Stand. Error | $\widehat{\mathrm{Var}}\,\xi_H$ | Stand. Error |
|---|---|---|---|---|
| 0.4 | 109.682 | 0.4698 | 151.707 | 0.2145 |
| 0.5 | 19.2544 | 0.0350 | 25.964 | 0.0367 |
| 0.6 | 6.52596 | 0.0163 | 8.63501 | 0.0386 |
| 0.7 | 3.16871 | 0.0066 | 4.08858 | 0.0182 |
| 0.81 | 1.82699 | 0.0032 | 2.24197 | 0.0101 |
| 0.91 | 1.28289 | 0.002 | 1.47782 | 0.0066 |

For simulation of increments of fBm we implemented the “circulant embedding method” (see [22]) which is recognized as one of the fastest methods for simulation of stationary Gaussian processes.

The qualitative behavior of $\mathrm{Var}\,(\zeta_H)$ and $\mathrm{Var}\,(\xi_H)$ while $H \in [0.4, 1)$ is displayed in Figure 1. Both $\mathrm{Var}\,(\zeta_H)$ and $\mathrm{Var}\,(\xi_H)$ are monotone functions taking large values for small values of $H$, $\mathrm{Var}(\zeta_H) < \mathrm{Var}(\xi_H)$. The results of calculations agree well with (5) and (18). A detailed discussion of accuracy of $\mathrm{Var}(\zeta_H)$ and $\mathrm{Var}(\xi_H)$ is provided in [16].

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