

# On zero-error communication via quantum channels in the presence of noiseless feedback<sup>a</sup>

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We initiate the study of zero-error communication via quantum channels when the receiver and sender have at their disposal a noiseless feedback channel of unlimited quantum capacity, generalizing Shannon's zero-error communication theory with instantaneous feedback.

We first show that this capacity is a function only of the linear span of Kraus operators of the channel, which generalizes the bipartite equivocation graph of a classical channel, and which we dub "non-commutative bipartite graph". Then we go on to show that the feedback-assisted capacity is non-zero (allowing for a constant amount of activating noiseless communication) if and only if the non-commutative bipartite graph is non-trivial, and give a number of equivalent characterizations. This result involves a far-reaching extension of the "conclusive exclusion" of quantum states [Pusey/Barrett/Rudolph, Nature Physics 8(6):475-478, 2012].

We then present an upper bound on the feedback-assisted zero-error capacity, motivated by a conjecture originally made by Shannon and proved later by Ahlswede. We demonstrate this bound to have many good properties, including being additive and given by a nice minimax formula. We also prove a coding theorem showing that this quantity is the entanglement-assisted capacity against an adversarially chosen channel from the set of all channels with the same Kraus span, which can also be interpreted as the feedback-assisted *unambiguous* capacity. The proof relies on a generalization of the "Postselection Lemma" (de Finetti reduction) [Christandl/König/Renner, PRL 102:020503, 2009] that allows to reflect additional constraints, and which we believe to be of independent interest. This capacity is a relaxation of the feedback-assisted zero-error capacity; however, we have to leave open the question whether they coincide in general.

We illustrate our ideas with a number of examples, including classical-quantum channels and Weyl diagonal channels, and close with an extensive discussion of open questions.

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## I. ZERO-ERROR COMMUNICATION ASSISTED BY NOISELESS QUANTUM FEEDBACK

Consider a quantum channel  $\mathcal{N} : \mathcal{L}(A) \rightarrow \mathcal{L}(B)$ , i.e. a completely positive and trace preserving (cptp) linear map from the operators on  $A$  to those of  $B$  (both finite-dimensional Hilbert spaces), with Kraus and Stinespring representations

$$\mathcal{N}(\rho) = \sum_j E_j \rho E_j^\dagger = \text{Tr}_C V \rho V^\dagger,$$

for linear operators  $E_j : A \rightarrow B$  such that  $\sum_j E_j^\dagger E_j = \mathbb{1}$ , and an isometry  $V : A \rightarrow B \otimes C$ , respectively. We will discuss a model of communication where Alice uses the channel  $n$  times in succession, allowing Bob after each round to send her back an arbitrary quantum system. They may also share an entangled state prior to the first round (if not, they can have it anyway from the second round since Bob could use the first feedback to create an arbitrary entangled state). Their goal is to allow Alice to send one of  $M$  messages down the channel such that Bob is able to distinguish them perfectly. More formally, the most general *quantum feedback-assisted code* consists of a state (w.l.o.g. pure)  $|\phi\rangle \in X_0 \otimes Y_0$  and for each message  $m = 1, \dots, M$  isometries for encoding and feedback decoding

$$\begin{aligned} U_t^{(m)} : X_{t-1} \otimes F_{t-1} &\rightarrow A_t \otimes X_t, \\ W_t : Y_{t-1} \otimes B_t &\rightarrow F_t \otimes Y_t, \end{aligned} \tag{1}$$

for  $t = 1, \dots, n$  and appropriate local quantum systems  $X_0$  (Alice) and  $Y_t$  (Bob), as well the feedback-carrying systems  $F_t$ ; see Fig. 1. For consistency (and w.l.o.g.),  $F_0 = F_n = \mathbb{C}$  are trivial.

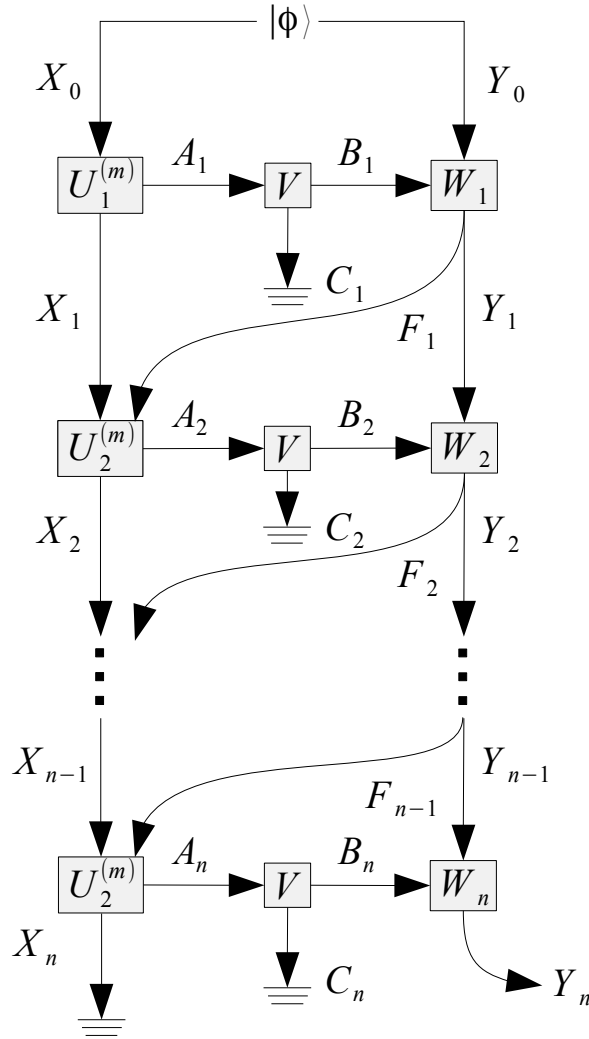


FIG. 1. Diagrammatic representation of a feedback-assisted code for messages  $m$  sent down a channel  $\mathcal{N}$  used  $n$  times, in the form of a schematic circuit diagram. All boxes are isometries (acting on suitably large input and output quantum registers), and the solid lines and arrows represent the “sending” of the respective register. Bob’s final output state  $\rho_m$  after  $n$  rounds of using the channel and feedback is in register  $Y_n$ .

We call it a *zero-error code* if there is a measurement on  $Y_n$  that distinguishes Bob’s output states  $\rho^{(m)} = \sum_{\underline{j}} \rho_{\underline{j}}^{(m)}$ , with certainty, where the sum is over the states

$$\rho_{\underline{j}}^{(m)} = \text{Tr}_{X_n} \left( \prod_{t=n}^1 (W_t E_{j_t} U_t^{(m)}) |\phi\rangle \langle \phi| \prod_{t=1}^n (U_t^{(m)\dagger} E_{j_t}^\dagger W_t^\dagger) \right), \quad (2)$$

the output states given a specific sequence  $\underline{j} = j_1 \dots j_n$  of Kraus operators. In other words, these states  $\rho_m$  have to have mutually orthogonal supports, i.e. for all  $m \neq m'$ , all  $\underline{j}, \underline{k}$  and all  $\xi \in \mathcal{L}(X_n)$ ,

$$0 = \langle \phi | \prod_{t=1}^n (U_t^{(m')\dagger} E_{j_t}^\dagger W_t^\dagger) \xi \prod_{t=n}^1 (W_t E_{k_t} U_t^{(m)}) | \phi \rangle =: \langle \phi_{\underline{j}}^{(m')} | \xi | \phi_{\underline{k}}^{(m)} \rangle.$$

By linearity, we see that this condition depends only on the linear span of the Kraus operators,

$$K = \mathcal{K}(\mathcal{N}) = \text{span}\{E_j : j\} \subset \mathcal{L}(A \rightarrow B),$$

in fact it can evidently be expressed as the orthogonality of a tensor defined as a function of  $|\phi\rangle$ , the  $U_t^{(m)}$  and  $W_t$ , to the subspace  $(K \otimes K^\dagger)^{\otimes n}$  – cf. similar albeit simpler characterisations of zero-error and entanglement-assisted zero-error codes in terms of the “non-commutative graph”  $S = K^\dagger K \subset \mathcal{L}(A)$  [18, 20, 21], and of no-signalling assisted zero-error codes in terms of the “non-commutative bipartite graph”  $K$  [22]. Thus we have proved

**Proposition 1** *A quantum feedback-assisted code for a channel  $\mathcal{N}$  being zero-error is a property solely of the Kraus space  $K = \mathcal{K}(\mathcal{N})$ . The maximum number of messages in a feedback-assisted zero-error code is denoted  $M_f(n; K)$ . Hence, the quantum feedback-assisted zero-error capacity of  $\mathcal{N}$ ,*

$$C_{0EF}(K) := \lim_{n \rightarrow \infty} \frac{1}{n} \log M_f(n; K) = \sup_n \frac{1}{n} \log M_f(n; K),$$

is a function only of  $K$ . □

In the case of a classical channel  $N : \mathcal{X} \rightarrow \mathcal{Y}$  with transition probabilities  $N(y|x)$ , assisted by classical noiseless feedback, the above problem was first studied – and completely solved – by Shannon [38]. Following Shannon, we introduce the (bipartite) *equivocation graph*  $\Gamma = \Gamma(N)$  on  $\mathcal{X} \times \mathcal{Y}$ , which has an edge  $xy$  iff  $N(y|x) > 0$ , i.e. the adjacency matrix is  $\Gamma(y|x) = [N(y|x)]$ ; furthermore the confusability graph  $G = G(N)$  on  $\mathcal{X}$ , with an edge  $x \sim x'$  iff there exists a  $y$  such that  $N(y|x)N(y|x') > 0$ , i.e., iff the neighbourhoods of  $x$  and  $x'$  in  $\Gamma$  intersect.

Note that for (the quantum realisation of) a classical channel, i.e.

$$\mathcal{N}(\rho) = \sum_{xy} N(y|x) |y\rangle\langle x| \rho |x\rangle\langle y|,$$

the corresponding subspace is given by

$$K = \text{span}\{|y\rangle\langle x| : xy \text{ is an edge in } \Gamma\},$$

so  $K$  should really be understood as the quantum generalisation of the equivocation graph (a *non-commutative bipartite graph*) [22], much as  $S = K^\dagger K$  was advocated in [21] as a quantum generalisation of an undirected graph.

Shannon proved

$$C_{0F}(N) = C_{0F}(\Gamma) = \begin{cases} \log \alpha^*(\Gamma) & \text{if } G \neq K_{\mathcal{X}} \text{ (the complete graph),} \\ 0 & \text{if } G = K_{\mathcal{X}} \text{ (iff } C_0(N) = 0). \end{cases} \quad (3)$$

Here,  $\alpha^*(\Gamma)$  is the so-called *fractional packing number* of  $\Gamma$ , defined as a linear programme, whose dual linear programme is the *fractional covering number* [35, 38]:

$$\begin{aligned} \alpha^*(\Gamma) &= \max \sum_x w_x \text{ s.t. } \forall x \ 0 \leq w_x, \forall y \ \sum_x w_x \Gamma(y|x) \leq 1, \\ &= \min \sum_y v_y \text{ s.t. } \forall y \ 0 \leq v_y, \forall x \ \sum_y v_y \Gamma(y|x) \geq 1. \end{aligned} \quad (4)$$

This number appears also in other zero-error communication problems, namely as the zero-error capacity of the channel assisted by no-signalling correlations [19].

The second case in eq. (3) of complete graph  $G$  is easy to understand: whatever the parties do, and regardless of the use of feedback, any two inputs may lead to the same output sequence, so not a single bit can be transmitted with certainty. In either case, Shannon showed that only some arbitrarily small rate of perfect communication (actually a constant amount, dependent only on  $\Gamma$ ) is sufficient to achieve what we might call the *activated capacity*  $\bar{C}_{0F}(N)$ , which is always equal to  $\log \alpha^*(\Gamma)$ . This was understood better in the work of Elias [24] who showed that the capacity of zero-error *list decoding* of  $N$  (with arbitrary but constant list size) is exactly  $\log \alpha^*(\Gamma)$ . Thus a coding scheme for  $N$  with feedback would consist of a zero-error list code with list size  $L$  and rate  $R \geq (1 - \frac{1}{L}) \log \alpha^*(\Gamma) - O(\frac{1}{L})$  for  $n$  uses of the channel  $N$ , followed by feedback in which Bob lets Alice know the list of  $L$  items in which he now knows the message falls, followed by a noiseless transmission of  $\log L$  bits of Alice to resolve the remaining ambiguity. Shannon's scheme [38] is based on a similar idea, but whittles down the list by a constant factor in each round, so Bob needs to update Alice on the remaining list each time. The constant noiseless communication at the end of this protocol can be transmitted using an unassisted zero-error code via the given channel  $N$  (at most  $\log L$  uses), or via an activating noiseless channel.

The dichotomy in eq. (3) has the following quantum channel analogue.

**Proposition 2** *For any non-commutative bipartite graph  $K = \mathcal{K}(N) < \mathcal{L}(A \rightarrow B)$ , the feedback-assisted zero-error capacity of  $K$  vanishes,  $C_{0EF}(K) = 0$ , if and only if the associated non-commutative graph is complete, i.e.  $S = K^\dagger K = \mathcal{L}(A)$ , which is equivalent to vanishing entanglement-assisted zero-error capacity,  $C_{0E}(S) = 0$ .*

**Proof** Clearly  $C_{0EF}(K) \geq C_{0E}(S)$  since on the right hand side we simply do not use feedback, but any code is still a feedback-assisted code. Hence, if the latter is positive then so is the former. It is well-known that if  $S \neq \mathcal{L}(A)$ , then  $C_{0E}(S) \geq 1 > 0$ , in fact each channel use can transmit at least one bit [20, 21].

Conversely, let us assume that  $C_{0E}(S) = 0$ , i.e.  $S = K^\dagger K = \mathcal{L}(A)$ . We will show by induction on  $t$  that for any two distinct messages, w.l.o.g.  $b = 0, 1$ , Bob's output states after  $t$  rounds,  $\rho_t^{(b)}$  on  $Y_t$ , cannot be orthogonally supported, meaning  $M_f(n; K) = 1$ . Here,

$$\rho_t^{(b)} = \sum_{j_1 \dots j_t} \text{Tr}_{X_t F_t} |\phi_{j_1 \dots j_t}^{(b)}\rangle \langle \phi_{j_1 \dots j_t}^{(b)}|, \text{ with}$$

$$|\phi_{j_1 \dots j_t}^{(b)}\rangle = \prod_{i=t}^1 W_i E_{j_i} W_i^{(b)} |\phi\rangle \in X_t \otimes F_t \otimes Y_t.$$

This is clearly true for  $t = 0$  since at that point Alice and Bob share only  $|\phi\rangle^{X_0 Y_0}$ , hence  $\rho_0^{(0)} = \rho_0^{(1)} = \text{Tr}_{X_0} \phi$ . For  $t > 0$ , let Bob after  $t - 1$  rounds have one of the states  $\rho_{t-1}^{(b)}$ ; by the induction hypothesis,  $\rho_{t-1}^{(0)} \not\perp \rho_{t-1}^{(1)}$  – by a slight abuse of notation meaning that the supports are not orthogonal, or equivalently that the operators are not orthogonal with respect to the Hilbert-Schmidt inner product. This means that there are indices  $j_1 \dots j_{t-1}$  and  $k_1 \dots k_{t-1}$  such that

$$(\phi_{j_1 \dots j_{t-1}}^{(0)})^{Y_{t-1}} := \left( \phi_{j_1 \dots j_{t-1}}^{(0)} \right)^{Y_{t-1}} \not\perp \left( \phi_{k_1 \dots k_{t-1}}^{(1)} \right)^{Y_{t-1}} =: (\phi_{t-1}^{(1)})^{Y_{t-1}}.$$

This can be expressed equivalently as

$$\text{Tr}_{Y_{t-1}} |\phi_0^{t-1}\rangle \langle \phi_1^{t-1}| \neq 0.$$

Now, in the  $t$ -th round, Alice applies the isometry  $U_t^{(b)} : X_{t-1} F_{t-1} \rightarrow X_t A$  to the  $X$  and  $F$  registers of  $|\phi_{t-1}^{(b)}\rangle$ , hence for  $|\psi_t^{(b)}\rangle = U_t^{(b)} |\phi_{t-1}^{(b)}\rangle$  (as we do not touch the  $Y_{t-1}$  register)

$$\text{Tr}_{Y_{t-1}} |\psi_t^{(0)}\rangle \langle \psi_t^{(1)}| = \text{Tr}_{Y_{t-1}} U_t^{(0)} |\phi_{t-1}^{(0)}\rangle \langle \phi_{t-1}^{(1)}| U_t^{(1)\dagger} \neq 0. \quad (5)$$

After that, the channel action consists in one of the Kraus operators  $E_j : A \rightarrow B$ . Let us assume, by contradiction, that Bob's states after the channel action were orthogonal, i.e. for all  $j$  and  $k$ ,

$$\mathrm{Tr}_{X_t} E_j \psi_t^{(0)} E_j^\dagger \perp \mathrm{Tr}_{X_t} E_k \psi_t^{(1)} E_k^\dagger.$$

In other words, for all  $j, k$  and operators  $\xi$  on  $X_t$ ,

$$\begin{aligned} 0 &= \langle \psi_t^{(1)} | \xi \otimes E_k^\dagger E_j \otimes \mathbb{1} | \psi_t^{(0)} \rangle \\ &= \mathrm{Tr} [(\xi \otimes E_k^\dagger E_j) \mathrm{Tr}_{Y_{t-1}} |\psi_t^{(0)}\rangle\langle\psi_t^{(1)}|]. \end{aligned}$$

But since  $\xi$  is arbitrary and the  $E_k^\dagger E_j$  span  $\mathcal{L}(A)$ , this would imply  $\mathrm{Tr}_{Y_{t-1}} |\psi_t^{(0)}\rangle\langle\psi_t^{(1)}| = 0$ , contradicting (5).

Thus, applying now also the isometry  $W_t : BY_{y-1} \rightarrow F_t Y_t$ , we find that there exist  $j_t$  and  $k_t$  such that

$$\left( \phi_{j_1 \dots j_t}^{(0)} \right)^{F_t Y_t} \not\perp \left( \phi_{k_1 \dots k_t}^{(1)} \right)^{F_t Y_t}, \text{ hence } \left( \phi_{j_1 \dots j_t}^{(0)} \right)^{Y_t} \not\perp \left( \phi_{k_1 \dots k_t}^{(1)} \right)^{Y_t},$$

and so finally  $\rho_t^{(0)} \not\perp \rho_t^{(1)}$ , proving the induction step.  $\square$

Motivated by  $\overline{C}_{0F}$  of a classical channel [38], see above, we define also feedback-assisted codes with  $n$  channel uses and up to  $b$  noiseless classical bits of forward communication. The setup is the same as in eq. (1) and Fig. 1 with  $n + b$  rounds,  $n$  of which sport the isometric dilation  $V$  of  $\mathcal{N}$ , and  $b$  the isometry  $V' : |i\rangle \mapsto |i\rangle|i\rangle$  ( $i = 0, 1$ ) corresponding to the noiseless bit channel  $\overline{\mathrm{id}}_2 : \rho \mapsto \sum_{i=0}^1 |i\rangle\langle i| \rho |i\rangle\langle i|$ . It is clear that the output states can be written in way similar to eq. (2), and that the maximum number of messages in a zero-error code depends only on  $n, b$  and  $K < \mathcal{L}(A \rightarrow B)$ , which we denote  $M_f^{+b}(n; K)$ . Clearly,  $M_f^{+0}(n; K) = M_f(n; K)$  and in general,  $M_f^{+b+1}(n; K) \geq 2 M_f^{+b}(n; K)$ . Furthermore, it can easily be verified that

$$2^{-b} M_f^{+b}(n; K) 2^{-c} M_f^{+c}(m; K) \leq 2^{-b-c} M_f^{+b+c}(n+m; K),$$

hence we can define the *activated feedback-assisted zero-error capacity*

$$\begin{aligned} \overline{C}_{0EF}(K) &:= \sup_b \sup_n \frac{1}{n} (\log M_f^{+b}(n; K) - b) \\ &= \sup_b \lim_{n \rightarrow \infty} \frac{1}{n} \log M_f^{+b}(n; K). \end{aligned}$$

Then the above Proposition 2 can be rephrased as

$$C_{0EF}(K) = \begin{cases} \overline{C}_{0EF}(K) & \text{if } S = K^\dagger K \neq \mathcal{L}(A), \\ 0 & \text{if } S = \mathcal{L}(A) \text{ (iff } C_{0E}(S) = 0), \end{cases} \quad (6)$$

motivating our focussing on  $\overline{C}_{0EF}(K)$  in the remainder of the paper.

The rest of the present paper is organized as follows: In Section II we start with a concrete example showing the importance of measurements “conclusively excluding” hypotheses from a list of options, and go on to show several concise characterizations of nontrivial channels, i.e. those for which  $\overline{C}_{0EF}(K) > 0$ . In Section III we first review a characterization of the fractional packing number in terms of the Shannon capacity minimized over a set of channels, which then motivates the definition of  $C_{\min E}(K)$  obtained as minimization of the entanglement-assisted capacity over

quantum channels consistent with the given non-commutative bipartite graph.  $C_{\min E}(K)$  represents our currently best upper bound on the feedback-assisted zero-error capacity. We illustrate the bound by showing how it allows us to determine  $\overline{C}_{0EF}(K)$  for Weyl diagonal channels, i.e.  $K$  spanned by discrete Weyl unitaries. We also show that  $C_{\min E}(K)$  is the ordinary (small error) capacity of the system assisted by entanglement, against an adversarial choice of the channel (proof in Appendix A, based on a novel Constrained Postselection Lemma, aka “de Finetti reduction”, in Appendix B). After that, we conclude in Section IV with a discussion of open questions and future work.

## II. CHARACTERIZATION OF VANISHING CAPACITY $\overline{C}_{0EF}(K)$

In this section, we will prove the following result.

**Theorem 3** *If the non-commutative bipartite graph  $K < \mathcal{L}(A \rightarrow B)$  contains a subspace  $|\beta\rangle \otimes A < K$  with a state vector  $|\beta\rangle \in B$ , meaning that the constant channel  $\mathcal{N}_0 : \rho \mapsto |\beta\rangle\langle\beta| \text{Tr } \rho$  has  $\mathcal{K}(\mathcal{N}_0) < K$ , then  $\overline{C}_{0EF}(K) = 0$ ; we call such  $K$  trivial.*

*Conversely, if  $K$  is nontrivial, then  $\overline{C}_{0EF}(K) > 0$ .*

**Proof (trivial  $\Rightarrow$  zero capacity)** We show the stronger statement  $M_f^{+b}(n; K) = 2^b$  for all  $n$  and  $b$ . Indeed, as the zero-error condition is only a property of  $K$ , we may assume a concrete constant channel  $\mathcal{N}_0$  with  $\mathcal{K}(\mathcal{N}_0) = |\psi_0\rangle \otimes A < K$ . The outputs of the  $n$  copies of  $\mathcal{N}_0$  in the feedback code do not matter at all as they are going to be  $\psi_0^{\otimes n}$ , which Bob can create himself. Hence the only information arriving at Bob’s from Alice is in the  $b$  classical bits in the course of the protocol. But even assisted by entanglement and feedback, Alice can convey at most  $b$  noiseless bits in this way, cf. [5].  $\square$

The opposite implication (“nontrivial  $\Rightarrow$  positive capacity”) will be the subject of the remainder of this section. We will start by looking at cq-channels first – Subsection II A for pure state cq-channels, Subsection II B for a mixed state example and Subsection II C for general cq-channels –, before completing the proof for general channels in Subsection II D.

### A. Pure state cq-channels

For a given orthonormal basis  $\{|i\rangle\}$  of the input space  $A$ , and pure states  $|\psi_i\rangle$  in the output space, consider the cq-channel

$$\mathcal{N}_0(\rho) = \sum_i |\psi_i\rangle\langle i|\rho|i\rangle\langle\psi_i|,$$

with Kraus subspace

$$K := \mathcal{K}(\mathcal{N}_0) = \text{span}\{|\psi_i\rangle\langle i|\}.$$

We shall demonstrate first the following result:

**Proposition 4** *For a pure state cq-channel,  $\overline{C}_{0EF}(K)$  is always positive unless  $K$  is trivial which is equivalent to all  $|\psi_i\rangle$  being collinear, i.e.  $K = |\psi\rangle \otimes A$ .*

**Proof** If  $K$  is trivial, then the above proof of the sufficiency of triviality in Theorem 3 shows  $\overline{C}_{0EF}(K) = 0$ .

Conversely, if  $K$  is non-trivial, then there are two output vectors, denoted  $|\psi_0\rangle$  and  $|\psi_1\rangle$ , that are not collinear, and we shall simply restrict the channel to the corresponding inputs 0 and 1. I.e., we focus only on  $K' = \text{span}\{|\psi_0\rangle\langle 0|, |\psi_1\rangle\langle 1|\}$ , and the corresponding channel

$$\mathcal{N}'(\rho) = |\psi_0\rangle\langle 0|\rho|0\rangle\langle\psi_0| + |\psi_1\rangle\langle 1|\rho|1\rangle\langle\psi_1|.$$

Consider using it three times, inputting only the code words 001, 010 and 100. This gives rise to output states

$$\begin{aligned} |u_a\rangle &= |\psi_0\rangle|\psi_0\rangle|\psi_1\rangle, \\ |u_b\rangle &= |\psi_0\rangle|\psi_1\rangle|\psi_0\rangle, \\ |u_c\rangle &= |\psi_1\rangle|\psi_0\rangle|\psi_0\rangle, \end{aligned}$$

which have the property that their pairwise inner products are all equal:  $\langle u_x|u_y\rangle = |\langle\psi_0|\psi_1\rangle|^2 =: \epsilon$ . By using the channel  $3n$  times, Alice can prepare the states

$$|t_x\rangle = |u_x\rangle^{\otimes n} \quad (x = a, b, c),$$

whose pairwise inner products are all equal and indeed  $\epsilon^n$ , i.e. arbitrarily close to 0. Now, if  $n$  is large enough (so that  $\epsilon^n \leq \frac{1}{2}$ ), there is a ctp map that Bob can apply to transform

$$\begin{aligned} |t_a\rangle &\longmapsto \frac{1}{\sqrt{2}}(|1\rangle + |2\rangle), \\ |t_b\rangle &\longmapsto \frac{1}{\sqrt{2}}(|2\rangle + |0\rangle), \\ |t_c\rangle &\longmapsto \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), \end{aligned}$$

(This follows from well-known results on pure-state transformations, see e.g. [11].) By now it may be clear where this is going: Bob measures the computational basis and overall we obtain a classical channel  $P : \{a, b, c\} \rightarrow \{0, 1, 2\}$  with exactly one 0-entry in each row and column:

$$P(0|a) = P(1|b) = P(2|c) = 0,$$

which has zero-error capacity 0, but assisted by feedback and a finite number of activating noiseless bits, it is  $\log \frac{3}{2}$  [38]. We conclude that  $\overline{C}_{0EF}(\mathcal{N}) \geq \frac{1}{3n} \log \frac{3}{2} > 0$ .  $\square$

## B. Mixed state cq-channel

To generalize the previous treatment to mixed states, let us first look at a specific simple example: Let  $|\psi_i\rangle$  ( $i = 0, 1, 2$ ) be three mutually distinct but non-orthogonal states in  $\mathbb{C}^3$ , and define a cq-channel  $\mathcal{N}_0$  with three inputs  $i = 0, 1, 2$ , mapping

$$\begin{aligned} 0 &\longmapsto \frac{1}{2}\psi_1 + \frac{1}{2}\psi_2, \\ 1 &\longmapsto \frac{1}{2}\psi_0 + \frac{1}{2}\psi_2, \\ 2 &\longmapsto \frac{1}{2}\psi_0 + \frac{1}{2}\psi_1. \end{aligned} \tag{7}$$

Thus,

$$K = \text{span}\{|\psi_1\rangle\langle 0|, |\psi_2\rangle\langle 0|, |\psi_0\rangle\langle 1|, |\psi_2\rangle\langle 1|, |\psi_0\rangle\langle 2|, |\psi_1\rangle\langle 2|\},$$



and the most general channel  $\mathcal{N}$  consistent with this  $K$  is a cq-channel of the form

$$i \mapsto \rho_i, \quad \rho_i \text{ supported on } \text{span}\{|\psi_j\rangle : j \in \{0, 1, 2\} \setminus i\}.$$

We shall show how to construct a zero-error scheme with feedback, achieving positive rate, at least for  $|\psi_i\rangle$  that are sufficiently close to being orthogonal. For the zero-error properties, we may as well focus on  $\mathcal{N}_0$ , which is easier to reason with. For the following, it may be helpful to think of eq. (7) in a partly classical way: any input  $i$  is mapped to a random  $|\psi_j\rangle$ , subject to  $j \neq i$ , so that for two uses of the channel, each pair  $i_1 i_2$  is mapped randomly to one of four  $|\psi_{j_1}\rangle|\psi_{j_2}\rangle$ , with  $j_1 \neq i_1, j_2 \neq i_2$ . Of course, vice versa each of these nine vectors is reached from exactly four inputs.

Now, assuming that the pairwise inner products of the  $|\psi_i\rangle$  are small enough, i.e.

$$|\langle\psi_0|\psi_1\rangle|, |\langle\psi_0|\psi_2\rangle|, |\langle\psi_1|\psi_2\rangle| \leq \epsilon,$$

there is a deterministic pure state transformation (by ctp map)  $|\psi_{j_1}\rangle|\psi_{j_2}\rangle \mapsto |\varphi_{j_1 j_2}\rangle$  [11], where

$$|\varphi_{j_1 j_2}\rangle = \frac{1}{\sqrt{8}} \sum_{\substack{j_1 j_2 \in I \in \{0,1,2\}^2 \\ |I|=2}} |I\rangle \in \mathbb{C}^{36}.$$

On these states, Bob performs a measurement in the computational basis of the  $|I\rangle$ , and we get an effective classical channel mapping  $i_1 i_2 \in \{0, 1, 2\}^2$  randomly to some  $\{j_1 j_2, k_1 k_2\} = I \subset \{0, 1, 2\}^2$ , subject to the constraint

$$(j_1 \neq i_1 \ \& \ j_2 \neq i_2) \text{ or } (k_1 \neq i_1 \ \& \ k_2 \neq i_2),$$

which means that each  $I$  is reached from at most eight out of the nine pairs  $i_1 i_2$ . In fact, the observation of  $I = \{j_1 j_2, k_1 k_2\}$  excludes at least two out of nine input symbols, namely  $j_1 k_2$  and  $k_1 j_2$ , meaning that this classical channel has zero-error capacity (plus feedback plus a finite number of noiseless bits) of  $\geq \log \frac{9}{7}$ . In conclusion, we achieve for  $\mathcal{N}_0$ , and hence for any  $\mathcal{N}$  with  $\mathcal{K}(\mathcal{N}) < K$ , a rate of  $\geq \frac{1}{2} \log \frac{9}{7} > 0$ .  $\square$

### C. General cq-channels

The above examples rely on measuring the output states  $\rho_i$  of the cq-channel  $\mathcal{N}$  by a POVM  $(M_j)$  such that the resulting classical(!) channel  $N : i \rightarrow j$  with  $N(J|i) = \text{Tr } \rho_i M_j$  has an equivocation graph  $\Gamma$  with  $\alpha^*(\Gamma) > 1$ , because then  $\overline{C}_{0EF}(K) \geq \overline{C}_{0F}(\Gamma) = \log \alpha^*(\Gamma) > 0$ . For this, cf. eq. (4), it is necessary and sufficient that each outcome  $j$  excludes at least one input  $i$ , i.e.  $N(j|i) = \text{Tr } \rho_i M_j = 0$ , or equivalently  $\rho_i \perp M_j$ . A POVM  $(M_j)$  with thus property is said to “conclusively exclude” the set  $\{\rho_i\}$  of states [4, 34]. It is clearly only a property of the support projections  $P_i$  of  $\rho_i$ , and w.l.o.g. the POVM is indexed by the same  $i$ ’s, i.e.  $(R_i)$  such that  $P_i R_i = 0$  for all  $i$ , as well as  $R_i \geq 0$  and  $\sum_i R_i = \mathbb{1}$ .

Our approach in the following will be to characterize when a set  $\{\rho_i\}$  of states, or one of its tensor powers  $\{\rho_{\underline{i}}\}^{\otimes n} = \{\rho_{i_1} \otimes \dots \otimes \rho_{i_n}\}$ , can be conclusively excluded. For instance, Pusey, Barrett and Rudolph [34] showed that for any two linearly independent pure states  $|\psi_0\rangle$  and  $|\psi_1\rangle$ , it is always possible to find an integer  $n$  and a  $2^n$ -outcome POVM  $(R_{\underline{i}} : \underline{i} \in \{0, 1\}^n)$  such that

$$\text{Tr } R_{\underline{i}} |\psi_{\underline{i}}\rangle\langle\psi_{\underline{i}}| = 0, \quad |\psi_{\underline{i}}\rangle = |\psi_{i_1}\rangle \otimes |\psi_{i_2}\rangle \otimes \dots \otimes |\psi_{i_n}\rangle.$$

I.e. we can design a quantum measurement that can conclusively exclude the  $n$ -fold states  $|\psi_{\underline{i}}\rangle$  with  $n$ -bit strings  $\underline{i} = i_1 \dots i_n$  as outcomes, even when  $|\psi_0\rangle$  and  $|\psi_1\rangle$  are not orthogonal.

We will employ the powerful techniques developed in the proof of [22, Prop. 14], allowing us to show a far-reaching generalization of the Pusey/Barrett/Rudolph result [34]. The version we need can be stated as follows; it is adapted to a cq-channel with  $a$ -dimensional input space  $A$  and output states  $\rho_i$  ( $i = 1, \dots, a$ ), whose support projectors are  $P_i$  and supports  $K_i$ , so that the non-commutative graph is  $K = \bigoplus_i K_i \otimes \langle i |$ .

**Proposition 5** *Let  $(P_i)_{i=1}^a$  be projectors on a Hilbert space  $B$ , with a transitive group action by unitary conjugation on the  $P_i$ , i.e. we have a finite group  $G$  acting transitively on the labels  $i$ , and a unitary representation  $U^g$  such that  $P_i^g = (U^g)^\dagger P_i U^g$  for  $g \in G$ .*

*Consider the isotypical decomposition of  $U^g$ ,*

$$B = \bigoplus_{\lambda} \mathcal{Q}_{\lambda} \otimes \mathcal{R}_{\lambda}$$

*into irreps  $\mathcal{Q}_{\lambda}$  of  $U^g$ , with multiplicity spaces  $\mathcal{R}_{\lambda}$  (cf. [27], see also [12, 28]). Denote the number of terms  $\lambda$  by  $L$ , and the largest occurring multiplicity by  $M = \max_{\lambda} |\mathcal{R}_{\lambda}|$ . If now*

$$\frac{a}{\|\sum_i P_i\|_{\infty}} > 16L^6 M^9,$$

*then there exists a POVM  $(R_i)$  with  $P_i R_i = 0$  for all  $i$ . In other words, any set  $\{\rho_i\}$  with  $\text{supp } \rho_i \subset K_i$  can be conclusively excluded.*

Before we prove it, we use it to derive the following general result. To state it, we need some notation: For a set  $\mathcal{E} = \{\rho_i\}_{i=1}^a$  of states, let

$$\mathcal{E}^{\otimes n} = \{\rho_{\underline{i}} = \rho_{i_1} \otimes \dots \otimes \rho_{i_n} : \underline{i} = i_1 \dots i_n \in [a]^n\}.$$

The strings  $\underline{i} = i_1 \dots i_n$  are classified according to *type*  $\tau$  [17], which is the empirical distribution of the letters  $i_t$ ,  $t = 1, \dots, n$ . There are only  $\binom{n+a-1}{a-1} \leq (n+1)^a$  many different types. The subset of  $\mathcal{E}^{\otimes n}$  corresponding to type  $\tau$  is denoted

$$\mathcal{E}_{\tau}^{(n)} = \{\rho_{\underline{i}} = \rho_{i_1} \otimes \dots \otimes \rho_{i_n} : \underline{i} = i_1 \dots i_n \text{ has type } \tau\}.$$

We also recall the definition of the *semidefinite packing number* [22] of a non-commutative bipartite graph  $K$  with support projection  $P_{AB}$  onto the Choi-Jamiołkowski range  $(\mathbb{1} \otimes K)|\Phi\rangle$ , where  $|\Phi\rangle = \frac{1}{\sqrt{|A|}} \sum_{i=1}^{|A|} |i\rangle|i\rangle$  is the maximally entangled state:

$$\begin{aligned} A(K) &= \max \text{Tr } S_A \text{ s.t. } 0 \leq S_A, \text{Tr}_A P_{AB}(S_A \otimes \mathbb{1}_B) \leq \mathbb{1}_B \\ &= \min \text{Tr } T_B \text{ s.t. } 0 \leq T_B, \text{Tr}_B P_{AB}(\mathbb{1}_A \otimes T_B) \geq \mathbb{1}_A. \end{aligned} \quad (8)$$

For the cq-channel case,  $P = \sum_i |i\rangle\langle i|^A \otimes P_i^B$ , this simplifies to

$$A(K) := \max \sum_i s_i \text{ s.t. } 0 \leq s_i, \sum_i s_i P_i \leq \mathbb{1}. \quad (9)$$

**Theorem 6** *Let  $\mathcal{E} = \{\rho_i\}_{i=1}^a$  be a finite set of quantum states with supports  $K_i = \text{supp } \rho_i$ , and let  $K$  be the associated non-commutative bipartite graph  $K = \bigoplus_i K_i \otimes \langle i |$ . Then the following are equivalent:*

- i.  $\overline{C}_{0EF}(K) > 0$ ;
- ii.  $K$  is nontrivial;

iii.  $\bigcap_i K_i = 0$ ;

iv.  $\|\sum_i P_i\|_\infty < a$ ;

v.  $A(K) > 1$ ;

vi. For sufficiently large  $n$  and a suitable type  $\tau$ , the set  $\mathcal{E}_\tau^{(n)}$  can be deterministically excluded.

**Proof**  $i. \Rightarrow ii.$  has been shown in the first part (necessity) of Theorem 3, at the start of this section.

$ii. \Leftrightarrow iii.$   $|\beta\rangle \otimes A < K = \bigoplus_i K_i \otimes \langle i|$  if and only if  $|\beta\rangle \in \bigcap_i K_i$ .

$iii. \Leftrightarrow iv.$   $\|\sum_i P_i\|_\infty \leq \sum_i \|P_i\|_\infty = a$  with equality if and only if there is a common eigenvector  $|\beta\rangle$  with eigenvalue 1 to all of the  $P_i$ , i.e.  $|\beta\rangle \in \bigcap_i K_i$ .

$iv. \Rightarrow v.$  We check that  $s_i = \frac{1}{\|\sum_i P_i\|_\infty}$  is feasible for  $A(K)$ ; indeed,

$$\sum_i s_i P_i = \frac{1}{\|\sum_i P_i\|_\infty} \sum_i P_i \leq \mathbf{1},$$

thus  $A(K) \geq \frac{a}{\|\sum_i P_i\|_\infty} > 1$ .

$v. \Rightarrow vi.$  Note that the non-commutative bipartite graph corresponding to  $\mathcal{E}^{\otimes n}$  is  $K^{\otimes n}$ . Let's denote the graph of  $\mathcal{E}_\tau^{(n)}$  by  $K_\tau^{(n)}$ . In [22] it is shown that  $A(K)$  is multiplicative,  $A(K^{\otimes n}) = A(K)^n$ ; indeed, for an optimal assignment of weights  $s_i$  feasible for  $A(K)$ ,  $s_{\underline{i}} = s_{i_1} \cdots s_{i_n}$  is feasible (and optimal) for  $A(K^{\otimes n})$ . Hence, there exists a type  $\tau$  such that

$$A(K_\tau^{(n)}) \geq \sum_{\underline{i} \in \tau} s_{\underline{i}} \geq \frac{1}{\text{poly}(n)} A(K)^n. \quad (10)$$

On the other hand, the symmetric group  $S_n$  acts transitively by permutation on the strings of type  $\tau$ , and equivalently by permutation of the  $n$  tensor factors of  $B^n$ . This representation is well-known to have only  $L \leq \text{poly}(n)$  irreps, each of which has multiplicity  $M \leq \text{poly}(n)$ . Thus, from eq. (10), we deduce that for sufficiently large  $n$ ,  $A(K_\tau^{(n)}) \geq 16L^6 M^9$ , which by Proposition 5 implies that the set  $\mathcal{E}_\tau^{(n)}$  can be conclusively excluded.

$vi. \Rightarrow i.$  By sending signals  $\underline{i} = i_1 \dots i_n \in \tau$  and measuring the output states  $\rho_{\underline{i}}$  with a conclusively excluding POVM  $(M_{\underline{i}} : \underline{i} \in \tau)$ , we simulate a classical channel whose bipartite equivocation graph  $\Gamma$  has  $\alpha^*(\Gamma) > 1$ , hence  $\overline{C}_{0EF}(K) \geq \frac{1}{n} \overline{C}_{0F}(\Gamma) > 0$ .  $\square$

**Proof (of Proposition 5)** Assume that we have a feasible  $s^* = s_i$  ( $i = 1, \dots, a$ ) for  $A(K)$  such that  $\sum_i s_i s^* a \geq 16L^6 M^9$ . Concretely, this means that  $\sum_i s_i P_i = s^* \sum_i P_i \leq \mathbf{1}$ .

We will show that a desired POVM  $(R_i)$  can be found, such that  $R_{ig} = (U^g)^\dagger R_0 U^g$  for all  $i$  and  $g$ . The problem of finding the POVM  $(R_i)$  then becomes equivalent to finding  $R_0 \geq 0$  and  $R_0 \leq \mathbf{1} - P_0$  such that

$$\frac{1}{a} \sum_{i=1}^a R_i = \frac{1}{|G|} \sum_{g \in G} (U^g)^\dagger R_0 U^g = \frac{1}{a} \mathbf{1}. \quad (11)$$

Schur's Lemma [27] tells us

$$\frac{1}{|G|} \sum_g (U^g)^\dagger R_0 U^g = \frac{1}{a} \sum_\lambda Q_\lambda \otimes \zeta_\lambda,$$

where  $Q_\lambda$  is the projection onto the irrep  $\mathcal{Q}_\lambda$ ,  $\zeta_\lambda$  is a semidefinite operator on  $\mathcal{R}_\lambda$ . The equality constraints (11) on  $R_0$  is equivalent to  $\zeta_\lambda = \Pi_\lambda$ , the projection onto  $\mathcal{R}_\lambda$ , for all  $\lambda$ .

Now, for each  $\lambda$  choose an orthogonal basis  $\{Z_\mu^{(\lambda)}\}$  of Hermitians over  $\mathcal{R}_\lambda$ , with  $Z_0^{(\lambda)} = \frac{1}{\text{Tr}\Pi_\lambda}\Pi_\lambda$  and  $\|Z_\mu^{(\lambda)}\|_2 = 1$  for  $\mu \neq 0$ . Then the operators  $\frac{1}{\text{Tr}Q_\lambda}Q_\lambda \otimes Z_\mu^{(\lambda)}$  form a basis of the  $U^g$ -invariant operators, hence our constraints on  $R_0$  can be rephrased as

$$0 \leq R_0 \leq \mathbb{1} - P_0, \quad \text{Tr} R_0 \left( \frac{Q_\lambda}{\text{Tr}Q_\lambda} \otimes Z_\mu^{(\lambda)} \right) = \frac{1}{a} \delta_{\mu 0} \forall \lambda \mu. \quad (12)$$

Notice that here, the semidefinite constraints on  $R_0$  leave quite some room, whereas we have “only”  $LM^2$  linear conditions to satisfy. Given  $s^*$  satisfying the constraint of  $A(K)$ , our strategy now will be to show that we can construct a  $0 \leq R_0 \leq \frac{2}{a}(\mathbb{1} - P_0)$  such that the above equations are true with  $s = s^*$  on the left hand side.

In detail, introduce a new variable  $X \geq 0$ , with

$$R_0 = \frac{1}{a}(\mathbb{1} - P_0)X(\mathbb{1} - P_0),$$

which makes sure that  $R_0$  is automatically supported on the complement of  $P_0$ . Now rewrite the conditions (12) in terms of  $X$ , introducing the notation

$$C_{\lambda\mu} = \frac{1}{\text{Tr}Q_\lambda}Q_\lambda \otimes Z_\mu^{(\lambda)}, \quad D_{\lambda\mu} = (\mathbb{1} - P_0)C_{\lambda\mu}(\mathbb{1} - P_0).$$

This gives the new form of the constraints as

$$\text{Tr} X D_{\lambda\mu} = \delta_{\mu 0}. \quad (13)$$

Our goal will be to find a “nice” dual set  $\{\widehat{D}_{\lambda\mu}\}$  to the  $\{D_{\lambda\mu}\}$ , i.e.  $\text{Tr} D_{\lambda\mu} \widehat{D}_{\lambda'\mu'} = \delta_{\lambda\lambda'} \delta_{\mu\mu'}$ , with which we can write a solution  $X = \sum_{\lambda\mu} \delta_{\mu 0} \widehat{D}_{\lambda\mu} = \sum_{\lambda} \widehat{D}_{\lambda 0}$ . To this end, we construct first the dual set  $\widehat{C}_{\lambda\mu}$  of the  $\{C_{\lambda\mu}\}$ , which is easy:

$$\widehat{C}_{\lambda\mu} = Q_\lambda \otimes \widehat{Z}_\mu^{(\lambda)} = \begin{cases} Q_\lambda \otimes \Pi_\lambda & \text{for } \mu = 0, \\ Q_\lambda \otimes Z_\mu^{(\lambda)} & \text{for } \mu \neq 0, \end{cases}$$

so that indeed  $\text{Tr} C_{\lambda\mu} \widehat{C}_{\lambda'\mu'} = \delta_{\lambda\lambda'} \delta_{\mu\mu'}$ . Now, consider the  $LM^2 \times LM^2$ -matrix  $T$ ,

$$\begin{aligned} T_{\lambda\mu, \lambda'\mu'} &= \text{Tr} D_{\lambda\mu} \widehat{C}_{\lambda'\mu'} \\ &= \text{Tr}(\mathbb{1} - P_0)C_{\lambda\mu}(\mathbb{1} - P_0)\widehat{C}_{\lambda'\mu'} \\ &= \delta_{\lambda\lambda'} \delta_{\mu\mu'} - \Delta_{\lambda\mu, \lambda'\mu'}, \end{aligned}$$

with the deviation

$$\Delta_{\lambda\mu, \lambda'\mu'} = \text{Tr} P_0 C_{\lambda\mu} (\mathbb{1} - P_0) \widehat{C}_{\lambda'\mu'} + \text{Tr} C_{\lambda\mu} P_0 \widehat{C}_{\lambda'\mu'}.$$

Here,

$$\begin{aligned} |\Delta_{\lambda\mu, \lambda'\mu'}| &\leq 2 \|P_0 C_{\lambda\mu}\|_1 \|\widehat{C}_{\lambda'\mu'}\|_\infty \\ &\leq 2 \|P_0 C_{\lambda\mu}\|_1 = 2 \|P_0 |C_{\lambda\mu}|\|_1 \\ &\leq 2 \sqrt{\text{Tr} P_0 |C_{\lambda\mu}|} \sqrt{\|C_{\lambda\mu}\|_1}, \end{aligned}$$

using  $\|\widehat{C}_{\lambda'\mu'}\|_\infty \leq 1$ , the unitary invariance of the trace norm, and Lemma 7 stated below. Since  $|C_{\lambda\mu}| = \frac{1}{\text{Tr} Q_\lambda} Q_\lambda \otimes |Z_\mu^{(\lambda)}|$  is invariant under the action of  $U^g$ , we have  $\text{Tr} P_0 |C_{\lambda\mu}| = \text{Tr} P_i |C_{\lambda\mu}|$  for all  $i$ , and using  $\sum_i s^* P_i \leq \mathbf{1}$  we get

$$|\Delta_{\lambda\mu, \lambda'\mu'}| \leq 2\sqrt{\frac{1}{s^*a}} \|C_{\lambda\mu}\|_1^2 \leq 2\sqrt{M}(s^*a)^{-1/2}. \quad (14)$$

With this and introducing a new parameter  $\beta$  we get that

$$\begin{aligned} \|T - \mathbf{1}\|_\infty &\leq \|T - \mathbf{1}\|_2 = \sqrt{\sum_{\lambda\mu\lambda'\mu'} |\Delta_{\lambda\mu, \lambda'\mu'}|^2} \\ &\leq \sqrt{L^2 M^4 4M (s^*a)^{-1}} \leq \frac{1}{\beta}, \end{aligned} \quad (15)$$

where  $s^*a \geq 4\beta^2 L^2 M^5$ . Assuming  $\beta \geq 2$  (which will be the case with our later choice), we thus know that  $T$  is invertible; in fact, we have  $T = \mathbf{1} - \Delta$  with  $\|\Delta\| \leq \frac{1}{\beta} \leq \frac{1}{2}$ , hence  $T^{-1} = \sum_{k=0}^{\infty} \Delta^k$  and so

$$\|T^{-1} - \mathbf{1}\|_\infty = \left\| \sum_{k=1}^{\infty} \Delta^k \right\|_\infty \leq \sum_{k=1}^{\infty} \|\Delta\|_\infty^k = \frac{1}{\beta - 1} \leq \frac{2}{\beta}.$$

I.e., writing  $T^{-1} = \mathbf{1} + \widetilde{\Delta}_{\lambda\mu, \lambda'\mu'}$  we get

$$|\widetilde{\Delta}_{\lambda\mu, \lambda'\mu'}| \leq \|\widetilde{\Delta}\|_\infty \leq \frac{2}{\beta}. \quad (16)$$

The invertibility of  $T$  implies that there is a dual set to  $\{D_{\lambda\mu}\}$  in  $\text{span}\{\widehat{C}_{\lambda\mu}\}$ . Indeed, from the definition of  $T_{\lambda\mu, \lambda'\mu'}$  and the dual sets,

$$\begin{aligned} \widehat{C}_{\lambda'\mu'} &= \sum_{\lambda\mu} T_{\lambda\mu, \lambda'\mu'} \widehat{D}_{\lambda\mu}, \text{ which can be rewritten as} \\ \widehat{D}_{\lambda\mu} &= \sum_{\lambda'\mu'} (T^{-1})_{\lambda'\mu', \lambda\mu} \widehat{C}_{\lambda'\mu'}. \end{aligned}$$

Now we can finally write down our candidate solution to Eq. (13):

$$\begin{aligned} X &= \sum_{\lambda\mu} \delta_{\mu 0} \widehat{D}_{\lambda\mu} \\ &= \sum_{\lambda\mu} \delta_{\mu 0} \sum_{\lambda'\mu'} (T^{-1})_{\lambda'\mu', \lambda\mu} \widehat{C}_{\lambda'\mu'} \\ &= \sum_{\lambda} \widehat{C}_{\lambda 0} + \sum_{\lambda\lambda'\mu'} \widetilde{\Delta}_{\lambda'\mu', \lambda 0} \widehat{C}_{\lambda'\mu'} \\ &= \mathbf{1} + \text{Rest}. \end{aligned}$$

The rest term can be bounded as follows:

$$\|\text{Rest}\|_\infty \leq \sum_{\lambda\lambda'\mu'} \frac{2}{\beta} = \frac{2}{\beta} L^2 M^2$$

using Eq. (16). Thus we find  $\|\text{Rest}\|_\infty \leq 1$  if  $\beta \geq 2L^2M^2$  and  $s^*a \geq 4\beta^2L^2M^5 \geq 16L^6M^9$ . In this case, we will have  $0 \leq X \leq 2$  and  $R_0 := \frac{1}{a}(\mathbb{1} - P_0)X(\mathbb{1} - P_0)$  satisfies

$$0 \leq R_0 \leq \frac{2}{a}(\mathbb{1} - P_0) \leq \mathbb{1} - P_0,$$

as well as

$$\frac{1}{|G|} \sum_{g \in G} (U^g)^\dagger R_0 U^g = \frac{1}{a} \mathbb{1}.$$

Thus we get the desired POVM  $(R_i = \frac{a}{|G|} \sum_{g \text{ s.t. } 0^g = i} (U^g)^\dagger R_0 U^g)$  such that

$$\sum_i R_i = \mathbb{1}, \quad R_i \geq 0, \quad \text{Tr } P_i R_i = 0,$$

and we are done.  $\square$

**Lemma 7 (Lemma 15 in [22])** *Let  $\rho$  be a state and  $P$  a projection in a Hilbert space  $\mathcal{H}$ . Then,*

$$\text{Tr } \rho P \leq \|\rho P\|_1 \leq \sqrt{\text{Tr } \rho P}.$$

*More generally, for  $X \geq 0$  and a POVM element  $0 \leq E \leq \mathbb{1}$ ,*

$$\text{Tr } X E \leq \|X E\|_1 \leq \sqrt{\text{Tr } X} \sqrt{\text{Tr } X E}. \quad \square$$

We even recover the Pusey/Barrett/Rudolph result [34] as a corollary: There,  $\mathcal{E} = \{|\psi_0\rangle, |\psi_1\rangle\}$  with (w.l.o.g.)  $|\psi_{0,1}\rangle = \alpha|0\rangle \pm \beta|1\rangle$  qubit states,  $1 > \alpha \geq \beta > 0$ . We have the unitary phase action of  $\mathbb{Z}_2 = \{\mathbb{1}, Z\}$ ,  $Z|\psi_{0,1}\rangle = |\psi_{1,0}\rangle$ , and hence on  $\mathcal{E}^{\otimes n}$  we have a transitive action of  $G = \mathbb{Z}_2^n \rtimes S_n$  (the semidirect product), the symmetric group  $S_n$  acting by permutation of the tensor factors and  $\mathbb{Z}_2^n$  as  $\bigotimes_{t=1}^n Z^{b_t}$ . It has  $L = n$  and  $M \leq n + 1$  [27], whereas

$$\frac{a}{\left\| \sum_i P_i \right\|_\infty} = \frac{2^n}{(2\alpha^2)^n} = \frac{1}{\alpha^{2n}}.$$

Hence, for large enough  $n$ , we have that the latter exceeds  $16L^6M^9 = \text{poly}(n)$ , and then Proposition 5 above implies that  $\mathcal{E}^{\otimes n}$  can be conclusively excluded.  $\square$

#### D. General case

We shall reduce the case of a general channel to that of a cq-channel. Indeed, recall that we allow Alice and Bob to share entanglement, so Alice can encode information into the Bell states

$$|\Phi_{uv}\rangle = (\mathbb{1} \otimes Z^v X^u)|\Phi\rangle = (X^u Z^v \otimes \mathbb{1})|\Phi\rangle,$$

with the maximally entangled state  $|\Phi\rangle = \frac{1}{\sqrt{|A|}} \sum_{i=1}^{|A|} |i\rangle|i\rangle$  and the discrete Weyl operators  $X$  and  $Z$  (basis and phase shift). This effectively constructs a cq-channel (with  $a = |A|$ )

$$\mathcal{M} : [a]^2 \ni uv \mapsto (\text{id} \otimes \mathcal{N})|\Phi_{uv}\rangle\langle\Phi_{uv}| = (X^u Z^v \otimes \mathbb{1})\rho_{00}(Z^{-v} X^{-u} \otimes \mathbb{1}) \in \mathcal{S}(AB), \quad (17)$$

with the Choi-Jamiołkowski state  $\rho_{00} = (\text{id} \otimes \mathcal{N})|\Phi\rangle\langle\Phi|$ . Applying Proposition 5 to this channel is the key to obtain the following result, which in turn directly implies the reverse direction (“nontrivial  $\Rightarrow$  positive capacity”) in Theorem 3, concluding its proof.

**Proposition 8** *A non-commutative bipartite graph  $K$  with support projection  $P_{AB}$  onto the Choi-Jamiołkowski range  $(\mathbb{1} \otimes K)|\Phi\rangle$  has positive activated feedback assisted zero-error capacity,  $\overline{C}_{0EF}(K) > 0$ , if and only if one of the following equivalent conditions hold:*

- i.  $K$  is non-trivial, i.e. there is no constant channel  $\mathcal{N}_0$  with  $\mathcal{K}(\mathcal{N}_0) < K$ ;*
- ii. There is no state  $|\beta\rangle \in B$  with  $|\beta\rangle \otimes A < K$ .*
- iii.  $\|P_B\|_\infty < |A|$ ;*
- iv.  $\text{Tr}_A(\mathbb{1} - P_{AB})$  has full rank;*
- v.  $A(K) > 1$ ;*

**Proof**  $\overline{C}_{0EF}(K) > 0 \Rightarrow i.$  has been shown in the first part (necessity) of Theorem 3, at the start of this section, likewise  $i. \Leftrightarrow ii.$

$ii. \Leftrightarrow iii.$   $P_{AB} \leq \mathbb{1}_A \otimes \mathbb{1}_B$ , hence  $P_B \leq |A|\mathbb{1}_B$ , i.e.  $\|P_B\|_\infty \leq |A|$ . Equality is attained if and only if there exists an eigenvector  $|\beta\rangle$  of  $P_B$  with eigenvalue  $|A|$ , which is equivalent to  $|A| = \text{Tr}|\beta\rangle\langle\beta|P_B = \text{Tr}(\mathbb{1}_A \otimes |\beta\rangle\langle\beta|)P_{AB}$ . But since  $\mathbb{1}_A \otimes |\beta\rangle\langle\beta|$  has trace  $|A|$  and  $P_{AB}$  is a projector, this is equivalent to  $\mathbb{1}_A \otimes |\beta\rangle\langle\beta| \leq P_{AB}$ , or again equivalently  $|\beta\rangle \otimes A < K$

$iii. \Leftrightarrow iv.$   $\|P_B\|_\infty < |A|$  if and only if  $P_B = \text{Tr}_A P_{AB} < |A|\mathbb{1}_B$ , if and only if  $\text{Tr}_A(\mathbb{1} - P_{AB}) > 0$ .

$iii. \Rightarrow v.$  Simply observe that  $S = \frac{1}{\|P_B\|_\infty} \mathbb{1}_A$  is feasible for  $A(K)$ , since  $\text{Tr}_A(S \otimes \mathbb{1})P_{AB} = \frac{1}{\|P_B\|_\infty} P_B \leq \mathbb{1}$ , hence  $A(K) \geq \text{Tr} S = \frac{|A|}{\|P_B\|_\infty} > 1$ .

$v. \Rightarrow ii.$  We show the contrapositive: If  $|\beta\rangle \otimes A < K$ , then  $\mathbb{1} \otimes |\beta\rangle\langle\beta| \leq P_{AB}$ . Now, if  $S$  is feasible for  $A(K)$ , we have  $\mathbb{1}_B \geq \text{Tr}_A(S \otimes \mathbb{1})P_{AB} \geq \text{Tr}_A(S \otimes \mathbb{1})(\mathbb{1} \otimes |\beta\rangle\langle\beta|) = (\text{Tr} S)|\beta\rangle\langle\beta|$ , hence  $\text{Tr} S \leq 1$ , and so  $A(K) = 1$ .

$iii. \Rightarrow \overline{C}_{0EF}(K) > 0$ . Consider the cq-channel  $\mathcal{M}$  in eq. (17). It has output state support projectors

$$P_{uv} = (X^u Z^v \otimes \mathbb{1})P_{AB}(Z^{-v} X^{-u} \otimes \mathbb{1}), \quad u, v = 1, \dots, a,$$

and we can verify directly that  $\sum_{uv} P_{uv} = |A|\mathbb{1}_A \otimes P_B$ , so its norm satisfies  $\|\sum_{uv} P_{uv}\|_\infty = |A|\|P_B\|_\infty < |A|^2$ . In other words, it satisfies the requirements of Proposition 5, hence  $\overline{C}_{0EF}(K) \geq \overline{C}_{0EF}(\mathcal{M}) > 0$ .  $\square$

### III. SHANNON THEORETIC UPPER BOUND ON $\overline{C}_{0EF}(K)$

In this section we will develop an upper bound on the feedback-assisted zero-error capacity via information theoretic ideas. For this purpose we first review the classical case, due to Shannon.

#### A. Shannon theoretic characterization of the fractional packing number: Shannon's Conjecture

The following characterization of the feedback-assisted zero-error capacity of a classical channel was conjectured by Shannon at the end of his seminal paper [38], and to our knowledge proved first by Ahlswede [2], in the context of his treatment of the capacity of arbitrarily varying (classical) channels with instantaneous feedback, and using his very general results in that theory. Our proof seems more direct, but then it is specially geared towards the zero-error setting.

**Proposition 9** For a bipartite graph  $\Gamma$  on  $\mathcal{X} \times \mathcal{Y}$  such that every  $x \in \mathcal{X}$  is adjacent to at least one  $y \in \mathcal{Y}$ ,

$$\log \alpha^*(\Gamma) = C_{\min}(\Gamma) := \min\{C(N) : \Gamma(N) \subseteq \Gamma\},$$

where  $C(N)$  is the usual Shannon capacity of a noisy classical channel [37].

**Proof** The left hand side is the zero-error capacity of  $\Gamma$ , assisted by feedback (plus some finite amount of communication),  $\overline{C}_{0F}(\Gamma)$  [38]. From this, and the fact that feedback does not increase the Shannon capacity of a channel [38] (which may also be proved invoking the Reverse Shannon Theorem [6]), it follows that  $C(N) \geq \log \alpha^*(\Gamma)$  for any eligible  $N$ , hence  $C_{\min}(\Gamma) \geq \log \alpha^*(\Gamma)$ .

There is also a direct proof of this that avoids operational arguments, relying instead only on elementary combinatorial notions. It goes via showing that for every eligible channel  $N$  and input probability distribution  $p$ ,

$$V(p) := \log \min_y \frac{1}{\sum_x \Gamma(y|x)p_x} \leq I(X : Y), \quad (18)$$

which is enough because the maximum of  $\max_p V(p) = \log \alpha^*(\Gamma)$ , while of course the maximum of  $I(X : Y)$  equals  $C(N)$ . Now, eq. (18) is easily seen to be true for uniform distribution  $p_x = \frac{1}{|\mathcal{X}|}$ . Namely, with the equivocation sets  $\mathcal{E}_y = \{x : \Gamma(y|x) = 1\}$  and the output probability distribution  $q_y = \sum_x p_x N(y|x)$ :

$$\begin{aligned} V(P) &= \log |\mathcal{X}| - \max_y \log |\mathcal{E}_y| \\ &\leq \log |\mathcal{X}| - \sum_y q_y \log |\mathcal{E}_y| \\ &\leq \log |\mathcal{X}| - \sum_y q_y H(X|Y=y) \\ &= H(X) - H(X|Y) = I(X : Y), \end{aligned}$$

where we have used the fact that  $P_{X|Y=y}$  is supported on  $\mathcal{E}_y$ , and the uniformity of the distribution of  $X$ . For non-uniform  $p$ , we use the method of types [17] to reduce the uniform case. In detail, consider the product distribution  $p^{\otimes n}$  and  $X^n \sim p^{\otimes n}$  as input to the i.i.d. channel  $N^{\otimes n}$ . Introducing the type  $T = T(X^n)$  of the string  $X^n$ , we have:

$$nI(X : Y) = I(X^n : Y^n) = I(TX^n : Y^n) = I(T : Y^n) + I(X^n : Y^n|T).$$

On the other hand, for every type  $\tau$ ,

$$\begin{aligned} \frac{1}{V(p)^n} &= \frac{1}{V(p^{\otimes n})} \\ &= \max_{y^n} \sum_{x^n} \Gamma(y^n|x^n)p_{x^n} \\ &\geq \max_{y^n} \sum_{x^n \in \tau} \Gamma(y^n|x^n)p_{x^n} \\ &= p^{\otimes n}(\tau) \max_{y^n} \sum_{x^n \in \tau} \frac{1}{|\tau|} \Gamma(y^n|x^n), \end{aligned}$$

since conditioned on  $T(X^n) = \tau$ ,  $X^n \sim u_\tau$  is uniformly distributed. Hence, using the uniform case of the inequality (18),

$$n \log V(P) \leq \log \frac{1}{p^{\otimes n}(\tau)} + \log V(u_\tau) \leq \log \frac{1}{p^{\otimes n}(\tau)} + I(X^n : Y^n|T = \tau),$$



and averaging over the different types this gives

$$n \log V(P) \leq H(T) + I(X^n : Y^n | T) \leq O(\log n) + nI(X : Y),$$

because there are only  $\text{poly}(n)$  many different types, and letting  $n \rightarrow \infty$  we are done.

So it remains only to show the opposite inequality. The proof uses the primal and dual linear programming [15] (LP) characterisations of  $\alpha^*(\Gamma)$  to construct an optimal channel  $N(y|x)$ , and in fact also an optimal input distribution  $p_x$ , such that  $C(N) = I(X : Y) = \log \alpha^*(\Gamma)$ .

Recall the fractional packing number, eq. (4), and choose optimal primal and dual solutions. Define an input distribution  $p_x := \frac{w_x}{\alpha^*(\Gamma)}$ . This is the one that appears in Shannon's [38, Thm. 7], and his  $\frac{1}{P_0}$  is the same as  $\alpha^*(\Gamma)$ . Now, by complementary slackness [15], if  $M_x := \sum_y \Gamma(y|x)v_y > 1$ , then  $w_x = p_x = 0$ ; per contrapositive, if  $p_x > 0$ , then  $M_x = \sum_y \Gamma(y|x)v_y = 1$ . Hence, we can define, for these latter  $x$ ,

$$N(y|x) := \Gamma(y|x)v_y,$$

and in general for all  $x$ ,

$$N(y|x) := \frac{1}{M_x} \Gamma(y|x)v_y.$$

This is our candidate channel, and we have to convince ourselves that indeed  $C(N) = \log \alpha^*(\Gamma)$ . First of all, let's confirm that with the above distribution  $p$ , the mutual information  $I(X : Y)$  equals  $\log \alpha^*(\Gamma)$ . Recall  $I(X : Y) = \sum_x p_x D(N(\cdot|x)||q)$ , with the output distribution

$$q_y = \sum_x p_x N(y|x) = \sum_x p_x \Gamma(y|x)v_y = \frac{v_y}{\alpha^*(\Gamma)},$$

using once more complementary slackness: the equality is trivial if  $v_y = 0$ , and if  $v_y > 0$  then  $\sum_x \Gamma(y|x)v_x = 1$ . Here,  $D(r||q) = \sum_y r_y \log \frac{r_y}{q_y}$  is the relative entropy, cf. [14]. In the present case, we calculate for all  $x$ ,

$$D(N(\cdot|x)||q) = \sum_y \frac{\Gamma(y|x)v_y}{M_x} \log \frac{\Gamma(y|x)v_y/M_x}{v_y/\alpha^*(\Gamma)} = \log \frac{\alpha^*(\Gamma)}{M_x},$$

which is  $\log \alpha^*(\Gamma)$  for all  $p_x > 0$  as then  $M_x = 1$ . So indeed  $I(X : Y) = \log \alpha^*(\Gamma)$ . But we see even more: While all the relative entropies  $D(N(\cdot|x)||q)$  with  $p_x > 0$  are equal to  $\log \alpha^*(\Gamma)$ , for  $p_x = 0$  instead,

$$D(N(\cdot|x)||q) = \log \frac{\alpha^*(\Gamma)}{M_x} \leq \log \alpha^*(\Gamma),$$

because  $M_x \geq 1$ . These two conditions (for  $p_x > 0$  and  $p_x = 0$ ) are well-known, classic characterizations of the Shannon capacity (cf. [16, 36]); they characterize an optimal input distribution for given channel  $N$ , so indeed we prove  $C(N) = \log \alpha^*(\Gamma)$ .  $\square$

**Remark.** Note that neither is  $C_{\min}$  altered by allowing the use of entanglement as well as feed-back [6], nor  $C_{0F}$  by allowing the use of entanglement and other no-signalling correlations [19].

## B. Quantum generalization of the Shannon bound

Recall that for a channel  $\mathcal{N} : \mathcal{S}(A) \rightarrow \mathcal{S}(B)$ , the entanglement-assisted classical capacity [6], i.e. the maximum rate of asymptotically error-free communication via many uses of the channel assisted by a suitable pre-shared entangled state, is given by

$$C_E(\mathcal{N}) = \max_{\rho} I(A : B)_{\sigma},$$

where  $\sigma_{AB} = (\text{id} \otimes \mathcal{N})\phi_{AA'}$  is the joint input-output state,  $\phi_{AA'}$  is a purification of  $\rho$ , and  $I(A : B) = S(\sigma_A) + S(\sigma_B) - S(\sigma_{AB})$  is the quantum mutual information. Using this, we define for a non-commutative bipartite graph  $K < \mathcal{L}(A \rightarrow B)$  such that  $\mathbb{1} \in K^\dagger K$  (these are precisely the possible Kraus subspaces of channels):

$$C_{\min E}(K) := \min\{C_E(\mathcal{N}) : \mathcal{K}(\mathcal{N}) < K\}.$$

That this is indeed a minimum follows from continuity of  $C_E$  and the fact that the eligible channels form a compact convex set. This definition is of course motivated by Proposition 9, suggesting  $2^{C_{\min E}(K)}$  as a possible quantum generalisation of the fractional packing number. For one thing, for the quantum realisation  $K$  of a classical equivocation graph  $\Gamma$ , it is easy to see that indeed  $C_{\min E}(K) = C_{\min}(\Gamma) = \log \alpha^*(\Gamma)$ , see the remark at the end of the preceding Subsection III A.

At least, this quantity is related to the feedback-assisted zero-error capacity: Indeed, the result of Bowen [10] (alternatively the Quantum Reverse Shannon Theorem [5, 7]) tells us that  $C_E(\mathcal{N})$  is not increased even by allowing feedback, so that  $C_{0EF}(K)$  (and actually even  $\overline{C}_{0EF}(K)$ ) is upper bounded by the entanglement-assisted capacity  $C_E(\mathcal{N})$  for any channel  $\mathcal{N}$  such that  $\mathcal{K}(\mathcal{N}) \subset K$ , hence

**Theorem 10**  $\overline{C}_{0EF}(K) \leq C_{\min E}(K)$  for any non-commutative bipartite graph  $K < \mathcal{L}(A \rightarrow B)$ .  $\square$

$C_{\min E}(K)$  shares many properties with  $C_{\min}(\Gamma)$ , to which it reduces for classical channels. First,  $C_{\min E}(K)$  is given by a minimax formula (min over channels and max over quantum mutual information – see below) to which the minimax theorem applies, so it is also given by a maximin (Lemma 11 below). Second, using this characterisation and properties of the von Neumann entropy, it can be shown that  $C_{\min E}$  is additive (Lemma 12 below). Third, thanks to the operational definition of  $C_E$ , it can be easily seen to be monotonic under pre- and post-processing (Lemma 13 below).

Before we begin, recall that for a given channel (cptp map)  $\mathcal{N}$ ,

$$C_E(\mathcal{N}) = \max_{\rho} \{S(\rho) + S(\mathcal{N}(\rho)) - S((\text{id} \otimes \mathcal{N})\phi_{\rho})\},$$

where the maximisation is over all input states  $\rho$  of the channel, and the quantity maximised is known as the quantum mutual information  $I(\rho; \mathcal{N}) := I(A : B)_{\sigma}$  of the state

$$\sigma^{AB} = (\text{id} \otimes \mathcal{N})\phi_{\rho},$$

with an arbitrary purification  $\phi_{\rho}$  of  $\rho$ . It is well-known that  $I(\rho; \mathcal{N})$  is concave in  $\rho$  and convex in  $\mathcal{N}$ , just like its classical counterpart [1, 6].

The convexity in  $\mathcal{N}$  follows from strong subadditivity: Let

$$\begin{aligned} \sigma^{AB} &= (\text{id} \otimes (\lambda \mathcal{N}_1 + (1 - \lambda) \mathcal{N}_2))\phi_{\rho} \\ &= \lambda \sigma_1^{AB} + (1 - \lambda) \sigma_2^{AB} \\ &= \text{Tr}_{B'} \tilde{\sigma}^{ABB'}, \end{aligned}$$

with  $\tilde{\sigma}^{ABB'} = \lambda\sigma_1^{AB} \otimes |1\rangle\langle 1|^{B'} + (1-\lambda)\sigma_2^{AB} \otimes |2\rangle\langle 2|^{B'}$ . Then,

$$\begin{aligned} I(\rho; \lambda\mathcal{N}_1 + (1-\lambda)\mathcal{N}_2) &= I(A : B)_\sigma \\ &\leq I(A : BB')_{\tilde{\sigma}} \\ &= \lambda I(A : B)_{\sigma_1} + (1-\lambda)I(A : B)_{\sigma_2} \\ &= \lambda I(\rho; \mathcal{N}_1) + (1-\lambda)I(\rho; \mathcal{N}_2). \end{aligned}$$

The concavity in  $\rho$  can be seen as follows, using strong subadditivity again: For states  $\rho_1, \rho_2$  with purifications  $\phi_1, \phi_2$ , respectively, and  $0 \leq \lambda \leq 1$ , we construct a purification of  $\lambda\rho_1 + (1-\lambda)\rho_2$ ,

$$|\phi\rangle = \sqrt{\lambda}|\phi_1\rangle|11\rangle^{A'A''} + \sqrt{1-\lambda}|\phi_2\rangle|22\rangle^{A'A''}.$$

With  $\sigma^{AA'A''B} = (\text{id}_{AA'A''} \otimes \mathcal{N})\phi$ , we have

$$\begin{aligned} I(\lambda\rho_1 + (1-\lambda)\rho_2; \mathcal{N}) &= I(AA'A'' : B)_\sigma \\ &\geq I(AA' : B)_\sigma \\ &\geq I(A : B|A')_\sigma \\ &= \lambda I(\rho_1; \mathcal{N}) + (1-\lambda)I(\rho_2; \mathcal{N}). \end{aligned}$$

**Lemma 11** For any non-commutative bipartite graph  $K < \mathcal{L}(A \rightarrow B)$ ,

$$\begin{aligned} C_{\min E}(K) &= \min_{\substack{\mathcal{N} \text{ s.t.} \\ \kappa(\mathcal{N}) < K}} \max_{\rho} I(\rho; \mathcal{N}) \\ &= \max_{\rho} \min_{\substack{\mathcal{N} \text{ s.t.} \\ \kappa(\mathcal{N}) < K}} I(\rho; \mathcal{N}). \end{aligned}$$

**Proof** The first equation is the definition of  $C_{\min E}(K)$ , with the formula for  $C_E(\mathcal{N})$  inserted. Above we saw that the argument  $I(\rho; \mathcal{N})$  is concave in the first and convex in the second argument. Hence von Neumann's minimax theorem, or rather its generalisation due to Sion [40] applies, allowing us to interchange the order of min and max.  $\square$

**Lemma 12** For non-commutative bipartite graphs  $K_1 < \mathcal{L}(A_1 \rightarrow B_1)$  and  $K_2 < \mathcal{L}(A_2 \rightarrow B_2)$ ,

$$C_{\min E}(K_1 \otimes K_2) = C_{\min E}(K_1) + C_{\min E}(K_2).$$

**Proof** We show this by separately demonstrating " $\leq$ " and " $\geq$ " in the above relation, using the two expressions for  $C_{\min E}$  from Lemma 11. In the following, choose optimal states  $\rho_1, \rho_2$  and channels  $\mathcal{N}_1, \mathcal{N}_2$  for  $K_1, K_2$ , respectively.

" $\leq$ ": By the first expression in Lemma 11,

$$\begin{aligned} C_{\min E}(K_1 \otimes K_2) &\leq \max_{\rho} I(\rho; \mathcal{N}_1 \otimes \mathcal{N}_2) \\ &= C_E(\mathcal{N}_1 \otimes \mathcal{N}_2) \\ &= C_E(\mathcal{N}_1) + C_E(\mathcal{N}_2) \\ &= C_{\min E}(K_1) + C_{\min E}(K_2), \end{aligned}$$

using the fact that the entanglement-assisted capacity is additive, proved in [6]. Note that  $\mathcal{K}(\mathcal{N}_1 \otimes \mathcal{N}_2) = \mathcal{K}(\mathcal{N}_1) \otimes \mathcal{K}(\mathcal{N}_2) \subset K_1 \otimes K_2$ .

" $\geq$ ": By the second expression in Lemma 11,

$$C_{\min E}(K_1 \otimes K_2) \geq \min_{\substack{\mathcal{N} \text{ s.t.} \\ \kappa(\mathcal{N}) < K_1 \otimes K_2}} I(\rho_1 \otimes \rho_2; \mathcal{N}),$$

and we need only to show that the minimum is attained at a product channel  $\mathcal{N} = \mathcal{N}_1 \otimes \mathcal{N}_2$  with  $\mathcal{K}(\mathcal{N}_i) \subset K_i$ . For this purpose, consider the state

$$\sigma^{A_1 A_2 B_1 B_2} = (\text{id}_{A_1} \otimes \text{id}_{A_2} \otimes \mathcal{N})(\phi_1 \otimes \phi_2),$$

for the purifications  $\phi_i$  of  $\rho_i$  ( $i = 1, 2$ ). Now observe that with respect to  $\sigma$ ,

$$\begin{aligned} I(A_1 A_2 : B_1 B_2) - I(A_1 : B_1) - I(A_2 : B_2) &= S(A_1 A_2) + S(B_1 B_2) - S(A_1 A_2 B_1 B_2) \\ &\quad - S(A_1) - S(B_1) + S(A_1 B_1) \\ &\quad - S(A_2) - S(B_2) + S(A_2 B_2) \\ &= I(A_1 B_1 : A_2 B_2) - I(B_1 : B_2) - I(A_1 : A_2) \geq 0, \end{aligned}$$

because  $I(A_1 : A_2) = 0$  and by strong subadditivity. In other words,

$$\begin{aligned} I(A_1 A_2 : B_1 B_2)_\sigma &\geq I(A_1 : B_1)_{\sigma_1} + I(A_2 : B_2)_{\sigma_2} \\ &= I(A_1 A_2 : B_1 B_2)_{\sigma_1 \otimes \sigma_2}, \end{aligned}$$

with the reduced states

$$\begin{aligned} \sigma_1^{A_1 B_1} &= \text{Tr}_{A_2 B_2} \sigma = (\text{id}_{A_1} \otimes (\text{Tr}_{B_2} \circ \mathcal{N}))(\phi_1 \otimes \rho_2), \\ \sigma_2^{A_2 B_2} &= \text{Tr}_{A_1 B_1} \sigma = (\text{id}_{A_2} \otimes (\text{Tr}_{B_1} \circ \mathcal{N}))(\rho_1 \otimes \phi_2). \end{aligned}$$

I.e.,

$$I(\rho_1 \otimes \rho_2; \mathcal{N}) \geq I(\rho_1; \text{Tr}_{B_2} \circ \mathcal{N}(\cdot \otimes \rho_2)) + I(\rho_2; \text{Tr}_{B_1} \circ \mathcal{N}(\rho_1 \otimes \cdot)).$$

Finally,  $\text{Tr}_{B_2} \circ \mathcal{N}(\cdot \otimes \rho_2)$  is eligible: If  $\mathcal{N}$  has Kraus operators  $E_i \in K_1 \otimes K_2 \subset \mathcal{L}(A_1 A_2 \rightarrow B_1 B_2)$ , and choosing an eigenbasis of  $\rho_2$  and an arbitrary basis of  $B_2$ ,

$$\mathcal{K}(\text{Tr}_{B_2} \circ \mathcal{N}(\cdot \otimes \rho_2)) = \text{span} \{ \langle j |^{B_2} E_i | k \rangle^{A_2} : i, j, k \} \subset K_1.$$

$\mathcal{K}(\text{Tr}_{B_1} \circ \mathcal{N}(\rho_1 \otimes \cdot)) \subset K_2$  is analogous, and we are done.  $\square$

**Lemma 13** *All of  $C_{0EF}$ ,  $\overline{C}_{0EF}$  and  $C_{\min E}$  are monotonic under pre- and post-processing of the channel: For non-commutative bipartite graphs  $K \subset \mathcal{L}(A \rightarrow B)$  and  $K_A \subset \mathcal{L}(U \rightarrow A)$ ,  $K_B \subset \mathcal{L}(B \rightarrow V)$ , the matrix-multiplied space  $K_B K K_A \subset \mathcal{L}(U \rightarrow V)$  is a non-commutative bipartite graph, and*

$$\begin{aligned} C_{0EF}(K) &\geq C_{0EF}(K_B K K_A), \\ \overline{C}_{0EF}(K) &\geq \overline{C}_{0EF}(K_B K K_A), \\ C_{\min E}(K) &\geq C_{\min E}(K_B K K_A). \end{aligned}$$

**Proof** For  $C_{0EF}$  and  $\overline{C}_{0EF}$  this follows directly from the operational definition: the pre- and post-processings may be absorbed into the input modulation and feedback-decoding, respectively, showing that a zero-error code for  $K_B K K_A$  yields one for  $K$ .

For  $C_{\min E}$ , the argument is similar using the fact that  $C_E(\mathcal{N})$  is the operational entanglement-assisted capacity of the channel  $\mathcal{N}$  [6].  $\square$

We can now give yet another characterization of the feasibility of  $\overline{C}_{0EF}(K) > 0$ , adding to the list of Theorem 3 and Proposition 8.

**Theorem 14** *For any non-commutative bipartite graph  $K$ ,  $\overline{C}_{0EF}(K) > 0$  if and only if  $C_{\min E}(K) > 0$ .*

**Proof** The only way in which  $C_{\min E}(K)$  can be 0 is that there is a channel  $\mathcal{N}$  with  $\mathcal{K}(\mathcal{N}) < K$  and  $C_E(\mathcal{N}) = 0$ , i.e.  $\mathcal{N}$  has to be constant. We have seen that this is equivalent to  $|\beta\rangle \otimes A < K$  for a state vector  $|\beta\rangle \in B$ . But by Theorem 3 this is precisely the characterization of  $\overline{C}_{0EF}(K)$  being 0.  $\square$

To illustrate the bound of Theorem 10, we consider the example of Weyl diagonal channels and the dependence on the output state geometry for cq-channels.

**Weyl diagonal channels.** Denoting by  $X$  and  $Z$  the discrete translation and phase shift (which generate a subgroup of the unitary group of cardinality  $d^3$ , thanks to the commutation relation  $XZ = \omega ZX, \omega = e^{2\pi i/d}$ ), consider the channel

$$\mathcal{N}_0(\rho) = \sum_{a,b=0}^{d-1} p_{ab} X^a Z^b \rho Z^{-b} X^{-a},$$

with probabilities  $p_{ab} \geq 0$  summing to 1. Clearly,

$$\mathcal{K}(\mathcal{N}_0) = \text{span}\{W_{ab} := X^a Z^b : p_{ab} > 0\},$$

i.e. this  $K$  is characterised by a subset  $\mathcal{S} \subset \mathbb{Z}_d \times \mathbb{Z}_d$ . It supports precisely those Weyl diagonal channels  $\mathcal{N}$  with  $p_{ab} = 0$  for  $ab \notin \mathcal{S}$  – and of course many channels that are not Weyl diagonal.

First, note that  $\mathcal{N}_0$  above is Weyl-covariant:

$$\mathcal{N}_0(W_{ab}\rho W_{ab}^\dagger) = W_{ab}\mathcal{N}_0(\rho)W_{ab}^\dagger$$

for all  $ab$ . From this, and the irreducibility of the action of the Weyl operators on  $\mathbb{C}^d$ , it follows that

$$C_E(\mathcal{N}_0) = I\left(\frac{1}{d}\mathbf{1}; \mathcal{N}_0\right) = 2 \log d - H(\vec{p}).$$

This means that for a  $k$ -element  $\mathcal{S} \subset \mathbb{Z}_d \times \mathbb{Z}_d$  and  $K = \text{span}\{W_{ab} : ab \in \mathcal{S}\}$ ,

$$\min_{\substack{\mathcal{N} \text{ Weyl-diag.} \\ \mathcal{K}(\mathcal{N}) < K}} C_E(\mathcal{N}) = 2 \log d - \log k, \quad (19)$$

the minimum being attained at the uniform distribution on  $\mathcal{S}$ :  $p_{ab} = \frac{1}{k}$  for  $ab \in \mathcal{S}$ , and 0 otherwise.

We will now show that  $2 \log d - \log k$  is an achievable rate of zero-error communication via this channel when assisted by feedback (plus a constant activating amount of noiseless communication). The key is the observation that if we use

$$\mathcal{N}_0(\rho) = \frac{1}{k} \sum_{ab \in \mathcal{S}} W_{ab} \rho W_{ab}^\dagger$$

with dense coding, i.e. with a maximally entangled state  $|\Phi_d\rangle$  and sender modulation by the very Weyl operators  $W_{ab}$ , the receiver making a Bell measurement in the basis  $(W_{ab} \otimes \mathbf{1})|\Phi_d\rangle$ , we obtain a generalised typewriter channel

$$T : \mathbb{Z}_d \times \mathbb{Z}_d \longrightarrow \mathbb{Z}_d \times \mathbb{Z}_d,$$

$$T(ab|cd) = \begin{cases} \frac{1}{k} & \text{if } (a-c, b-d) \in \mathcal{S}, \\ 0 & \text{otherwise.} \end{cases}$$

(And choosing a different  $\mathcal{N}$  supported by  $K$  changes only the non-zero transition probabilities.)  $T$  is easily seen to have fractional packing number  $d^2/k$ , so its activated feedback-assisted zero-error capacity is  $2 \log d - \log k$ . Hence  $\overline{C}_{0EF}(K) \geq 2 \log d - \log k$ , and together with eq. (19), we conclude

$$\overline{C}_{0EF}(K) = C_{\min E}(K) = \overline{C}_{0F}(T) = 2 \log d - \log k.$$

Finally, this is also the minimal zero-error communication cost to simulate a channel supported by  $K$  (using entanglement and shared randomness), making use of an idea in [6]: By the results of [19], one can simulate  $T$  with free shared randomness at communication rate  $2 \log d - \log k$ . Now, if in the teleportation protocol using a maximally entangled state and the Weyl unitaries  $W_{ab}$ , we replace the noiseless channel of  $d^2$  messages by this  $T$ , one simulates exactly  $\mathcal{N}_0$ .  $\square$

**Nontrivial dependence of  $\overline{C}_{0EF}$  on the channel geometry.** Consider a non-commutative bipartite graph corresponding to a pure state cq-channel,  $K = \text{span}\{|\psi_i\rangle\langle i|\}$ . We can see that  $\overline{C}_{0EF}(K)$  depends nontrivially on the geometry of the vector arrangement of the  $|\psi_i\rangle$ , even if they are all pairwise non-orthogonal: Indeed, when they are close to parallel,  $\overline{C}_{0EF}(K)$  is arbitrarily close to 0, but when they are sufficiently close to being mutually orthogonal,  $\overline{C}_{0EF}(K)$  is arbitrarily close to  $\log |A|$ .

Clearly, the closer to being parallel the  $|\psi_i\rangle$  are, the larger the required  $n$  in the argument in Subsection II A becomes, so the lower bound moves closer to 0. On the other hand, this is really necessary, since

$$C_{\min E}(K) = \max_{(p_i)} S \left( \sum_i p_i |\psi_i\rangle\langle\psi_i| \right)$$

converges to 0 as the  $|\psi_i\rangle$  get closer to being collinear.

In the other extreme, to show that  $C_{0EF}(K) \rightarrow \log |A|$  when  $C_{\min E}(K) \rightarrow \log |A|$ , i.e. when the  $\psi_i$  become closer and closer to being orthogonal, we use once more the ideas from Subsection II A: Assume that for all  $i \neq j$ ,  $|\langle\psi_i|\psi_j\rangle| \leq \epsilon$ , which is a more convenient expression for  $C_{\min E}(K) \geq \log |A| - \delta$ .

We claim that if  $\epsilon$  is small enough, we can use  $K$  to simulate a “random superset channel” (cf. [19]): for integers  $t < a = |A|$  define the classical channel  $S_{1,t}^a : [a] \rightarrow \binom{[a]}{t}$  such that

$$S_{1,t}^a : i \mapsto J \in \binom{[a]}{t} \text{ randomly with } i \in J.$$

Indeed, we use the characterization of [11], which will show that there is a deterministic transformation of the set  $\{|\psi_i\rangle\}$  to the set  $\{|\varphi_i\rangle\}$ , with

$$|\varphi_i\rangle = \frac{1}{\sqrt{\binom{a-1}{t-1}}} \sum_{i \in J \in \binom{[a]}{t}} |J\rangle \in \mathbb{C}^{\binom{[a]}{t}}.$$

Once this is achieved, Bob measures the states  $|\varphi_i\rangle$  in the computational basis, resulting in an output of the channel  $S_{1,t}^a$ . To see this in detail, let us focus on the smallest possible case  $t = 2$ , for which we see that for  $i \neq j$ ,  $\langle\varphi_i|\varphi_j\rangle = \frac{1}{a-1}$ . The necessary and sufficient condition required in [11] for the existence of a ctp map transforming  $\{|\psi_i\rangle\}$  into  $\{|\varphi_i\rangle\}$  is that there exists a positive semidefinite  $a \times a$ -matrix  $M$  such that  $\Psi = \Phi \circ M$ , where  $\Psi = [\langle\psi_i|\psi_j\rangle]$  and  $\Phi = [\langle\varphi_i|\varphi_j\rangle]$  are the Gram matrices of the two input/output state sets, and  $\circ$  denotes the elementwise (Hadamard/Schur) product. In other words,

$$M = \Psi \circ \Phi^{\circ-1} \geq 0, \quad \text{i.e. } (a-1)\Psi \geq (a-2)\mathbb{1}.$$

However, all eigenvalues of  $\Psi$  are lower bounded by  $1 - (a - 1)\epsilon$ , which is  $\geq \frac{a-2}{a-1}$  as soon as  $\epsilon \leq \frac{1}{(a-1)^2}$ . In this case, we find  $\overline{C}_{0EF}(K) \geq \overline{C}_{0F}(S_{1,2}^a) = \log a - 1$ . Applying the same to multiple copies of the channel, this reasoning shows that if  $\epsilon \leq (|A| - 1)^{-2n}$ , then  $\overline{C}_{0EF}(K) \geq \log a - \frac{1}{n}$ .  $\square$

We do not know whether in general  $\overline{C}_{0EF}$  equals  $C_{\min E}$  or not. However, we can show that the latter is a genuine capacity, as per the following theorem, whose proof however we relegate to Appendix A because it would detract from our principal, zero-error argument.

**Theorem 15** *For any non-commutative bipartite graph  $K$ , the adversarial entanglement-assisted classical capacity of  $K$  is given by  $C_{*E}(K) = C_{\min E}(K)$ .*

The definition of this capacity is as follows: An entanglement-assisted  $n$ -block code consists of an entangled state (w.l.o.g. pure)  $|\phi\rangle^{A_0B_0}$ ,  $N$  modulation cptp maps  $\mathcal{E}_i : \mathcal{L}(A_0) \rightarrow \mathcal{L}(A^n)$  ( $m = 1, \dots, N$ ), and a POVM  $(D_i)_{i=1}^N$  on  $B_0B^n$ . The code is said to have error  $\epsilon$  for  $K^{\otimes n}$  if the (average) error probability,

$$P_{\text{err}}(\mathcal{N}^{(n)}) = \frac{1}{N} \sum_{i=1}^N \left( 1 - \text{Tr}((\mathcal{N}^{(n)} \circ \mathcal{E}_i \otimes \text{id})\phi) D_i \right),$$

is  $\leq \epsilon$  for every channel  $\mathcal{N}^{(n)}$  with  $\mathcal{K}(\mathcal{N}^{(n)}) < K^{\otimes n}$ . In this case, we call the collection  $(\phi; \mathcal{E}_i, D_i)$  an  $(n, \epsilon)$ -code for  $K^{\otimes n}$ . Denoting the largest number  $N$  of messages of an  $(n, \epsilon)$ -code as  $N(n, \epsilon; K)$ , the *adversarial entanglement-assisted classical capacity* is defined as

$$C_{*E}(K) := \inf_{\epsilon > 0} \liminf_{n \rightarrow \infty} \frac{1}{n} \log N(n, \epsilon; K).$$

In Appendix A we shall actually show that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log N(n, \epsilon; K) = C_{\min E}(K)$$

for every  $0 < \epsilon < 1$  (this is known as a strong converse). There we will see that even allowing entanglement and arbitrary feedback in the communication protocol does not increase the capacity  $C_{*E}(K)$  beyond  $C_{\min E}(K)$ , hence we may also address it as *feedback-assisted adversarial capacity*  $C_{*EF}(K)$ .

#### IV. CONCLUSION

We have introduced the problem of determining the zero-error capacity of a quantum channel assisted by noiseless feedback. We showed that the capacity only depends on the “non-commutative bipartite graph”  $K$  of the channel, and that every nontrivial  $K$  has positive capacity.

Motivated by Shannon’s treatment of the classical case, we considered the minimisation of entanglement-assisted classical capacities over all channels with the same non-commutative bipartite graph and proved several properties of this definition: it is an upper bound on the activated feedback-assisted zero-error capacity, it is given by a minimax/maximin formula, and is additive. It is also shown to be equal to the adversarial entanglement-assisted capacity.

Note that when restricting all statements above to classical channels, which are given by a bipartite equivocation graph  $\Gamma$ , all of these quantities boil down to the fractional packing number:

$$2^{C_{\min E}(K)} = 2^{C_{\min}(\Gamma)} = 2^{\overline{C}_{0F}(\Gamma)} = \alpha^*(\Gamma),$$

which furthermore quantifies the zero-error capacity and simulation cost of  $\Gamma$  when assisted by general no-signalling correlations [19],  $2^{C_{0,\text{NS}}(\Gamma)} = 2^{S_{0,\text{NS}}(\Gamma)} = \alpha^*(\Gamma)$ . However, for quantum channels and non-commutative bipartite graphs these notions start diverging, so none of them can be considered as a preferred “quantum fractional packing number”: In [22], no-signalling assisted zero-error capacity and simulation cost were determined for cq-channels,  $C_{0,\text{NS}}(K) = \log A(K)$  and  $S_{0,\text{NS}}(K) = \log \Sigma(K)$ , with the semidefinite packing number  $A(K)$  and another SDP  $\Sigma(K)$ , and while in general (for cq-channels)

$$\log A(K) \leq C_{\min E}(K) \leq \log \Sigma(K),$$

both inequalities can be strict [22]. It remains an open question how  $\overline{C}_{0EF}(K)$  fits into this picture, and in particular whether it is equal to or sometimes strictly smaller than  $C_{\min E}(K)$ . We believe that pure state cq-channels offer a good testing ground for ideas; we might take encouragement from [41], where it was shown that the *unambiguous capacity* of a pure state cq-graph  $K$  equals  $C_{\min E}(K)$ . Other interesting  $K$  are those that admit only one channel  $\mathcal{N}$ , for instance channels extremal in the set of cptp maps, cf. [22], an example of which is the amplitude damping channel; in this case,  $C_{\min E}(K) = C_E(\mathcal{N})$ .

Next, motivated by the fact that both  $A(K)$  and  $\Sigma(K)$  are SDPs (at least for cq-graphs), we ask if there a manifestly semidefinite programming (or even just convex optimisation) characterisation of  $2^{C_{\min E}(K)}$ ? To make progress, we need at least to understand some properties of an optimal  $\mathcal{N}$  for given  $K$ , and potentially also an optimal input state.

To offer a concrete approach to the question whether  $C_{\min E}(K)$  is an achievable rate for pure state cq-graph  $K$ , we suggest to look at the possible use of conclusive exclusion to implement a list-decoding protocol, by excluding more than one state by each outcome – cf. [4].

*List-decoding from approximate decoding?* Given state vectors  $|\psi_1\rangle, \dots, |\psi_N\rangle \in B$  (w.l.o.g.  $|B| = N$ ) that are sufficiently orthogonal in the sense that there exists an orthonormal basis  $\{|v_1\rangle, \dots, |v_N\rangle\}$  of  $B$  such that

$$\forall i \quad |\langle v_i | \psi_i \rangle|^2 \geq 1 - \epsilon. \quad \left( \text{For instance, this holds if for each } i, \sum_{j \neq i} |\langle \psi_i | \psi_j \rangle|^2 \leq \epsilon, \text{ by [29].} \right)$$

Then, does there exist a subset of  $N' \geq \Omega(N^{1-\delta})$  of these states,  $\{|\psi_{i_j}\rangle : j = 1, \dots, N'\}$ ,  $L \leq O(N^\delta)$  ( $\delta \rightarrow 0$  with  $\epsilon \rightarrow 0$  uniformly) and a POVM  $(M_S : S \in \binom{[N']}{L})$ , such that  $\{j : \langle \psi_{i_j} | M_S | \psi_{i_j} \rangle \neq 0\} \subset S$  for all  $S \in \binom{[N']}{L}$ ?

Note that a positive answer would imply that by preparing  $\psi_{i_j}$  and measuring the POVM elements  $M_S$ , we construct a classical channel/hypergraph  $\Gamma$  with  $\alpha^*(\Gamma) \geq \frac{N'}{L}$ . To see this, observe that each output  $S$  is reached from at most  $L$  inputs  $j$ , namely those  $j \in S$ , so the weight distribution  $w_j = \frac{1}{L}$  for all  $i$  is admissible in the definition of  $\alpha^*(\Gamma)$ . Thus we would obtain

$$\overline{C}_{0EF}(K) \geq \overline{C}_{0F}(\Gamma) \geq \log \frac{N'}{L} \geq (1 - 2\delta) \log N - O(1),$$

which is at least consistent with  $C(\mathcal{N})$  being of the order  $(1 - \epsilon) \log N - O(1)$ , by the existence of the basis  $\{|v_1\rangle, \dots, |v_N\rangle\}$  and Fano’s inequality.

By Hausladen et al. [29] this would imply that we can asymptotically achieve the rate  $C(\mathcal{N}) = C_{\min E}(K)$  as activated feedback-assisted zero-error capacity, where  $K = \text{span}\{|\psi_i\rangle\langle i| : i = 1, \dots, N\}$ . It would also imply a new proof of the result of [41], since we could use the Shannon scheme [38] to get arbitrarily close to the rate  $\log \alpha^*(\Gamma)$  by a deterministic list-decoding with



constant list size, and then constant activating communication, which we clearly can realize in an inambiguous fashion with constant overhead.

Finally, there is another generalization of the instantaneous feedback considered by Shannon, which was dubbed “coherent feedback” in [5], and which consist in the channel environment  $C$  from the Stinespring isometry  $V : A \hookrightarrow B \otimes C$  to be handed back to Alice. More like Shannon’s model, it is completely passive as it doesn’t involve any action of Bob’s. The resulting zero-error capacity,  $C_{0|F}(V)$  is not even obviously a function of  $K$  only, nor is it clear whether additional free entanglement or free active feedback from Bob to Alice will increase it, though it is clear from the Quantum Reverse Shannon Theorem that all of  $C_{0|F}(V)$  and its variants are upper bounded by  $C_E(\mathcal{N})$ .

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### Appendix A: $C_{\min E}(K)$ equals the adversarial entanglement-assisted capacity

Here we give a complete proof of the following theorem from Section III.

**Theorem 15** For any non-commutative bipartite graph  $K$ , the *adversarial entanglement-assisted classical capacity* of  $K$  is given by  $C_{*E}(K) = C_{\min E}(K)$ .

Before proving it, we show a simpler statement on so-called compound channels, which will be pivotal for the general proof, however. For a non-commutative bipartite graph  $K < \mathcal{L}(A \rightarrow B)$ , and a pure state  $|\phi\rangle \in AA'$  such that  $\phi^A = \phi^{A'} = \rho$ , define  $X = (\mathbb{1} \otimes K)|\phi\rangle < A \otimes B$  and the sets of states,

$$\mathcal{S}_{K,\rho} := \{(\text{id} \otimes \mathcal{N})\phi : \mathcal{K}(\mathcal{N}) < K\} = \{\sigma \in \mathcal{S}(AB) : \text{supp } \sigma < X, \sigma^A = \rho\},$$

as well as, for  $\epsilon > 0$ ,

$$\mathcal{S}_{K,\rho}^{(\epsilon)} = \{\sigma \in \mathcal{S}(AB) : \exists \sigma' \in \mathcal{S}_{K,\rho} \text{ s.t. } \|\sigma - \sigma'\|_1 \leq \epsilon\}.$$

**Proposition 16** For any non-commutative bipartite graph  $K < \mathcal{L}(A \rightarrow B)$ , a test state  $\rho$  on  $A$ , and parameters  $\epsilon > 0$  and an integer  $k$ , consider the family of cq-channels  $[W^\sigma : S_k \rightarrow \mathcal{S}(A^k \otimes B^k) : \sigma \in \mathcal{S}_{K,\rho}^{(\epsilon)}]$ , with

$$W^\sigma : \pi \longmapsto (\mathbb{1} \otimes U_\pi)\sigma^{\otimes k}(\mathbb{1} \otimes U_\pi)^\dagger.$$

Then, for sufficiently large  $\ell$ , there is an  $\ell$ -block code of  $N = 2^{nR}$  messages ( $n = k\ell$ ) and decoding POVM  $(D_i)_{i=1}^N$ , with rate

$$R \geq \min_{\sigma \in \mathcal{S}_{K,\rho}} I(A : B)_\sigma - 2\delta,$$

and uniformly bounded error probability

$$P_{\text{err}}((W^\sigma)^{\otimes \ell}) = \frac{1}{N} \sum_{i=1}^N \left(1 - \text{Tr}(W_{\pi_1(i)}^\sigma \otimes \cdots \otimes W_{\pi_\ell(i)}^\sigma) D_i\right) \leq c^\ell$$

for all  $\sigma \in \mathcal{S}_{K,\rho}^{(\epsilon)}$ . Here,  $c < 1$  and  $\delta = 2\epsilon \log(|A||B|) + \frac{3}{k} + 2|B|^2 \frac{\log(k+|B|)}{k}$ .

**Proof** The family of cq-channels  $[W^\sigma : S_k \rightarrow \mathcal{S}(A^k \otimes B^k) : \sigma \in \mathcal{S}_{K,\rho}^{(\epsilon)}]$  generates a compound channel, meaning that on block length  $\ell$ , the communicating parties face one of the i.i.d. channels  $(W^\sigma)^{\otimes \ell}$ ,  $\sigma \in \mathcal{S}_{K,\rho}^{(\epsilon)}$ , but they do not know beforehand which one, so they need to use a code that is good for all of them.

For this we invoke the general result of Bjelakovic and Boche [8], which states that there are such codes with rate

$$\min_{\sigma \in \mathcal{S}_{K,\rho}^{(\epsilon)}} \chi \left( \left\{ p_\pi = \frac{1}{k!}, W_\pi^\sigma = (\mathbf{1} \otimes U_\pi) \sigma^{\otimes k} (\mathbf{1} \otimes U_\pi)^\dagger \right\} \right) - k\delta$$

for any  $\delta > 0$  and with error probability uniformly bounded by  $c^\ell$ ,  $c = c(\delta) < 1$ .

By Lemma 17 below,

$$\chi \left( \left\{ p_\pi = \frac{1}{k!}, W_\pi^\sigma = (\mathbf{1} \otimes U_\pi) \sigma^{\otimes k} (\mathbf{1} \otimes U_\pi)^\dagger \right\} \right) \geq k I(A : B)_\sigma - 2|B|^2 \log(k + |B|),$$

and because there is  $\sigma' \in \mathcal{S}_{K,\rho}$  with  $\|\sigma - \sigma'\|_1 \leq \epsilon$ , Fannes' inequality [25] shows that the rate (over  $n = k\ell$ ) is

$$\geq \min_{\sigma \in \mathcal{S}_{K,\rho}} I(A : B)_\sigma - 2\epsilon \log(|A||B|) - \frac{3}{k} - 2|B|^2 \frac{\log(k + |B|)}{k} - \delta,$$

and we are done, choosing  $\delta$  as advertised.

We end this proof pointing out a rather nice feature of the code: each message is encoded as an  $\ell$ -tuple of permutations from  $S_k$ ,  $i \mapsto \underline{\pi}(i) = \pi_1(i) \dots \pi_\ell(i)$ , which we may view naturally as an element of  $S_k \times \cdots \times S_k \subset S_n$ , acting on  $B^n$  by permuting the tensor factors, each  $\pi_j(i)$  on its own block of  $k$ , hence message  $i$  is mapped to the state  $W_{\underline{\pi}(i)}^\sigma = (\mathbf{1} \otimes U_{\underline{\pi}(i)}) \sigma^{\otimes n} (\mathbf{1} \otimes U_{\underline{\pi}(i)})^\dagger$  on  $A^n B^n$ .  $\square$

**Lemma 17 (Cf. Shor [39])** For any channel  $\mathcal{N} : \mathcal{L}(A) \rightarrow \mathcal{L}(B)$  and a state  $\rho$  on  $A$  with purification  $|\phi\rangle \in AA'$ , and let  $\sigma^{AB} = (\text{id} \otimes \mathcal{N})\phi$ . Then, for any integer  $k$ ,

$$\chi \left( \left\{ p_\pi = \frac{1}{k!}, W_\pi^\sigma = (\mathbf{1} \otimes U_\pi) \sigma^{\otimes k} (\mathbf{1} \otimes U_\pi)^\dagger \right\} \right) \geq k I(A : B)_\sigma - 2|B|^2 \log(k + |B|),$$

where  $\pi$  ranges over the symmetric group  $S_k$ , acting on  $B^k$  by permuting the tensor factors.

**Proof** With the average state

$$\Omega^{A^k B^k} = \frac{1}{k!} \sum_{\pi \in S_k} (\mathbf{1} \otimes U_\pi) \sigma^{\otimes k} (\mathbf{1} \otimes U_\pi)^\dagger,$$

we have

$$\begin{aligned} \chi \left( \left\{ \frac{1}{k!}, (\mathbb{1} \otimes U_\pi) \sigma^{\otimes k} (\mathbb{1} \otimes U_\pi)^\dagger \right\} \right) &= S(\Omega^{A^k B^k}) - S(\sigma^{\otimes k}) \\ &= S(\Omega^{A^k}) + S(\Omega^{B^k}) - I(A^k : B^k)_\Omega - S(\sigma^{\otimes k}) \\ &= kI(A : B)_\sigma - I(A^k : B^k)_\Omega, \end{aligned}$$

where we have used that all ensemble members are just unitary transformed versions of  $\sigma^{\otimes k}$  (first line), the definition of the mutual information (second line), the fact that  $\Omega^{A^k} = (\sigma^A)^{\otimes k}$  and  $\Omega^{B^k} = (\sigma^B)^{\otimes k}$  as well as additivity of the von Neumann entropy (third line).

Now we use the representation theory of  $S_k$  acting on  $B^k$  to bound the mutual information remaining: From Schur-Weyl duality [27] it is known that

$$B^k = \bigoplus_{\lambda} Q_{\lambda}^b \otimes P_{\lambda},$$

where  $\lambda$  are Young diagrams with at most  $b = |B|$  rows,  $P_{\lambda}$  are the corresponding irreps of  $S_k$  and  $Q_{\lambda}^b$  is the multiplicity space, which is an irrep of the commutant representation,  $SU(b)$ . With the maximally mixed state  $\tau_{\lambda}$  on  $P_{\lambda}$ , Schur's Lemma implies that

$$\Omega^{A^k B^k} = \bigoplus_{\lambda} q_{\lambda} \omega_{\lambda}^{A^k Q_{\lambda}^b} \otimes \tau_{\lambda}^{P_{\lambda}}.$$

Now observe that  $\Omega^{A^k B^k}$  can by local reversible operations  $B^k \leftrightarrow D := \bigoplus_{\lambda} Q_{\lambda}^b$  be transformed into

$$\tilde{\Omega}^{A^k D} = \bigoplus_{\lambda} q_{\lambda} \omega_{\lambda}^{A^k Q_{\lambda}^b},$$

hence

$$I(A^k : B^k)_\Omega = I(A^k : D)_{\tilde{\Omega}} \leq 2 \log |D| \leq 2b^2 \log(k + b).$$

The latter because it is known that there are only  $L \leq (k + 1)^b$  Young diagrams and each  $SU(b)$  irrep has dimension  $|Q_{\lambda}^b| \leq M = (k + b)^{\frac{1}{2}b^2}$ , hence  $|D| \leq LM = (k + 1)^b (k + b)^{\frac{1}{2}b^2} \leq (k + b)^{b^2}$ , as we only need to consider the case  $b \geq 2$ .  $\square$

**Proof (of Theorem 15)** First we show the upper bound, to be precise the strong converse. Because among the eligible channels is  $\mathcal{N}^{\otimes n}$  with  $\mathcal{K}(\mathcal{N}) < K$  attaining the minimum in  $C_{\min E}(K)$ , we see immediately that  $C_{*E}(K) \leq C_E(\mathcal{N}) = C_{\min E}(K)$ . In fact, the Quantum Reverse Shannon Theorem for  $\mathcal{N}^{\otimes n}$  [5, 7] implies the strong converse as well, i.e. for all  $\epsilon < 1$ ,

$$\limsup_{n \rightarrow \infty} \frac{1}{n} N(n, \epsilon; K) \leq C_E(\mathcal{N}) = C_{\min E}(K).$$

A direct proof of this can be found in [23]. Furthermore, Bowen [10] (alternatively again the Quantum Reverse Shannon Theorem) showed that feedback does not increase the entanglement-assisted capacity.

It remains to show achievability of  $C_{\min E}(K)$ ; for this it will be enough to show that for any test state  $\rho$  on  $A$ ,  $C_{*E}(K) \geq \min_{\mathcal{K}(\mathcal{N}) < K} I(\rho; \mathcal{N})$ , by exhibiting a sequence of codes with this rate and error probability going to 0, exponentially in  $n$ . Choose a purification  $|\phi\rangle^{AA'}$  of  $\rho$  and let Alice and Bob share  $\phi^{\otimes n}$  as well as a maximally entangled state of Schmidt rank  $n!$ , which is measured

by both parties in the computational basis to obtain a shared random permutation  $\tau \in S_n$ . Alice's encoding will be to subject her  $n$  input  $A'$ -systems to a permutation  $\underline{\pi}(i)$  for each each message  $i = 1, \dots, N$ , then apply  $\tau$  and send the resulting state through the channel  $\mathcal{N}^{(n)}$ ; Bob will apply the permutation  $\tau^{-1}$  to his  $n$  output  $B$ -systems. The state this prepares for Bob is

$$\begin{aligned}\omega(i)^{A^n B^n} &= \frac{1}{n!} \sum_{\tau \in S_n} (\mathbf{1} \otimes U_\tau)^\dagger \left[ (\text{id} \otimes \mathcal{N}^{(n)}) \left( (\mathbf{1} \otimes U_\tau U_{\underline{\pi}(i)}) \phi^{\otimes n} (\mathbf{1} \otimes U_\tau U_{\underline{\pi}(i)})^\dagger \right) \right] (\mathbf{1} \otimes U_\tau) \\ &= (\mathbf{1} \otimes U_{\underline{\pi}(i)}) [(\text{id} \otimes \overline{\mathcal{N}}^{(n)}) \phi^{\otimes n}] (\mathbf{1} \otimes U_{\underline{\pi}(i)})^\dagger \\ &=: (\mathbf{1} \otimes U_{\underline{\pi}(i)}) \sigma^{(n)} (\mathbf{1} \otimes U_{\underline{\pi}(i)})^\dagger,\end{aligned}$$

with the permutation-symmetrized channel

$$\overline{\mathcal{N}}^{(n)}(\rho) = \frac{1}{n!} \sum_{\tau \in S_n} U_\tau^\dagger \mathcal{N}^{(n)}(U_\tau \rho U_\tau^\dagger) U_\tau.$$

Note that as  $\mathcal{K}(\mathcal{N}^{(n)}) < K^{\otimes n}$ , the same holds for  $\overline{\mathcal{N}}^{(n)}$ . The permutations  $\underline{\pi}(i)$  form a code for the compound channel

$$\left[ W_\pi^\sigma = (\mathbf{1} \otimes U_\pi) \sigma^{\otimes k} (\mathbf{1} \otimes U_\pi)^\dagger : \sigma^{AB} \in \mathcal{S}_{K,\rho}^{(\epsilon)} \right]$$

according to Proposition 16 and its proof; here,  $n = k\ell$ , and we will determine  $k$  and  $\epsilon$  later. Bob will use the very decoding POVM  $(D_i)$  from the same proposition.

To analyze the performance of this strategy, we apply the Constrained Postselection Lemma 18 to the permutation-symmetric state  $\sigma^{(n)} = (\text{id} \otimes \overline{\mathcal{N}}^{(n)}) \phi^{\otimes n}$ ,  $X = (\mathbf{1} \otimes K) |\phi\rangle\langle\phi| < A \otimes B$  and  $\mathcal{R} = \text{Tr}_B$ :

$$\sigma^{(n)} \leq (n+1)^{3|A|^2|B|^2} \int d\sigma \sigma^{\otimes n} F(\sigma^A, \rho^A)^{2n},$$

where the integral is over states  $\sigma^{AB}$  supported on  $X < AB$ . We split the integral into two parts, a first where  $F(\sigma^A, \rho^A) < 1 - \alpha$  and a second one where  $F(\sigma^A, \rho^A) \geq 1 - \alpha$ . Choosing  $\alpha$  small enough ensures that those  $\sigma^{AB}$  are in  $\mathcal{S}_{K,\rho}^{(\epsilon)}$ . Thus,

$$\sigma^{(n)} \leq (n+1)^{3|A|^2|B|^2} (1-\alpha)^{2n} \sigma_0 + (n+1)^{3|A|^2|B|^2} \int_{F(\sigma^A, \rho^A) \geq 1-\alpha} d\sigma \sigma^{\otimes n},$$

with some state  $\sigma_0$ . At this point we can evaluate the error probability:

$$\begin{aligned}P_{\text{err}} &= \frac{1}{N} \sum_{i=1}^N \text{Tr} \left( (\mathbf{1} \otimes U_{\underline{\pi}(i)}) \sigma^{(n)} (\mathbf{1} \otimes U_{\underline{\pi}(i)})^\dagger (\mathbf{1} - D_i) \right) \\ &\leq \text{poly}(n) \left[ (1-\alpha)^{2n} + \max_{\sigma \in \mathcal{S}_{K,\rho}^{(\epsilon)}} \text{Tr} \left( (\mathbf{1} \otimes U_{\underline{\pi}(i)}) \sigma^{(n)} (\mathbf{1} \otimes U_{\underline{\pi}(i)})^\dagger (\mathbf{1} - D_i) \right) \right] \\ &\leq \text{poly}(n) \left( (1-\alpha)^{2n} + c^{n/k} \right),\end{aligned}$$

showing that for every  $n$  and  $\epsilon$  the error probability goes to zero exponentially – in fact, at the same rate as the corresponding compound channel, except for the additional term  $(1-\alpha)^{2n}$ .

The rate, according to Proposition 16 is  $\geq \min_{\mathcal{K}(\mathcal{N}) < K} I(\rho; \mathcal{N}) - 2\delta$ , where  $\delta = 2\epsilon \log(|A||B|) + \frac{3}{k} + 2|B|^2 \frac{\log(k+|B|)}{k}$  can be made arbitrarily small by choosing  $\epsilon$  small enough and  $k$  large enough.  $\square$

**Remark** Along the same lines, the use of permutation-symmetrization and the Postselection Lemma allow to give a new proof of the coding theorem for *arbitrarily varying cq-channels* [9], by reducing it to a compound cq-channel [8], cf. also [32].

Observe however that what we treated here is not an “arbitrarily varying quantum channel” in any sense previously considered [3, 9], going beyond the model in [32], too.

### Appendix B: A Constrained Post-Selection Lemma

Here we show the following extension of the main technical result of [13] (albeit with a worse polynomial prefactor).

**Lemma 18** *For given Hilbert space  $X$  with dimension  $d$ , denote by  $d\sigma$  the measure on the quantum states  $\mathcal{S}(X)$  obtained by drawing a pure state from  $X \otimes X'$  uniformly at random (i.e., from the unitary invariant probability measure) and tracing out  $X'$ .*

*Then, for any cptp map  $\mathcal{R} : \mathcal{L}(X) \rightarrow \mathcal{L}(Y)$ , state  $\eta \in \mathcal{S}(Y)$  and  $S_n$ -invariant state  $\rho^{(n)}$  on  $X^n$  with  $\mathcal{R}^{\otimes n}(\rho^{(n)}) = \eta^{\otimes n}$ ,*

$$\rho^{(n)} \leq \binom{n+d^2-1}{d^2-1}^3 \Omega^{(n)} \leq (n+1)^{3d^2} \Omega^{(n)},$$

*with the de Finetti state*

$$\Omega^{(n)} := \int d\sigma \sigma^{\otimes n} F(\mathcal{R}(\sigma), \eta)^{2n},$$

*which depends only on  $X$ ,  $\mathcal{R}$ ,  $\eta$  and  $n$ . The measure  $d\sigma$  is universal in the sense that it depends only on the space  $X$ .*

Here,  $F(\xi, \eta) = \|\sqrt{\xi}\sqrt{\eta}\|_1$  is the fidelity between (mixed) states  $\xi, \eta \in \mathcal{S}(X)$  [26, 31, 42].

**Remark** Note that in  $\Omega^{(n)}$ , the contribution of states  $\sigma$  with  $F(\mathcal{R}(\sigma), \eta) < 1 - \epsilon$  is exponentially small in  $n$ . I.e., for a symmetric state with an additional constraint, expressed by  $\mathcal{R}$  and  $\eta$ , the universal de Finetti state from [13] may be chosen in such a way that almost all its contributions also approximately obey the constraint.

**Proof** Denoting the uniform (i.e. unitarily invariant) probability measure over *pure* states  $\zeta = |\zeta\rangle\langle\zeta|$  on  $X \otimes X'$  by  $d\zeta$ , it is well-known that

$$\int d\zeta \zeta^{\otimes n} = \frac{1}{\binom{n+d^2-1}{d^2-1}} \Pi_{\text{Sym}^n(X \otimes X')},$$

with  $\Pi_{\text{Sym}^n(X \otimes X')}$  denoting the projector onto the (Bose) symmetric subspace of  $(X \otimes X')^{\otimes n}$ . The reason is that the latter is an irrep of the  $U^{\otimes n}$ -representation for  $U \in \text{SU}(d^2)$ , so Schur’s Lemma applies. Now we apply Caratheodory’s Theorem, which says that  $d\zeta$  can be convex-decomposed into measures with finite support, more precisely ensembles  $\{q_i, \zeta_i\}_{i=1}^{D^2}$ , with  $D = \binom{n+d^2-1}{d^2-1}$ , the dimension of the Bose symmetric subspace of  $(X \otimes X')^{\otimes n}$ , and

$$\sum_i q_i \zeta_i^{\otimes n} = \frac{1}{D} \Pi_{\text{Sym}^n(X \otimes X')}.$$

For the moment we shall focus one one of these measures/ensembles.

It is also well-known that one can purify  $\rho^{(n)}$  in a Bose symmetric way, i.e.  $\rho^{(n)} = \text{Tr}_{X'^n} \varphi^{(n)}$ , with  $\varphi^{(n)} = |\varphi^{(n)}\rangle\langle\varphi^{(n)}|$  a pure state supported on the Bose symmetric subspace. Thus, with the operator  $A := \sum_i |\zeta_i\rangle^{\otimes n} \langle i|$ ,

$$\begin{aligned}
\varphi^{(n)} &= \Pi_{\text{Sym}^n(X \otimes X')} \varphi^{(n)} \Pi_{\text{Sym}^n(X \otimes X')} \\
&= D^2 \sum_{ij} q_i q_j \zeta_i^{\otimes n} \varphi^{(n)} \zeta_j^{\otimes n} \\
&= D^2 A \left( \sum_{ij} q_i q_j |i\rangle\langle j| \langle \zeta_i |^{\otimes n} \varphi^{(n)} | \zeta_j \rangle^{\otimes n} \right) A^\dagger \\
&\leq D^4 A \left( \sum_i q_i^2 |i\rangle\langle i| \langle \zeta_i |^{\otimes n} \varphi^{(n)} | \zeta_i \rangle^{\otimes n} \right) A^\dagger \\
&\leq D^3 A \left( \sum_i q_i |i\rangle\langle i| F(\zeta_i^{\otimes n}, \varphi^{(n)})^2 \right) A^\dagger \\
&\leq D^3 \sum_i q_i \zeta_i^{\otimes n} F((\text{Tr}_{X'} \zeta_i)^{\otimes n}, \rho^{(n)})^2 \\
&\leq D^3 \sum_i q_i \zeta_i^{\otimes n} F(\mathcal{R}(\text{Tr}_{X'} \zeta_i), \eta)^{2n},
\end{aligned}$$

where in the fourth line we have used Hayashi's pinching inequality [30], and in the fifth  $q_i \leq \frac{1}{D}$ ; in lines six and seven we have invoked the monotonicity of the fidelity under cptp maps (partial trace and  $\mathcal{R}$ ), as well as  $\text{Tr}_{X'^n} \varphi^{(n)} = \rho^{(n)}$  and  $\mathcal{R}^{\otimes n}(\rho^{(n)}) = \eta^{\otimes n}$ .

Now we remember that  $\{q_i, \zeta_i\}$  was just one of the Caratheodory components of the uniform measure  $d\zeta$ , so by convex combination,

$$\varphi^{(n)} \leq D^3 \int d\zeta \zeta^{\otimes n} F(\mathcal{R}(\text{Tr}_{X'} \zeta), \eta)^{2n},$$

hence by partial trace over  $X'^n$ , and recalling the definition of  $d\sigma$ , we arrive at

$$\rho^{(n)} \leq D^3 \int d\sigma \sigma^{\otimes n} F(\mathcal{R}(\sigma), \eta)^{2n},$$

as desired. □

**Remark** It is the trick to sandwich the Bose-symmetric state  $\varphi^{(n)}$  between symmetric subspace projectors – rather than bounding it directly by that projector –, which allows the introduction of fidelities between the state and “test” product states.

Here we have used this to enforce a linear constraint valid for  $\rho^{(n)}$  on the components of the de Finetti state on the right hand side. It turns out, perhaps unsurprisingly, that also other convex constraints (with a “good” behaviour linking  $n = 1$  with the general case) are amenable to the same treatment, for instance membership in the convex set of separable states for a multipartite space  $X = X_1 \otimes \cdots \otimes X_k$ , and other similar sets, or even non-convex constraints. Such generalizations and their applications are discussed in [33].

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