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# *Modelling Sustainable Spending Plans for Family Offices, Foundations and Trusts*

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A critical question for many high net worth individuals (HNWIs) is how to best adjust spending rates when the investment outlook changes. This chapter sets out optimal spending plans for HNWIs or family offices. Unlike standard approaches, the plans derived here allow for the fact that family offices often decide spending plans and investment strategy separately. The plans also account for the fact that many such trusts have high risk tolerance but a preference for steady consumption streams.

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Using a sophisticated recursive utility function (Weil 1990; Epstein and Zin 1989), we first model the optimal disbursement rate for a perpetual entity with a predetermined asset allocation. Then, under general assumptions about investment returns, we use properties of stochastic dominance to estimate modifications to the current consumption rate when the investment outlook changes. The capacity of the decision-maker to vary spending ~~rather than risk tolerance~~ through time (elasticity of intertemporal substitution) is key to understanding optimal increases or decreases in consumption.

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We begin by setting out the background needed to model optimal consumption paths under conditions of uncertainty, showing how the recursive utility set-up is well adapted to modelling HNWIs' decisions, and briefly foreshadowing the main results. We address adjustments to the set-up needed when planning for a real family with finite survival prospects, and then describe and solve the analytical model for optimal consumption before giving a simple empirical example illustrating the important results. We derive principles of scenario analysis, covering changes in both expected returns

and volatility. Overall, in this chapter we set out a sophisticated but robust approach to planning disbursement rates from trusts or endowments, including how to modify the optimal spending rule for revisions in the investment outlook.

### BACKGROUND AND MODELLING FRAMEWORK

Globally, in 2011, there were 11 million individuals with over US\$1 million in investable wealth, amounting to more than US\$42 trillion in assets. More than 3 million HNWI's resided in each of Asia-Pacific, Europe and North America with the highest concentration of wealth in North America (CapGemini and RBC Wealth Management 2012, p. 5). HNWI's often employ skilled professionals in family offices, foundations and trusts to manage their wealth for current and future generations (Martiros and Millay 2006).<sup>1</sup> While concerns for privacy make it difficult to measure the size of the HNWI sector separately, Martiros and Millay infer that it is substantial. The discussion of HNWI plans in this chapter extends naturally to many similar types of organisations, including perpetual charities and foundations. The charity sector is known to be very large: in 2012, the UK Charity Commission reported over 161,000 charities, holding investments in excess of £78 billion with annual spending over £53 billion,<sup>2</sup> and, for the US, Standard & Poor's Money Market Directories reported over 5,000 endowments and foundations, controlling more than US\$946 billion in assets.<sup>3</sup>

The wealth management problem for these organisations has some important non-standard features that are addressed. Specifically, the setting combines the Epstein–Zin–Weil (EZW) recursive utility model (Epstein and Zin 1989; Weil 1990), which allows a separation of tastes for risk from tastes for consumption smoothing, with general assumptions on investment returns, and in an infinite time horizon. The model is particularly suited to HNWI individuals or families who plan investment separately from spending. For example, if assets are concentrated in operating businesses, and/or if investment decisions are delegated to managers, investment goals are likely to be made largely independently of the spending interests of family members. Amit *et al* (2008) reported that 58% of family offices in their survey sample operated family businesses, with 77.5% of those holding a controlling interest. On average, around one quarter of wealth is tied to operating business in the US and Europe. Such

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HNW individuals or families are likely to be more constrained in asset allocation than in spending decisions. Moreover, most family offices have investment management committees. While the family offices surveyed reported that trans-generational wealth management was their overarching purpose, investment goals ranged from aggressive wealth growth to conservative preservation. Non-expected utility models such as EZW give the flexibility needed to match this disjunction in tastes between investment management and spending.

In addition, when investment returns can be treated as independent and identically distributed (iid) the asset allocation and consumption decisions in this model are theoretically separable, reflecting a division between investment and spending policies. So the analysis to follow derives spending plans under alternative scenarios for returns that are conditional on a separately managed portfolio. The framework also allows family office managers and trustees to carry out general scenario analysis without assuming that returns processes are lognormal. HNWI portfolios include private equity, hedge funds and real estate in significant quantities in both Europe and North America (Amit *et al* 2008) and since there is evidence that returns to these asset classes are typically non-normal more general distributional assumptions are needed.<sup>4</sup> Consequently, the analysis is robust to many of the irregularities of financial returns processes.

The general problem of an entity making spending and investment plans over a finite or infinite horizon, subject to uncertainty, has generated a huge literature. Standard models usually comprise time-additive von Neumann–Morgenstern utility, often with uncertainty generated by lognormal diffusions, with only a few cases where explicit solutions can be derived. Merton's seminal model (Merton 1969) analyses an infinitely lived entity with a constant relative risk aversion utility function.<sup>5</sup> In the case where all asset returns are lognormally distributed and some regularity conditions on the rate of discounting of future utility are satisfied, the optimal rate of consumption is constant, and optimal wealth is lognormal and bounded below. In Merton's case all calculations are done continuously rather than discretely. Although the key features of the Merton solution (a constant disbursement rate and strictly positive wealth) are interesting and the solution is relatively easy to compute, it only partly addresses the problem of this chapter.



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First, a continuous time framework does not fit the decision-making of a family office or trust, where boards may meet quarterly or less often (Amit *et al* 2008). So this analysis solves for annual spending rates over an infinite horizon. (The survival prospects of families, and how the problem could be modified in the light of variation in survival, are also discussed below.) Second, joint lognormality seems an excessively restrictive assumption for returns, given the asset classes invested in by HNWIs, family offices and foundations. Third, as noted above, Amit *et al* (2008) and Martiros and Millay (2006) describe processes of investment management that are delegated to groups of in-house or external managers, so the model below allows asset allocation decisions to be decided separately from disbursement rates.

Fourth, beneficiaries and/or family members may want smooth spending paths. Models which apply the usual time-separable expected utility functions limit the scope of analysis by constraining relative risk aversion to be the inverse of the elasticity of intertemporal substitution, so that agents who have low risk aversion must also be willing to transfer consumption through time. However, for HNW individuals or families, risk aversion and aversion to intertemporal substitution are likely to be conceptually and practically distinct: many family trusts can tolerate considerable uncertainty over returns while aiming for fairly smooth payments to beneficiaries over time.

Recursive or non-expected-utility preferences as proposed by Kreps and Porteus (1978, 1979) allow a partial separation of tastes for risk and intertemporal consumption. Whereas the von Neumann–Morgenstern agent is interested only in the conditional expectation of all future consumption (the timing of the resolution of uncertain outcomes does not matter), the Kreps–Porteus agent also cares how soon uncertainty over consumption will be resolved. If an entity is highly risk averse but willing to redistribute consumption through time, then they prefer an early resolution of uncertainty, but if an entity is tolerant of risk and, relatively speaking, dislikes transferring consumption through time, then later resolution is better. As Weil (1990) points out, this amounts to a trade-off between the safety and stability of utility, where safety is improved by early resolution of risk and stability by late resolution. The model used here adapts

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Weil's version of the Kreps–Porteus preferences to the dynamic consumption problem of HNWI's, and explores the properties of the model under scenario analysis.<sup>6</sup>

Giovannini and Weil (1989) and Weil (1990) showed that the optimal constant disbursement rate for an HNWI with EZW utility is set by the rule

$$m = 1 - (\delta\varphi^{(1-\rho)/(1-\alpha)})^{1/\rho} \quad (12.1)$$

where  $m$  is the proportion of wealth spent each year,  $\delta$  is a parameter which is a component of time preference,<sup>7</sup>  $\alpha$  is relative risk aversion,  $1/\rho$  is the elasticity of intertemporal substitution and  $\varphi$  is the expected value of  $\tilde{Z}^{1-\alpha}$ , the risk-adjusted return to wealth, where  $\tilde{Z}$  is the gross return to the investment portfolio or family business.

Under constant relative risk aversion (CRRA) preferences, the spending rule simplifies to  $m = 1 - (\delta\varphi)^{1/\alpha}$ . The analysis to follow also gives the conditions for the convergence of the value function for this problem. Given plausible parameter values and historical estimates of investment returns, optimal spending rates might lie between 1% and 3% of wealth per annum in real terms.

Shifting probability mass from the lower to the upper tail of the returns density, and working with mean-preserving spreads, allows for an analytical and numerical calculation of the trade-off between income and substitution effects and the ensuing changes to disbursement rates when the investment returns distribution changes. Further, while superficial intuition might predict that spending out of an endowment will be positively related to an optimistic investment outlook and negatively linked to pessimism, this is true only for a sub-set of preferences and the reverse reaction can be optimal. Estimation of these effects indicates that optimal consumption rates are remarkably sensitive to small changes in beliefs about future returns distributions. The direction of revisions to optimal consumption depends on whether the elasticity of intertemporal substitution is greater than or less than 1, not on tastes for risk, but the size of revisions will be sensitive to relative risk aversion. Stochastic dominance arguments confirm, extend and illustrate analytical results found in Giovannini and Weil (1989), Weil (1990) and Bhamra and Uppal (2006), which demonstrate the pivotal role of the elasticity of intertemporal substitution for consumption paths.

Having set out the reasons for our choice of utility function and approach to returns distributions, we now address the question of

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the planning period, or expected lifetime of the foundation, family office or trust.

### MODELLING FOR PERPETUAL ENTITIES AND FAMILIES

A crucial question for disbursement decisions is the planning horizon of the endowment or trust. Embedded in the planning horizon are questions of intra- and inter-generational equity, so important to the management of family offices and foundations. Two main strands of economic literature have studied the management of wealth over generations. The first relates to perpetual foundations such as university endowments, and the second relates to families.

#### Perpetual foundations and endowments

Studies of university endowment behaviour look for a disbursement rate rule that satisfies "inter-generational equity" while preserving capital over the long horizon (see, for example, Tobin 1974; Litvack *et al* 1974; Nichols 1974). Most are not interested in deriving optimal portfolio allocations for endowments. Tobin proposed consuming out of permanent foundation income (ie, from the long-run rate of return on assets). However, Woglom (2003) showed that Tobin's definition of inter-generational equity (fixed real consumption through time) implied a zero rate of intertemporal substitution. For agents with CRRA utility functions this means infinite risk aversion, an assumption that is contradicted by endowment investment patterns. Using a deterministic, continuous-time model, Woglom argued that endowments should consume from recurrent capital gains, but he relaxed the inter-generational equity constraint to allow optimal real consumption to vary over time.

University spending and investment were readdressed in later papers by Merton (1990, 2003), who considered optimal consumption and portfolio allocation at the whole university level, rather than the endowment level. When income streams (gifts, bequests, etc) and the costs of university activities co-vary with investment returns, he argued that university portfolio managers can hedge against future cost changes and adjust to non-tradeable income sources by employing replicating strategies. Dybvig (1995, 1999) viewed the inter-generational equity question differently, proposing that most endowments will want short-run spending certainty while maintaining long-run viability. He argued for a dynamic portfolio



insurance strategy where the institution creates a riskless perpetuity matched to the current minimum spending level while maintaining exposure to risky returns.<sup>8</sup> This strategy is probably too conservative for most HNWI, and the analysis to follow allows more flexible intertemporal consumption and investment plans.

**Families and family survival**

Studies of university endowments treat the decision-making entity as a unified whole with one set of preferences. The second line of literature studying family utility maximisation explores intra- and inter-generational preferences. As Xu (2007) notes, the family is a place of both conflict and cooperation: Amit *et al* (2008, p. 10) reported that the average single family office in their survey sample served "13 households, 40 family members and two to three generations". A simplifying approach to inter-generational transfers is to treat the head (altruist or dictator) as deciding consumption among current and future members of the family so that the welfare of the family is indistinguishable from the welfare of the head (Becker 1974, 1981).

For family trusts and family offices, a trust deed or constitution can stand in the place of a family "head", deciding on allocations between beneficiaries. The fact that most HNWI families use formal agreements is evidence that cooperative bargaining (Manser and Brown 1980; McElroy and Horney 1981) and exchange between family members (Cigno 1993, 2006, 2007) are not effective or stable structures. Further, the utility of future generations is often valued, as well as the interests of the current family members. Becker and Tomes (1986) and Becker and Barro (1988) discussed cases where consumption is divided equally among children in each time period and then aggregated. In this case, trusts and foundations have to plan for expected survival rates.

The survival of families has been a question of interest to mathematicians and demographers for hundreds of years but few empirical estimates of family line survival are available (Albertsen 1995). ~~Early estimates of family survival from the 20th century put the probability of family extinction below 1 (Keyfitz 1968; Hull 1998) but since the 1960s, fertility rates in many countries have declined and are below replacement rates. For some countries, including the UK and Australia, this decline seems to have slowed or reversed (Office for National Statistics 2011, Table 1.4), but rates are still below~~



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**Table 12.1** Estimated probability of family survival and expected family size

Number of generations	Single-branch family		Triple-branch family	
	Probability of survival	Expected family size	Probability of survival	Expected family size
0	1.000	1.000	1.000	3.000
1	0.620	0.949	0.945	2.847
2	0.452	0.901	0.836	2.702
3	0.354	0.855	0.731	2.564
4	0.289	0.811	0.641	2.433
5	0.243	0.770	0.566	2.309
10	0.126	0.592	0.333	1.777
25	0.037	0.285	0.105	0.854
50	0.008	0.077	0.024	0.024
150	≈ 0.000	≈ 0.000	≈ 0.000	≈ 0.000

~~replacement~~. For England and Wales, for example, the average number of live daughters a woman of child-bearing age of a particular cohort can expect to have in her lifetime (gross reproduction rate) rose to 0.96 in 2008 from a low point of 0.80 in 2000.<sup>9</sup> Since the probability that a family eventually reaches extinction depends on the average number of daughters (~~sons~~) born to women (~~men~~) in the family, the gross reproduction rate of 0.96 implies that a UK family traced from an average mother along the female line would become extinct ~~in a finite number of generations~~.

Satchell and Thorp (2011) showed how the theory of branching processes and birth statistics can be used to estimate family survival functions. The pattern of family survival depends on overall fertility, the probabilities of particular numbers of births and the number of branches in the original family. Table 12.1 reports the estimated probability of survival and expected family size (in terms of one gender only) using average fertility patterns of mothers born in England and Wales in 1960. The data shows that, while not expected to survive forever, families have positive probabilities of survival over several hundred years. The three-branch family in the table is not expected to have a survival probability of less than 50% for six generations. Another interesting implication of the model, explored in

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Satchell and Thorp, is that optimal spending plans by family foundations and trusts will vary with survival prospects, being optimally higher in years when survival is more likely. For a single-branch family, this result implies a hyperbolic discount function, while for a multiple-branch family the function is non-monotonic.

Modelling in the remainder of this chapter assumes a constant discount factor since including a time-varying discount rate would rule out analytical solutions. While the assumption of constant discounting is a simplification of survival prospects, assuming an infinite horizon is a reasonable approximation to the very long horizon of any multiple-branch family aiming for sustainable inter-generational wealth transfers.

### ANALYTICAL APPROACHES TO OPTIMAL DISBURSEMENT RATES

The standard problem for intertemporal utility maximisation is to find the optimal functional form for consumption and the set of asset allocations that will maximise the expected multi-period utility of wealth through time. Here, the decision-maker is infinitely lived but makes annual consumption plans. Proofs for all the propositions that follow appear in Appendix A.

#### Recursive utility

Giovannini and Weil (1989) and Weil (1990) find the closed-form solution for the optimal consumption path of an infinitely lived entity that maximises a discrete-time recursive utility function.<sup>10</sup> The aggregator function for utility has two arguments; the first represents the value of current consumption and the second represents expected future utility over uncertain future consumption

$$\begin{aligned}
 L_t &= U[C_t, E_t L_{t+1}] \\
 &\equiv \frac{1}{(1 - \delta)(1 - \alpha)} \\
 &\quad \times \{(1 - \delta)C_t^{1-\rho} \\
 &\quad + \delta[1 + (1 - \delta)(1 - \alpha)E_t L_{t+1}]^{(1-\rho)/(1-\alpha)}\}^{(1-\alpha)/(1-\rho)} - 1
 \end{aligned}
 \tag{12.2}$$

where  $\delta \in (0, 1)$ ,  $\alpha > 0$  and  $\rho > 0$ , and where  $C_t$  is consumption in the form of spending by beneficiaries and costs.<sup>11</sup>

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Time preference in Equation 12.2 is represented by the aggregator function, so that the derivative of Equation 12.2 with respect to expected future utility can be viewed as a subjective discount factor. If the aggregator is convex with respect to expected future utility, the agent prefers early resolution of uncertainty, or safety over stability. If Equation 12.2 is concave with respect to its second argument, then the agent prefers a stable certainty equivalent path of future consumption. As Weil points out,  $\delta$  is the subjective discount factor in the case of certainty and in the linear CRRA case where  $\alpha = \rho$ . It is straightforward to show that the convexity or concavity of the aggregator function depends on the relative sizes of  $\alpha$  and  $\rho$ , being convex when  $\alpha > \rho$  and concave when  $\alpha < \rho$ . Convexity implies more rapidly increasing patience and concavity more slowly increasing patience as expected future utility rises. Agents who are more risk tolerant and value smoothness ( $\alpha < \rho$ ) prefer late resolution, and agents who dislike risk but tolerate larger swings in certainty equivalent utility ( $\alpha > \rho$ ) prefer early resolution.

Another way to view the parameters of the model is to recognise that the coefficient of relative risk aversion for timeless gambles is  $\alpha$  and the constant elasticity of intertemporal substitution for deterministic consumption paths is  $\rho$ . If either parameter approaches unity, then preferences become logarithmic in that dimension, so that we get logarithmic risk preferences when  $\alpha \rightarrow 1$  and logarithmic intertemporal substitution preferences when  $\rho \rightarrow 1$ . Under the special case where  $\alpha = \rho$ , the utility function represents the preferences of an individual with CRRA and for whom the inverse of the risk aversion parameter is the elasticity of intertemporal substitution.<sup>12</sup>

**Wealth**

The HNWI's optimisation problem also depends on the wealth generated by investment income and donations. The amount of money available for investment,  $I_t$ , is given by

$$I_t = W_t - C_t \tag{12.3}$$

where  $W_t$  is the wealth at time  $t$ . If  $I_t$  is invested in  $n$  assets, buying  $N_{i,t}$  shares in the  $i$ th asset at a price  $P_{i,t}$ , then

$$I_t = \sum_{i=1}^n N_{i,t} P_{i,t} \tag{12.4}$$

If we define the random return to the  $i$ th asset as the random variable

$$\tilde{z}_{i,t} \equiv \frac{\tilde{P}_{i,t+1}}{P_{i,t}} \quad (12.5)$$

then the stochastic wealth of the charity at time  $t + 1$  is

$$W_{t+1} = I_t \sum_{i=1}^n w_{i,t} \tilde{z}_{i,t} \quad (12.6)$$

where  $w_{i,t} \equiv N_{i,t}P_{i,t}/I_t$  represents the relative weights of the assets in the portfolio, so that  $\sum_{i=1}^n w_{i,t} = 1$  and saving from wealth is fully invested in each period.

The charity consumes  $C_t$  by spending on administration and providing funding to beneficiaries. Setting aside questions of portfolio allocation, and assuming for now that no donation income is received, the budget constraint is

$$W_{t+1} = (W_t - C_t)\tilde{Z}_t \quad (12.7)$$

where  $\tilde{Z}_t \equiv \sum_{i=1}^n w_{i,t}\tilde{z}_{i,t}$  is the random growth in investments from  $t$  to  $t + 1$ . If  $C_t = m_t W_t$ , then Equation 12.7 is

$$W_{t+1} = (1 - m_t)W_t\tilde{Z}_t \quad (12.8)$$

This gives us a difference equation

$$\begin{aligned} W_t &= W_0 \prod_{i=0}^{t-1} (1 - m_i)\tilde{Z}_i \\ &= W_0 \tilde{V}_{t-1} \prod_{i=0}^{t-1} (1 - m_i) \end{aligned} \quad (12.9)$$

where  $\tilde{V}_{t-1}$  is the accumulated value of one unit of wealth invested at  $t = 0$  and held until time  $t$ ; it is random and assumed to be non-negative.

**Proposition 12.1.** If  $\tilde{Z}_i$  is a positive iid random variable and  $\tilde{Z}_i^{1-\alpha}$  is a well-defined random variable such that  $\mathbb{E}(\tilde{Z}_i^{1-\alpha}) = \varphi$  exists for  $0 < \alpha < \infty$ , it follows that  $\mathbb{E}(\tilde{V}_{t-1}^{1-\alpha}) = \varphi^t$  for all integer  $t > 0$ .

#### Income and new contributions

Many HNWI's and family offices rely entirely on investment income after a foundational business has been sold (Amit *et al* 2008). However, it is possible to generalise to the case where new sources of

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income or new contributions  $Y_t$  are received during the time period  $t - 1$  to  $t$  but invested at the end of the period. Some family foundations may also receive charitable donations, for example. (Income received during the period cannot be invested in this discrete-time framework until the market opens in integer time.) This means that the wealth equation (Equation 12.8) needs to be adjusted to



$$W_t = (1 - m_t)W_t \tilde{Z}_{t-1} + Y_{t-1} \quad (12.10)$$

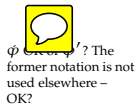
Then

$$W_t = W_0 \tilde{V}_{t-1} \prod_{i=0}^{t-1} (1 - m_i) + \sum_{j=0}^{t-1} Y_j \left( \frac{\tilde{V}_{t-1}}{\tilde{V}_j} \right) \prod_{i=0}^{t-1-j} (1 - m_i) \quad (12.11)$$

where  $\tilde{V}_0$  is assumed to equal 1.

It is apparent that no closed-form solution to the optimisation problem described by Equation 12.2 exists for additive income for general distributions. However, using the fact that new sources of income must be positive, they can be expressed as a multiplicative addition to wealth. Define the cumulative growth in income from new sources as a proportion of wealth  $\tilde{Y}_{t-1} = \prod_{i=0}^{t-1} \tilde{y}_i$ , and rewrite the wealth constraint as

$$W_t = W_0 \tilde{V}_{t-1} \tilde{Y}_{t-1} \prod_{i=0}^{t-1} (1 - m_i) \quad (12.12)$$



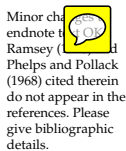
In this case the new interpretation of the risk-adjusted expected return to wealth,  $\varphi$ , is

$$\mathbb{E}(\tilde{Z}_{t-1}^{1-\alpha} \tilde{Y}_{t-1}^{1-\alpha}) = \mathbb{E}(\tilde{Z}_{t-1}^{1-\alpha}) \mathbb{E}(\tilde{Y}_{t-1}^{1-\alpha}) + \text{cov}(\tilde{Z}_{t-1}^{1-\alpha}, \tilde{Y}_{t-1}^{1-\alpha}) = \varphi \quad (12.13)$$

We now go on to solve the optimisation problem for consumption or the disbursement rate subject to the wealth process set out above.

**Optimal consumption path**

Giovannini and Weil (1989) and Weil (1990) show that the optimal disbursement rate will be a constant proportion of wealth when the horizon is infinite and the return to invested wealth  $\tilde{Z}_i$  is positive iid.<sup>13</sup>



**Proposition 12.2 (Giovannini and Weil 1989; Weil 1990).** The consumption-to-wealth ratio  $m$  that maximises aggregated utility

(Equation 12.2) for  $t = 0, \dots, \infty$ , subject to the wealth constraint (Equation 12.9), where  $\tilde{Z}_i$  is positive iid, is constant and given by

$$m = 1 - (\delta\varphi^{(1-\rho)/(1-\alpha)})^{1/\rho} \quad (12.14)$$

and the optimised value of Equation 12.2 is

$$L(W) = \frac{(\psi W)^{1-\alpha} - 1}{(1-\delta)(1-\alpha)} \quad (12.15)$$

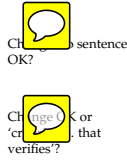
for  $\psi = [(1-\delta)m^{-\rho}]^{1/(1-\rho)}$ .

For CRRA utility, when  $\alpha = \rho$  the optimal disbursement rate simplifies to  $m = 1 - (\delta\varphi)^{1/\alpha}$ . In the case of logarithmic risk preferences when  $\alpha \rightarrow 1$ , the consumption-to-wealth ratio is myopic over investment risk even when the investment opportunity set is non-constant, and in the case of logarithmic intertemporal substitution preferences when  $\rho \rightarrow 1$  the consumption-to-wealth ratio is constant for all values of  $\alpha$ . Thus, in either logarithmic case, the optimal disbursement rate is independent of our assumption about  $\mathbb{E}(\tilde{V}_{t-1}^{1-\alpha})$ . In other words, the consumption rate is independent of asset allocation, although the amount of wealth drawn down is not.<sup>14</sup>

The feasibility and dynamic stability of this plan can be ensured by placing conditions on model parameters. The dynamic spending plan in Equation 12.14 is feasible (satisfying strictly positive wealth and consumption constraints) when the rate of disbursement is positive so that  $\delta\varphi^{(1-\rho)/(1-\alpha)} < 1$ , or for the CRRA case, when  $\delta\varphi < 1$ . Since the (gross) return to wealth is always non-negative so that  $\varphi \geq 0$ , and given that the discount rate  $\delta \in (0, 1)$ , a consumption-to-wealth ratio strictly less than 1 is sufficient to ensure feasibility. Dynamic stability, such that the expected value of optimised utility is bounded at the infinite horizon, is also satisfied by  $\delta\varphi < 1$  in the CRRA case, but the conditions for feasibility and dynamic stability do not always coincide in the non-linear recursive utility case.<sup>15</sup> Proposition 12.3 sets out a sufficient condition for dynamic stability that utilises the binomial form of the aggregator function (Equation 12.2).

**Proposition 12.3.** Under Newton's Generalised Binomial Theorem (Graham *et al* 1994), the aggregator function in Equation 12.2 is the sum of a convergent infinite series if  $m/(1-m) < 1$  so that the spending rate is less than the saving rate.

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The proof of this proposition is set out in Appendix A, and begins by inserting the value function (Equation 12.15) into the aggregator function (Equation 12.2). An inspection of the result shows its similarity to a generalised binomial form. Newton’s Generalised Binomial Theorem thus gives the criteria for convergence, which verifies the transversality condition, and it follows that the value function converges to zero in the limit.

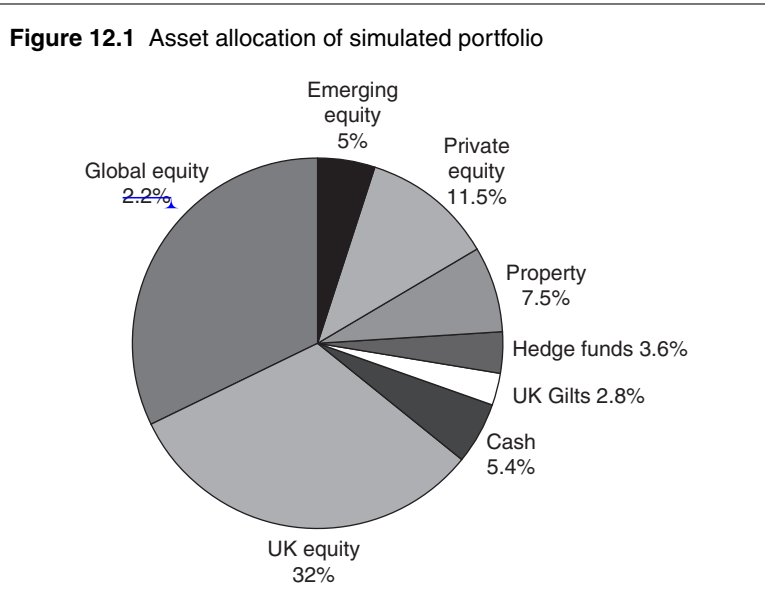
The convergence condition  $m/(1 - m) < 1$  applies where the discounted value of expected future utility (the second argument in the aggregator function) exceeds the value of current consumption (the first argument in the aggregator function), and amounts to the requirement that the optimal spending rate be less than the saving rate. If the reverse is true and the value of current consumption exceeds discounted expected utility, then the rate of spending needs to exceed the rate of saving to achieve dynamic stability. For most of the empirical applications to follow, where the optimal spending response to moderate changes to the investment outlook is modelled, the spending rate must be less than the saving rate. This condition is equivalent to  $m < \frac{1}{2}$ , which is not a binding constraint for most conventional parameterisations of the problem.

The conditions for optimal portfolio selection for this problem are well known and we do not repeat them here. Importantly, when returns are iid, portfolio choice is dependent only on tastes for risk, not preferences over intertemporal substitution, and does not depend on expectations of the future consumption path.<sup>16</sup>

In the following sections, we take advantage of this separation and treat the portfolio allocation as predetermined (although not necessarily optimal). But, for any given asset allocation, however determined, it is possible to calculate the impact on the ideal disbursement rate caused by changes in the distribution of future returns, risk attitudes and/or portfolio weights. The scenario analysis is given below (see p. 230).

**EMPIRICAL ILLUSTRATION**

The empirical implications of the explicit solution for the optimal consumption-to-wealth ratio (Equation 12.14) can be illustrated using a simulated sample of portfolio returns. Amit *et al* (2008) report that HNWIs and family offices hold capital in public and private



equity, hedge funds, real estate, fixed income, commodities and collectibles. The actual holdings of HNWI's are not available, so we construct a returns series using an asset allocation similar to the Wellcome Trust in 2005 as a long-lived trust established by an HNWI.<sup>17</sup> Figure 12.1 shows the proportions of total funds invested in each asset class for simulated portfolio returns and these weights are fixed for the whole sample period.<sup>18</sup>

The simulated data are monthly real portfolio returns over the period January 1990 to June 2006 (198 observations), based on individual asset class returns from standard indexes and deflated using consumer prices and earnings data. It is reasonable to expect that wages are an important cost for a family office, and deflation using consumer prices alone will overstate real spending power, so inflation is computed as 50% consumer price driven and 50% purely due to wage increases.

The mean (log) real annualised return to this portfolio is 4.75% with volatility of 13% per year. Summary statistics in Table 12.2 show that the (monthly) data is significantly non-normal: negatively skewed and leptokurtic. However, the autocorrelation structure of the de-measured returns and squared de-measured returns supports an assumption that real portfolio returns are iid. Ljung-Box Q statistics, not reported here, are insignificant to at least 50 lags for

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QUANTITATIVE APPROACHES TO HIGH NET WORTH INVESTMENT

**Table 12.2** Summary statistics, real annualised portfolio returns, January 1990–June 2006

Mean (%)	4.75
Standard deviation (%)	13.02
Skewness	-0.69
Kurtosis	3.88
Jarque–Bera	21.87
( $p$ -value)	(0.000)

the de-meaned returns. The squared residuals have one significant autocorrelation at lag 10.

Equation 12.14 is the optimal rate of disbursement rate for an infinitely lived charity under a fixed asset allocation, given time preference parameter  $\delta$ , consumption smoothing parameter  $\rho$  and relative risk aversion  $\alpha$ . Another key determinant is the mean of the risk-aversion-scaled portfolio return,  $\mathbb{E}(\tilde{Z}_i^{1-\alpha}) = \varphi$ . To estimate  $\varphi$ , the monthly portfolio returns were bootstrapped using 120,000 random draws, then summed to get 10,000 annual real (gross) returns and the sample mean was computed

$$\hat{\varphi} = \frac{1}{j} \sum_{i=1}^j \tilde{Z}_i^{1-\alpha} = 10,000 \tag{12.16}$$

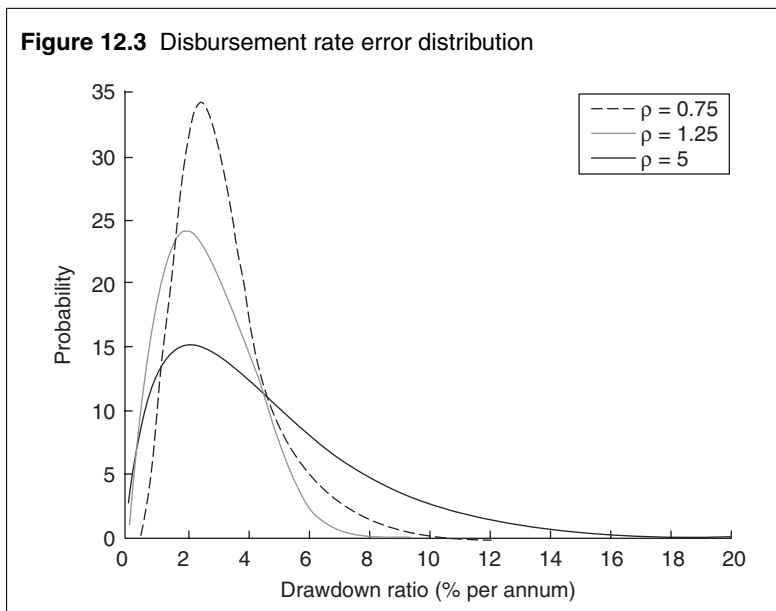
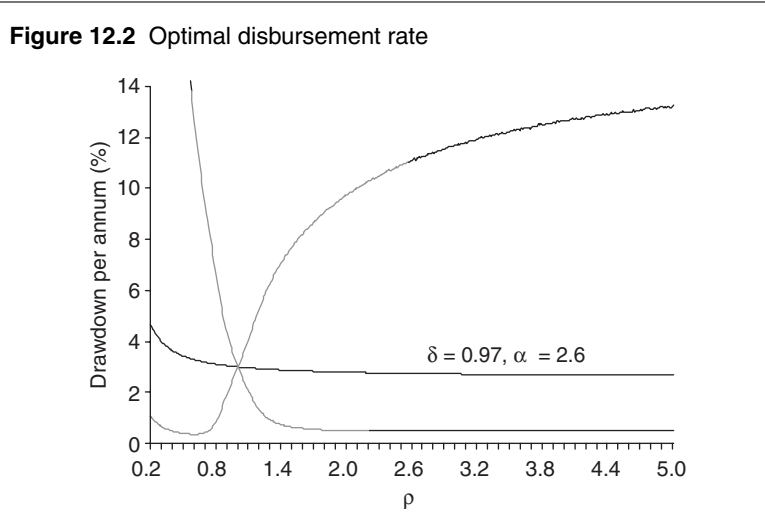
for  $\delta = 0.97$  and  $\alpha > 0$ .

The estimated optimal disbursement rate  $\hat{m}$  is shown in Figure 12.2 for values of the intertemporal substitution parameter  $\rho$  between 0.2 and 5, and with risk aversion  $\alpha = 2.6$ . This value for risk aversion is estimated from the portfolio weights of the Wellcome Trust, assuming that the portfolio is optimal.<sup>19</sup> The estimated risk aversion parameter is only indicative, and serves as a reference point for numerical illustrations.

The light grey curves in Figure 12.2 give an approximate 95% error range for the estimate of  $m$ . A beta distribution is fitted to 1,000 bootstrapped estimates of  $\hat{\nu}_\alpha := (\delta \varphi^\rho / (1-\alpha))^{1/\rho}$  by maximum likelihood, after filtering out values that do not meet the feasibility and boundary conditions. From the estimated beta parameters, quantiles  $1 - \hat{\nu}_{\alpha,0.025} = 1 - F^{-1}(0.025)$  and  $1 - \hat{\nu}_{\alpha,0.975} = 1 - F^{-1}(0.975)$  can be inferred as a guide to the accuracy of  $\hat{m}$ . Consistent with the solution for logarithmic intertemporal substitution preferences, the optimal consumption rate,  $\hat{m} = (1 - \delta)$ , is 3% per annum when  $\rho = 1$ . As



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tolerance for consumption transfer through time decreases and  $\rho$  increases, the disbursements fall from around 4.7% when  $\rho = 0.2$ , reaching 2.8% when  $\rho = 5$ .<sup>20</sup>

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Figure has been released and rechecked according to the style. Please check legend has been reproduced correctly – this was unclear from your manuscript. Also, change from "13" to "14" OK on horizontal axis?

The error range around  $\hat{m}$  widens rapidly as the elasticity of intertemporal substitution (EIS) diverges from 1 in either direction. Figure 12.3 graphs the estimated beta distributions of the optimal disbursement rate at three indicative values of the intertemporal substitution parameter. When the EIS is relatively high, at 1.33 ( $\rho = 0.75$ ), the error distribution is more right-skewed and dispersed than when the EIS falls to 0.8 ( $\rho = 1.25$ ), where the distribution is more tightly packed around the 3% logarithmic disbursement rate. However, as the EIS moves away from 1, falling to 0.2 ( $\rho = 5$ ), the probability distribution becomes more right-skewed again, and uncertainty over the optimal spending rate increases. This pattern indicates the increasing importance of the stochastic risk-scaled-returns parameter  $\varphi$  to optimal consumption paths as the EIS diverges from 1, since at  $\rho = 1$  consumption depends only on the discount parameter  $\delta$ , which is assumed to be known with certainty.

Hence, a moderately risk averse HNWI will spend between 5% and 2% of wealth each year, but the uncertainty surrounding that optimal solution is very large and increasing as the EIS diverges from 1.

### SCENARIO ANALYSIS

'use historical data to set? Do you mean the outlooks are relative to the historical benchmark or to recent experience? Or that recent experience is tested against the (historical) benchmark to build the outlook? Please clarify sentence.

HNWIs need a way of assessing whether their chosen disbursement rate is robust to changes in beliefs about future returns, an exercise usually called scenario analysis. A natural approach is to set past history as the benchmark and build optimistic or pessimistic outlooks ~~relative to recent experience~~. Alternatively, a range of drawdown rates for a cross-section of foundations or family offices with different beliefs about the returns distribution could be estimated.

In this section we set out a simple procedure to conduct scenario analysis that is not highly dependent on specific assumptions about distributions of returns. The analysis directly connects the desired consumption-to-wealth ratio with stochastic dominance properties of alternative returns distributions.

The influence of the returns distribution on optimal spending rates is via the expectation of risk-scaled portfolio returns,  $\varphi$ . To gauge the optimal spending response to optimistic and pessimistic investment scenarios, consider changes in the expected risk-scaled

portfolio return  $\varphi$ , keeping constant tastes for risk,  $\alpha$ , and intertemporal substitution,  $\rho$ , fixed but varying general distributional characteristics. The change in optimal disbursement rate as  $\varphi$  varies depends on the relative sizes of  $\alpha$  and  $\rho$

$$\frac{\partial m}{\partial \varphi} = -\delta^{1/\rho} \frac{(1-\rho)}{(1-\alpha)} \varphi^{(1-\rho)/((1-\alpha)\rho)-1} \quad (12.17)$$

Since  $\alpha$ ,  $\rho$  and  $\varphi$  are positive, the response of the optimal disbursement rate to an increase in  $\varphi$  will be positive when  $\rho > 1$  and  $\alpha < 1$ , and when  $\rho < 1$  and  $\alpha > 1$ . If both  $\alpha$  and  $\rho$  are greater than 1 or less than 1, then the response of the optimal disbursement rate to an increase in  $\varphi$  will be negative. However, the influence of relative risk aversion on  $\varphi$  itself needs to be accounted for. It turns out that this can be done using the properties of stochastic dominance.

### First-order stochastic dominance

**Proposition 12.4.** If  $Z_i^\Delta$  first-order stochastically dominates  $Z_i$ , then  $\varphi$  is increased if  $0 < \alpha < 1$  and decreased if  $\alpha > 1$ .

First-order stochastic dominance (FSD) implies that  $E_\Delta[G(Z)] \geq E[G(Z)]$  for  $G(\cdot)$  any increasing function. Now apply the result for  $\varphi = E(\tilde{Z}^{1-\alpha})$ , to see that  $G(Z) = Z^{1-\alpha}$  is positive increasing for  $0 < \alpha < 1$  (hence,  $\varphi_\Delta > \varphi$ ), and positive decreasing for  $\alpha > 1$ , so that  $\varphi_\Delta < \varphi$ , where  $\Delta$  is an FSD transformation.

Consider now the change in  $m$  under an FSD shift  $F_\Delta(Z)$ , for each of four combinations of values for relative risk aversion and the elasticity of intertemporal substitution. Two effects interact to determine the response of the consumption-to-wealth ratio to FSD transformations of the returns density. The first depends on the properties of the function  $G(Z)$ , and the second on the sign of the derivative of  $m$  with respect to  $\varphi$  given in Equation 12.17. The outcomes are summarised in Table 12.3.

So, regardless of the size of the relative risk aversion parameter, transformations of the returns distribution that are described by first-order stochastic dominance mean a decrease in the consumption-to-wealth ratio whenever  $0 < \rho < 1$  and an increase in the consumption-to-wealth ratio when  $\rho > 1$ . Weil (1990) showed this result for lognormally distributed portfolio returns, but here it is generalised to the case of any well-behaved continuous returns distribution. The former case,  $0 < \rho < 1$  fits decision-makers with



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QUANTITATIVE APPROACHES TO HIGH NET WORTH INVESTMENT

**Table 12.3** Revisions to drawdown rates under first-order transformations

Risk aversion	Intertemporal substitution parameter	
	$0 < \rho < 1$	$\rho > 1$
$0 < \alpha < 1$	$\frac{\partial G}{\partial Z} > 0 \Rightarrow \varphi$ increases	$\frac{\partial G}{\partial Z} > 0 \Rightarrow \varphi$ increases
	$\frac{\partial m}{\partial \varphi} < 0 \Rightarrow m$ decreases	$\frac{\partial m}{\partial \varphi} > 0 \Rightarrow m$ increases
$\alpha > 1$	$\frac{\partial G}{\partial Z} < 0 \Rightarrow \varphi$ decreases	$\frac{\partial G}{\partial Z} < 0 \Rightarrow \varphi$ decreases
	$\frac{\partial m}{\partial \varphi} > 0 \Rightarrow m$ decreases	$\frac{\partial m}{\partial \varphi} < 0 \Rightarrow m$ increases

high elasticities of intertemporal substitution, and the latter case,  $\rho > 1$ , agents with low elasticities of intertemporal substitution. For optimistic returns scenarios and where the willingness to transfer consumption over time is high, the substitution effect dominates the income effect and the HNWI reduces spending rates, whereas for HNWIs with low elasticities of intertemporal substitution, the income effect dominates the substitution effect, and they increase spending rates. These effects are independent of tastes for risk when returns are iid.

Now consider reshaping the returns distribution to reflect optimistic scenarios for investment that are consistent with FSD transformations. For optimistic outlooks, the aim is to make extremely poor payouts unlikely relative to the recent past by shifting tail mass from the left tail to the right tail of the distribution. For an arbitrary positive continuous density, pdf( $x$ ) we consider two points  $x_l$  and  $x_u$  and the probabilities

$$P_l = \int_0^{x_l} \text{pdf}(x) dx, \quad P_u = \int_{x_u}^{\infty} \text{pdf}(x) dx, \quad P_{md} = \int_{x_l}^{x_u} \text{pdf}(x) dx \tag{12.18}$$

and clearly  $P_u + P_l + P_{md} = 1$ .

Construct a new density by the following shift

$$P'_u = P_u + \Delta, \quad P'_l = P_l - \Delta, \quad P_{md} = P_{md} \tag{12.19}$$

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where  $0 < \Delta < \min(P_u, P_l)$  and

$$\text{pdf}'(x) = \begin{cases} \frac{P'_u \text{pdf}(x)}{P_u} & \text{for } x_u < x < \infty \\ \text{pdf}(x) & \text{for } x_1 < x < x_u \\ \frac{P'_l \text{pdf}(x)}{P_l} & \text{for } 0 < x < x_1 \end{cases}$$

It is easy to check that  $\text{pdf}'(x)$  is still a well-defined density, although it is no longer continuous at  $x = x_1$  or  $x = x_u$ . Note that since a continuous density with zero probability mass at any point was assumed, the discontinuities induced by our transformation will not affect the existence of the integrals. Furthermore, the above transformation can be called "optimistic" in that it transfers probability from the lower tail to the upper tail of the density, while a "pessimistic" transformation does the reverse. For an optimistic transformation

$$\int_0^\infty \text{pdf}'(x) dx < \int_0^\infty \text{pdf}(x) dx$$

which satisfies FSD, and the following corollary holds.

**Corollary 12.5.** If  $G(x)$  is a positive increasing function, then

$$\int_0^\infty G(x) \text{pdf}'(x) dx > (<) \int_0^\infty G(x) \text{pdf}(x) dx, \quad (12.21)$$

for  $\text{pdf}'(x)$  the result of an optimistic (pessimistic) transformation, respectively. An opposite result applies to positive decreasing functions.

*Empirical illustration*

Figures 12.4 and 12.5 show the impact on the optimal spending rate of a range of transformations of the distribution of  $\tilde{Z}_i$ , the portfolio return. Panels (a) and (b) in Figure 12.4 show graphs for the optimal spending rate when  $\alpha = 0.5$  and  $\alpha = 2.6$ , respectively, and  $\rho$  ranges from 0.4 to 1. A positive rescaling of the returns distribution of size, say, 0.02, shifts 2% of the total probability mass from the left tail to the right tail of the distribution and matches an optimistic outlook for investment returns. In the same way, a negative rescaling of 0.02 shifts the same probability mass from the right tail to the left tail, when the investment outlook is bleak. Whenever  $\rho = 1$

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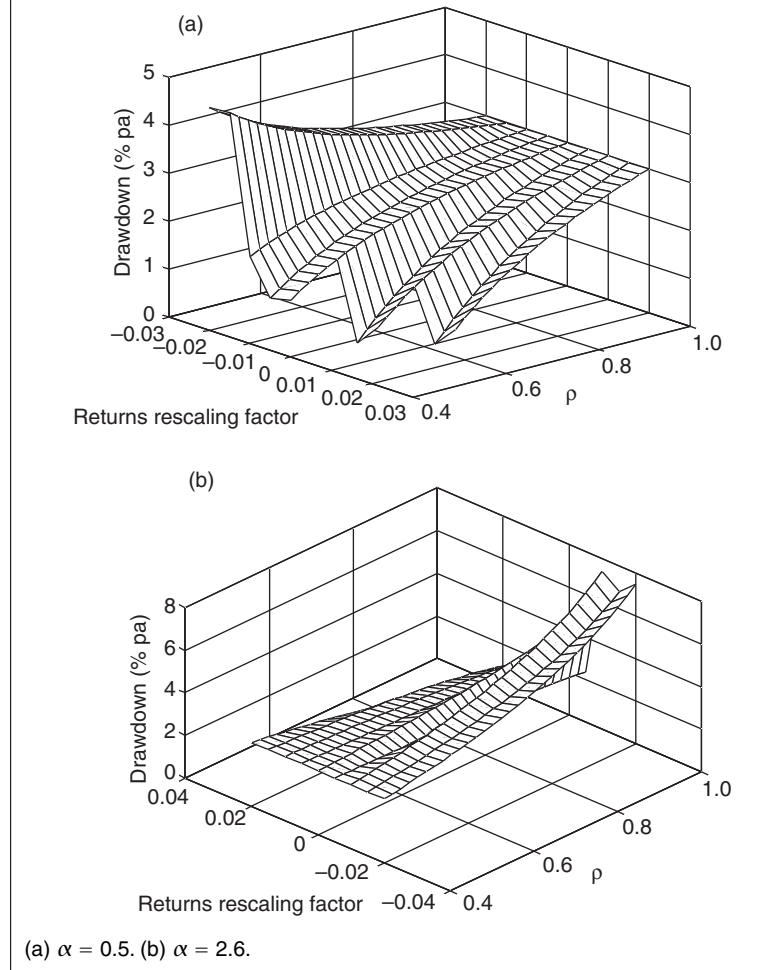
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QUANTITATIVE APPROACHES TO HIGH NET WORTH INVESTMENT

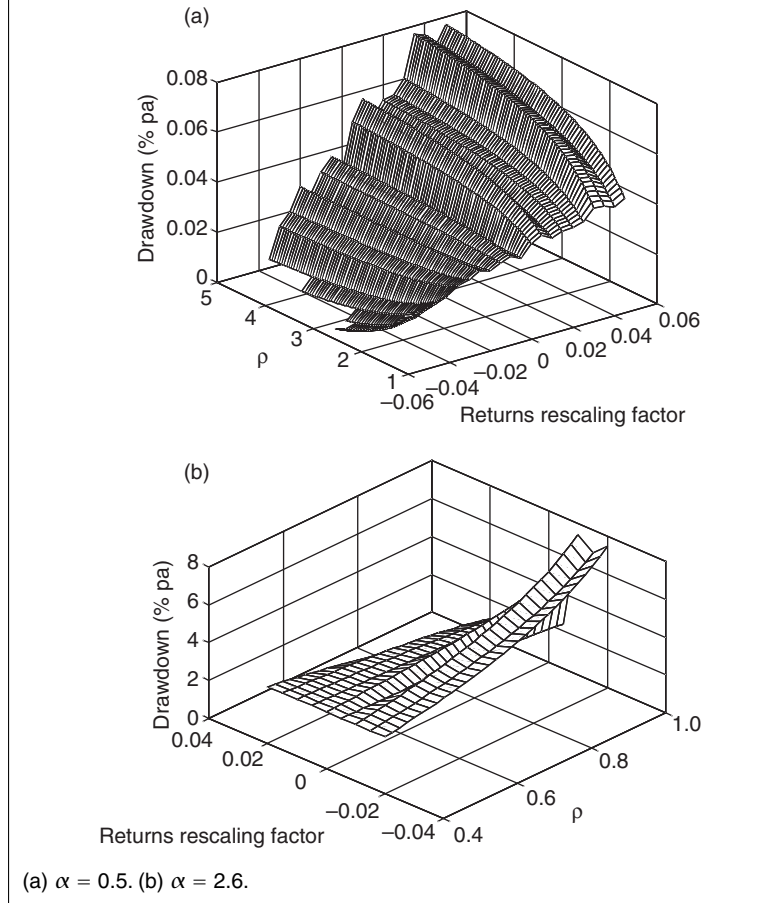
**Figure 12.4** Optimal disbursement rate under transformations of the portfolio returns distribution,  $0 < \rho \leq 1$



the optimal spending rate is 3% per year, but, as  $\rho$  shrinks, EIS increases and spending falls with optimistic expectations and rises with pessimistic expectations.<sup>21</sup> A very flexible foundation or family office facing better prospects does best by decreasing current spending rates in favour of future consumption, with substitution effects dominating income effects.

For HNWIs, foundations or family offices with low elasticities of intertemporal substitution, where  $\rho > 1$ , optimistic transformations

**Figure 12.5** Optimal disbursement rate under transformations of the portfolio returns distribution,  $1 \leq \rho \leq 5$



of the portfolio returns distribution increase the optimal disbursement rate, as they enjoy higher income in the current period rather than favouring future consumption. Figure 12.5 graphs changing spending rates as optimism increases and EIS decreases.

Table 12.4 shows specific examples of the numerical scale of changes in disbursement rates. While the size of the EIS relative to 1 determines the direction of revisions to disbursement rates, relative risk aversion influences the scale of the change. When  $\rho = 2$  (EIS = 0.5) and  $\alpha = 2.6$ , for example, optimal spending at the historical average return is 2.8% per year. Reducing the probability of

**Table 12.4** Numerical example of changes to drawdown rates (FSD)

(a) $\rho = 2, \text{EIS} = 0.5$			
RRA	Benchmark (%) $\Delta = 0.00$	Optimistic (%) $\Delta = 0.04$	Pessimistic (%) $\Delta = -0.04$
$\alpha = 2.6$	2.8	4.6	0.4
$\alpha = 0.5$	3.5	5.7	0.7

(b) $\rho = 0.8, \text{EIS} = 1.25$			
RRA	Benchmark (%) $\Delta = 0.00$	Optimistic (%) $\Delta = 0.04$	Pessimistic (%) $\Delta = -0.04$
$\alpha = 2.6$	3.1	1.9	4.3
$\alpha = 0.5$	2.7	1.9	3.6

left tail returns by 4 percentage points raises spending by 180 basis points (bp) from 2.8% to 4.6% per year. The same size shift in the direction of pessimism reduces spending by 240bp from 2.8% to 0.4% per year. For lower risk aversion, the revisions to benchmark spending are an increase of 220bp for the optimistic scenario and a decrease of 180bp for the pessimistic scenario.

**Second-order stochastic dominance**

The first discussion considered first-order transformations of the returns distribution. Now we consider second-order changes.

**Proposition 12.6.** If  $Z_i^\omega$  second-order stochastically dominates  $Z_i$ , then  $\varphi$  is increased if  $0 < \alpha < 1$  and decreased if  $\alpha > 1$ .

Second-order stochastic dominance (SSD) implies that

$$E_\omega[G(Z)] \geq E[G(Z)]$$

for  $G(\cdot)$  any increasing, concave function. Applying this result for  $\varphi = E(\tilde{Z}^{1-\alpha})$ , we see that  $G(Z) = Z^{1-\alpha}$  is positive increasing and concave for  $0 < \alpha < 1$  (hence,  $\varphi_\omega > \varphi$ ), and positive decreasing and convex for  $\alpha > 1$ , so that  $\varphi_\omega < \varphi$ , where  $\omega$  is an SSD transformation.



The change in  $m$  for each of four combinations of  $\alpha$  and  $\rho$ , given an optimistic transformation of the returns density, is summarised in Table 12.5. If our transformation creates SSD over the original distribution, then the optimal consumption-to-wealth ratio  $m$  decreases



**Table 12.5** Revisions to drawdown rates under second-order transformations

Risk aversion	Intertemporal substitution parameter	
	$0 < \rho < 1$	$\rho > 1$
$0 < \alpha < 1$	$\frac{\partial G}{\partial Z} > 0, \frac{\partial^2 G}{\partial Z^2} < 0 \implies \varphi$ increases	$\frac{\partial G}{\partial Z} > 0, \frac{\partial^2 G}{\partial Z^2} < 0 \implies \varphi$ increases
	$\frac{\partial m}{\partial \varphi} < 0 \implies m$ decreases	$\frac{\partial m}{\partial \varphi} > 0 \implies m$ increases
$\alpha > 1$	$\frac{\partial G}{\partial Z} < 0, \frac{\partial^2 G}{\partial Z^2} > 0 \implies \varphi$ decreases	$\frac{\partial G}{\partial Z} < 0, \frac{\partial^2 G}{\partial Z^2} > 0 \implies \varphi$ decreases
	$\frac{\partial m}{\partial \varphi} > 0 \implies m$ decreases	$\frac{\partial m}{\partial \varphi} < 0 \implies m$ increases

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whenever  $0 < \rho < 1$ . If  $\rho > 1$ , SSD implies the opposite effect, where  $m$  increases as risk shrinks and decreases as risk rises (for a constant expected return). This result confirms the reasoning in Weil (1990) that responses to mean-preserving spreads of the returns distribution depend only on the value of  $\rho$ , but it is shown that this result holds for any second-order stochastic dominance transformation of the returns distribution.



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To illustrate the result, consider a mean-preserving spread of the distribution as a special case of SSD. For an arbitrary positive continuous density,  $\text{pdf}(x)$ , where  $x_i = \mu_x + \varepsilon_i$ ,  $\varepsilon_i \sim \text{iid}(0, \sigma_\varepsilon^2)$ , construct a mean-preserving spread by the following transformation of  $x_i$

$$x'_i = \mu_x + (1 + \omega)\varepsilon_i, \quad 0 < \omega < \infty \tag{12.22}$$

The mean of both distributions is

$$\mathbb{E}(x'_i) = \mathbb{E}(x_i) = \mu_x \tag{12.23}$$

and, for  $0 < \omega < \infty$ , the variance of the transformed variable  $x'_i$  is greater than the variance of  $x_i$

$$\text{var}(x'_i) = (1 + \omega)^2 \sigma_\varepsilon^2 > \text{var}(x_i), \tag{12.24}$$



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These are sufficient conditions for the second-order stochastic dominance of  $\text{pdf}(x)$  over  $\text{pdf}'(x)$ . The variance of  $x_i$  can be shrunk by choosing an optimistic transformation such that  $-1 < \omega < 0$ , so that the transformed distribution  $\text{pdf}'(x)$  dominates the original distribution,  $\text{pdf}(x)$ .

**Corollary 12.7.** If  $G(x)$  is a positive increasing, concave function, then

$$\int_0^\infty G(x) \text{pdf}'(x) dx > (<) \int_0^\infty G(x) \text{pdf}(x) dx \tag{12.25}$$



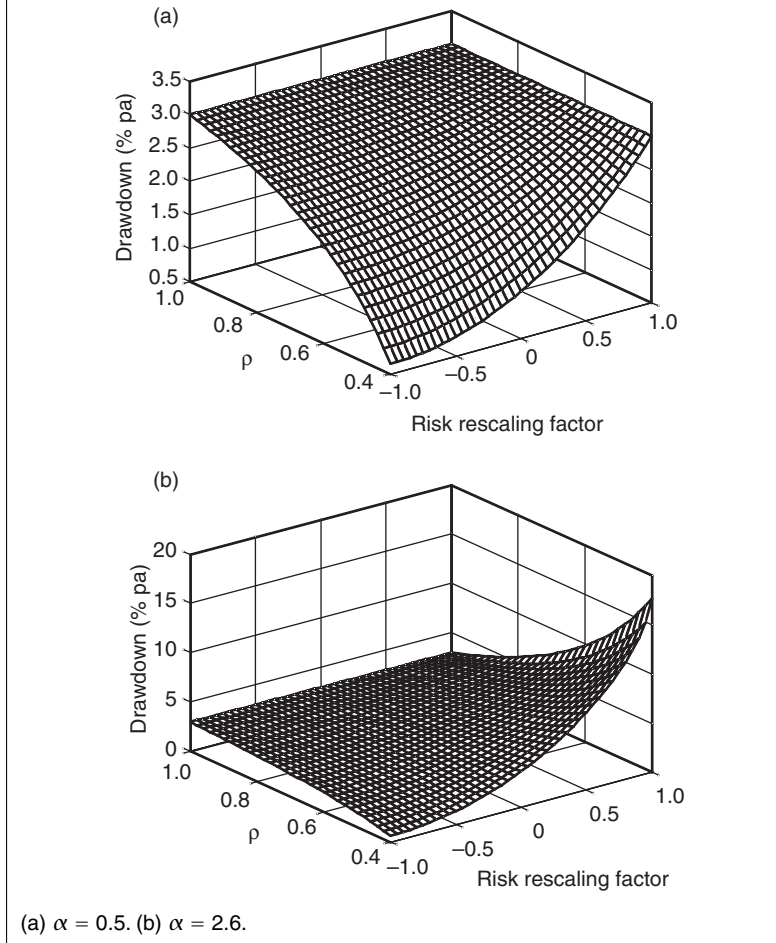
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for  $\text{pdf}'(x)$  the result of an optimistic (pessimistic) transformation. The opposite result applies to positive decreasing, convex functions.

*Empirical illustration*

Figures 12.6 and 12.7 graph the optimal disbursement rate when the variance, but not the mean, of the distribution of  $\tilde{Z}_i$  is increased or decreased. In Figure 12.6 the standard deviation is shrunk from its historical value to almost zero (rescaling to  $-1$ ), or pessimistically raised to twice the historical size (rescaling to  $1$ ), while setting  $\alpha =$

**Figure 12.6** Optimal disbursement rate under mean-preserving spread transformations of the portfolio returns distribution,  $0 < \rho \leq 1$

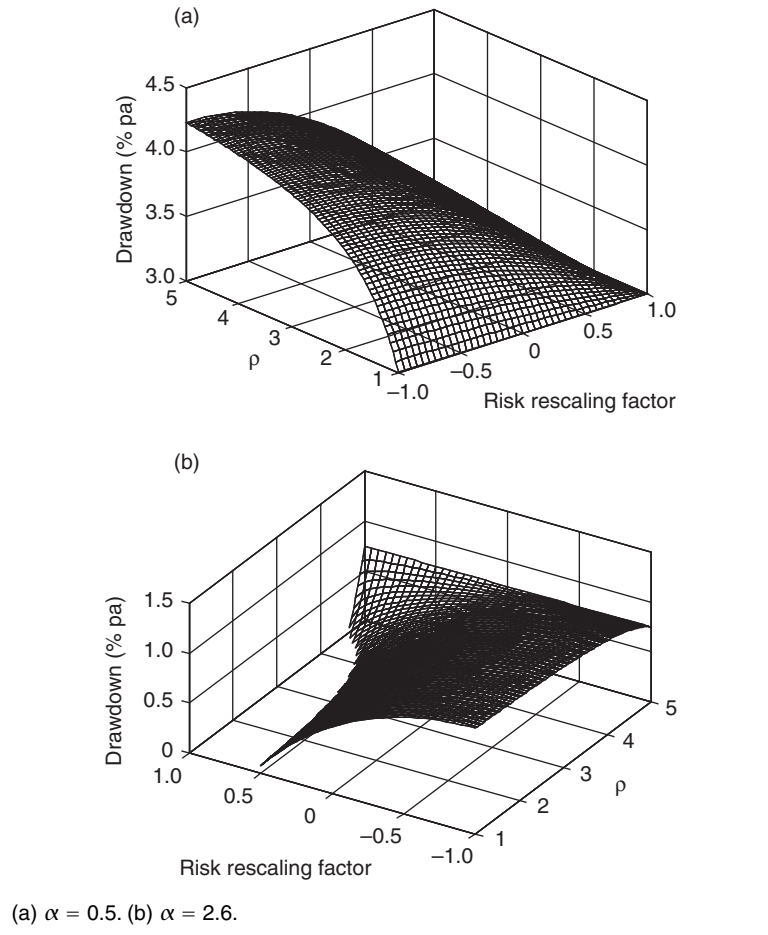


0.5 or  $\alpha = 2.6$ , and allowing  $\rho$  to range from 0.4 to 1. For HNWIs with low elasticities of intertemporal substitution, when  $\rho > 1$ , increases in risk lower optimal spending rates, with the effect becoming more dramatic as EIS shrinks. Figure 12.7 graphs these changing spending rates as optimism over volatility increases and EIS decreases.

Numerically, changes in spread create relatively small revisions to the disbursement rates, as can be seen in Table 12.6. Again the direction of changes depends on the EIS, but the size and original benchmark are also influenced by risk aversion.

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**Figure 12.7** Optimal disbursement rate under mean-preserving spread transformations of the portfolio returns distribution,  $1 \leq \rho \leq 5$



For  $\rho = 2$  and  $\alpha = 0.5$ , the optimal spending rate based on historical returns is around 3.5%, and shrinking volatility by 50% causes a small increase in disbursements towards 3.7%, while increasing volatility by 50% decreases disbursements by about the same amount. For  $\rho = 2$  and  $\alpha = 2.6$ , the same changes on the optimistic side raise spending by 70bp and on the pessimistic side decrease spending by 150bp.

Lower current spending as a reaction to improved prospects is not necessarily irrational or irresponsible. On the contrary, such episodes

**Table 12.6** Numerical example of changes to drawdown rates (SSD)

(a) $\rho = 2$ , EIS = 0.5			
RRA	Benchmark (%) $\omega = 0.00$	Optimistic (%) $\omega = -0.5$	Pessimistic (%) $\omega = 0.5$
$\alpha = 2.6$	2.8	3.5	1.3
$\alpha = 0.5$	3.5	3.7	3.4
(b) $\rho = 0.8$ , EIS = 1.25			
RRA	Benchmark (%) $\omega = 0.00$	Optimistic (%) $\omega = -0.5$	Pessimistic (%) $\omega = 0.5$
$\alpha = 2.6$	3.1	2.7	3.8
$\alpha = 0.5$	2.7	2.6	2.8

could be evidence for a high level of willingness to transfer disbursements into the future. However, if an HNWI favours smoother consumption, then unwillingness to shift consumption towards the future dominates, and optimal spending rises and falls as the outlook brightens or blackens. Somewhat surprisingly, this is true whatever the degree of risk aversion. Preferences for early or late resolution of uncertainty do not determine the direction of response. While the benchmark level of spending,  $m$ , will be sensitive to both risk aversion and the intertemporal elasticity, whether spending decreases or increases from that level in response to scenario changes depends only on whether the elasticity of intertemporal substitution is less than or greater than 1.

**CONCLUSION**

In this chapter we built and solved a model of the ideal constant disbursement rate for a foundation or trust. The specific features of management decisions for an HNWI or family office were built in. First, the ideal rate of spending depends on preferences for safety and smoothness in expected consumption, tastes which can be represented in an EZW utility framework. The EZW, or recursive, utility separates risk tolerance from intertemporal consumption preferences, so that if returns to investment are iid, then the asset allocation and consumption decisions are separable, and spending rates can be treated as contingent on a pre-set portfolio. Descriptions of



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the governance structures of family offices and foundations indicate that investment choices are not always made simultaneously with choice of spending rates, so flexibility between spending and investment planning are critical.



The ideal spending rate depends on investment returns and risk, the risk preferences of the decision-maker and their capacity for transferring consumption from the present to the future, or the elasticity of intertemporal substitution. Using a simulated returns distribution, we derived an ideal spending rate of 3% per annum when the elasticity of intertemporal substitution was  $\rho = 1$ . As tolerance for consumption transfer through time decreases and  $\rho$  increases, the disbursements fall from around 4.7% when  $\rho = 0.2$  to 2.8% when  $\rho = 5$ . However, this result is fixed for stipulated investment returns.



In order to investigate the responsiveness of disbursement policies to changes in the shape of very general returns distributions, we considered scenario analysis. The effects of optimistic and pessimistic transformations of the returns distribution were identified using the properties of stochastic dominance. Analytical results were derived for revisions to expected returns (FSD) and for revisions to diffusion of returns (SSD).

Without assuming a specific functional form for the probability density, the effects on optimal spending due to a transfer of probability mass from the lower to the upper tail (FSD), and vice versa, and the effects of mean-preserving spread (SSD) were derived, incorporating important idiosyncratic features of actual returns distributions. These experiments can represent either revisions to the beliefs of an HNWI, or a cross-section of beliefs about investment opportunities from a sample of such individuals or families.

While the optimal drawdown rate depends on both tastes for risk and the elasticity of intertemporal substitution (EIS), scenario analysis shows that whether optimal spending rates increase or decrease in response to first and second-order dominance changes in returns depends entirely on the EIS. Whenever the EIS is less than 1, income effects dominate substitution effects, and optimistic changes to returns (FSD and/or SSD) raise current spending. The reverse holds when the EIS is greater than 1, and when the EIS is unitary spending rates are immune to revision and depend only on time preference.

Foundations, trusts and family offices have been treated here as always having an interest in future regardless of time horizon. This approach is an approximation for an HNWI or family office for which survival is at least likely into the distant future, even though eventual extinction is inevitable, as discussed above. Moreover, the importance of internal family relationships have been subsumed into an assumption of unitary preferences, which is very likely an over-simplification. Amit *et al* (2008) report that single family offices (SFOs), especially later generation SFOs, commonly perform family education, counselling services and relationship management, emphasising the limitations of the framework used here.



### APPENDIX A

*Proof of Proposition 12.1* Since  $\tilde{Z}_i$  is iid,  $\tilde{Z}_i^{1-\alpha}$  is iid and

$$\begin{aligned} \mathbb{E}_0(\tilde{V}_{t-1}^{1-\alpha}) &= \mathbb{E}\left[\left(\prod_{i=0}^{t-1} \tilde{Z}_i\right)^{1-\alpha}\right] \\ &= \mathbb{E}\left[\prod_{i=0}^{t-1} \tilde{Z}_i^{1-\alpha}\right] \\ &= \prod_{i=0}^{t-1} \mathbb{E}(\tilde{Z}_i^{1-\alpha}) \\ &= \prod_{i=0}^{t-1} \varphi = \varphi^t \end{aligned}$$

□

*Proof of Proposition 12.2* Substituting Equation 12.15 into Equation 12.2 and using the expressions for consumption and the wealth constraint (Equation 12.9) gives

$$\begin{aligned} L_t &= \frac{1}{(1-\delta)(1-\alpha)} \\ &\quad \times \{(1-\delta)(m_t W_t)^{1-\rho} \\ &\quad + \delta[\mathbb{E}_t(\psi \tilde{Z}_t (1-m_t) W_t)^{1-\alpha}]^{(1-\rho)/(1-\alpha)}\}^{(1-\alpha)/(1-\rho)} - 1 \end{aligned} \tag{12.26}$$

Maximising Equation 12.26 over  $m_t$  is the same as maximising over consumption, and gives the first-order condition as a function

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of  $\psi$

$$\begin{aligned} \frac{\partial L_t}{\partial m_t} &= (1 - \rho)(1 - \delta)m_t^{-\rho}W_t^{1-\rho} \\ &\quad - (1 - \rho)\delta\psi^{1-\rho}[\mathbb{E}_t(\tilde{Z}_t^{1-\alpha})]^{(1-\rho)/(1-\alpha)}(1 - m_t)^{-\rho}W_t^{1-\rho} \\ &= 0 \end{aligned} \quad (12.27)$$

Rearranging Equation 12.27 gives

$$m_t = \left\{ 1 + \left[ \frac{\delta}{1 - \delta} \psi^{1-\rho} ([\mathbb{E}_t(\tilde{Z}_t^{1-\alpha})]^{1/(1-\alpha)})^{1-\rho} \right]^{1/\rho} \right\}^{-1} \quad (12.28)$$

If  $\psi = [(1 - \delta)m_t^{-\rho}]^{1/(1-\rho)}$ , then Equation 12.28 becomes

$$m_t = \left\{ 1 + \left[ \frac{\delta}{1 - \delta} (1 - \delta)m_t^{-\rho} ([\mathbb{E}_t(\tilde{Z}_t^{1-\alpha})]^{1/(1-\alpha)})^{1-\rho} \right]^{1/\rho} \right\}^{-1} \quad (12.29)$$

and rearranging confirms that

$$m_t = 1 - [\delta([\mathbb{E}_t(\tilde{Z}_t^{1-\alpha})]^{1/(1-\alpha)})^{1-\rho}]^{1/\rho} \quad (12.30)$$

and if  $\mathbb{E}_t(\tilde{Z}_t^{1-\alpha}) = \mathbb{E}(\tilde{Z}_t^{1-\alpha}) = \varphi$  then

$$m_t = m = 1 - [\delta(\varphi^{(1-\rho)/(1-\alpha)})]^{1/\rho}$$

□

*Proof of Proposition 12.3* Newton's Generalised Binomial Theorem states that, for any  $r \in \mathbb{R}$ , if  $|a| < 1$ , then

$$\sum_{t \geq 0} \binom{r}{t} a^t$$

converges to  $(1 + a)^r$ . This result implies, for  $a = x/y$

$$(y + x)^r = \sum_{t \geq 0} \binom{r}{t} y^{r-t} x^t = \sum_{t \geq 0} \binom{r}{t} a^t = (1 + a)^r \quad (12.31)$$



Using the aggregator function (Equation 12.2) and substituting the value function (Equation 12.15), we obtain

$$\begin{aligned} L_t &= \frac{1}{(1 - \delta)(1 - \alpha)} \\ &\quad \times \{ (1 - \delta)C_t^{1-\rho} \\ &\quad \quad + \delta[\mathbb{E}_t([(1 - \delta)m^{-\rho}]^{1/(1-\rho)}W_{t+1}^{1-\alpha}]^{(1-\rho)/(1-\alpha)})^{(1-\alpha)/(1-\rho)} \} \\ &\quad - 1 \end{aligned}$$



$$\begin{aligned}
 &= \frac{1}{(1-\delta)(1-\alpha)} \\
 &\quad \times \{ (1-\delta)C_t^{1-\rho} \\
 &\quad \quad + \delta[(1-\delta)m^{-\rho}][\mathbb{E}_t(W_{t+1})^{1-\alpha}]^{(1-\rho)/(1-\alpha)}\}^{(1-\alpha)/(1-\rho)} - 1 \\
 &= \frac{1}{(1-\delta)(1-\alpha)} \\
 &\quad \times \{ (1-\delta)m^{1-\rho}W_t^{1-\rho} \\
 &\quad \quad + \delta(1-\delta)m^{-\rho}(1-m)^{1-\rho}W_t^{1-\rho}\varphi^{(1-\rho)/(1-\alpha)}\}^{(1-\alpha)/(1-\rho)} - 1
 \end{aligned} \tag{12.32}$$

and Equation 12.32 will be the convergent sum of the generalised binomial expansion above if

$$\begin{aligned}
 a &= \frac{x}{y} = \frac{(1-\delta)m^{1-\rho}W_t^{1-\rho}}{\delta(1-\delta)m^{-\rho}(1-m)^{1-\rho}W_t^{1-\rho}\varphi^{(1-\rho)/(1-\alpha)}} \\
 &= \frac{m}{1-m} < 1
 \end{aligned}$$

and  $r = (1-\alpha)/(1-\rho)$  is a real number. (Note that this condition restricts  $\rho \neq 1$ .)

The generalised binomial expansion in this case is

$$\sum_{t \geq 0} \binom{r}{t} a^t = \sum_{t \geq 0} \frac{r(r-1)(r-2) \cdots (r-t+1)}{t!} \left( \frac{m}{1-m} \right)^t \tag{12.33}$$

In each period, the first ratio in Equation 12.33 grows by a factor  $(r-t+1)/t$ , which in the limit as  $t \rightarrow \infty$  goes to  $-1$ , ~~so as long as~~  $m/(1-m) < 1$  gives convergence to a bounded value for the summation in Equation 12.33. □

Changes to sentence OK?

*Proof of Proposition 12.4* Note that if  $Z_i^\Delta$  first-order stochastic dominates  $Z_i$ , then  $F_\Delta(Z) \leq F(Z)$ , where  $F_\Delta(Z)$  and  $F(Z)$  are the respective distribution functions. We denote expectations with respect to them by  $\mathbb{E}_\Delta(\cdot)$  and  $\mathbb{E}(\cdot)$ . FSD implies that  $\mathbb{E}_\Delta[G(Z)] \geq \mathbb{E}[G(Z)]$  for  $G(\cdot)$  any increasing function (see Huang and Litzenberger (1988) for the proof). If  $G(Z) = Z^{1-\alpha}$ ,  $0 < \alpha < 1$ , then  $G(Z)$  is increasing in  $Z$ , and hence, under  $F_\Delta(Z)$ ,  $\varphi$  is increased. If  $\alpha > 1$ , then  $G(Z)$  is a decreasing function in  $Z$  and under  $F_\Delta(Z)$  the reverse happens:  $\varphi$  decreases. □

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Table 12.7 Data sources for each returns series

	Asset class	Data	Mnemonic/source	Weight (%)
1	UK equity	FTSE All-Share	FTALLSH(RI)	32.2
2	Global equity	MSCI World ex UK	MSWFUK\$(RI) ~ U\$, to BPN using BBGBPSP(ER)	32.0
3	Overseas equity	MSCI Emerging Markets	MSEMKF\$(RI) ~ U\$, to BPN using BBGBPSP(ER)	5.0
4	UK gilts	FTA	FTBGTTF(RI) ~ £	2.8
5	Property	IPD	UKIPDRI.F	7.5
6	Hedge funds	CSFB/Tremont hedge fund	CSTHEDG ~ £	3.6
7	Private equity	UK Trusts Private equity	ITVCAPT(RI) ~ £	11.5
8	Cash	Three-month CD rate	Bank of England	5.4
9	Inflation	Average of CPI and earnings	CPI: UKCPHARMF Wages: UKWAGES.E	

*Proof of Proposition 12.6* Note that if  $Z_i^\omega$  second-order stochastic dominates  $Z_i$ , then

$$\int_0^Z F_w(s) ds \leq \int_0^Z F(s) ds \quad \text{for all } Z \in (0, \infty)$$

where  $F_w(Z)$  and  $F(Z)$  are the respective distribution functions. We denote expectations with respect to them by  $\mathbb{E}_\omega(\cdot)$  and  $\mathbb{E}(\cdot)$ . SSD implies that  $\mathbb{E}_\omega[G(Z)] \geq \mathbb{E}[G(Z)]$  for  $G(\cdot)$  any increasing, concave function (see Huang and Litzenberger (1988) for proof). If  $G(Z) = Z^{1-\alpha}$ ,  $0 < \alpha < 1$ , then  $G(Z)$  is increasing and concave in  $Z$ , and hence, under  $F_\omega(Z)$ ,  $\varphi$  is increased. If  $\alpha > 1$ , we have  $G(Z)$  a decreasing and convex function in  $Z$ , and under  $F_\omega(Z)$  the reverse happens:  $\varphi$  decreases. □

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## APPENDIX B

Table 12.7 lists data sources for each returns series. A consistent series of returns to hedge funds was not available prior to January 1994, so from January 1990 to December 1993 the allocations to UK, global, emerging and private equity were each increased by 0.9% and hedge funds set to zero. Total portfolio return is the weighted sum of log changes in each returns index and the cash rate (expressed on a monthly basis) minus the log change in the inflation rate

$$\dot{p} = \frac{1}{2} \left[ \ln \left( \frac{\text{CPI}_t}{\text{CPI}_{t-1}} \right) + \ln \left( \frac{\text{earnings}_t}{\text{earnings}_{t-1}} \right) \right]$$

All the series are from DataStream, apart from the cash rate, which is from the Bank of England database.

- 1 There is surprisingly little quantitative research published in the area of family trusts, endowments and foundations. Studies of family offices include Amit *et al* (2008) and references therein, and, from a sociological perspective, Gilding (2004). The literature on the optimal spending and investment plans for university endowments originates with Tobin (1974), Litvack *et al* (1974), Nichols (1974), but also features in Merton (1990) and more recently in Dybvig (1999), Woglom (2003) and Merton (2003). Empirical studies of the structure and investment performance of endowments include Brown (1999), Lerner *et al* (2005), Brown *et al* (2010), Dimmock (2007) and Acharya and Dimson (2007).
- 2 See <http://www.charity-commission.gov.uk/showcharity/registerofcharities/registerhome page.aspx>.
- 3 See <http://www.mmdwebaccess.com/SPContent/Endowment>.
- 4 Returns to hedge funds, real estate and private equity can be serially correlated. See Satchell *et al* (2012) for analysis of spending in the EZW model when returns are not iid.
- 5 Merton also addresses the problem for a finite horizon.

Sentence added here, as Risk Books style is to cite all figures and tables in the text. OK? Also, change \$ to US\$ in table (the latter is house style for dollars)? Does the ~ denote 'approximates to' here? Please clarify.

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- 6 The analysis also links to the large asset pricing and life-cycle literature using the EZW utility function (see, for example, Campbell 1993; Tallarini 2000; Campbell and Viceira 2002; Campbell and Vuolteenaho 2004; Bansal and Yaron 2004; ~~Gomes and Michaelides 2005~~; Hansen *et al* 2005). Many of these studies either fix the EIS at 1 (or at some calibrated value), or use linearisations of the problem that constrain the EIS to be close to 1, and most assume conditional or unconditional lognormality in the returns process.
- 7 In a time-additive utility model  $\delta$  would simplify to the rate of time preference, but time preference is generally endogenous in non-expected utility settings. See Backus *et al* (2005, pp. 321–90) for a general discussion of recursive preferences.
- 8 Constant proportion portfolio insurance is the optimal investment strategy of an investor or endowment protecting a fixed minimum level of consumption, a result implicit in Merton (1971) and explicit in Kingston (1989).
- 9 ONS data covers the decade 1998 to 2008.
- 10 Bhamra and Uppal (2006) solve a related problem for a finite horizon with non-constant investment opportunity set.
- ~~11 Professor James Sefton suggested this approach.~~
- 12 Giovannini and Weil (1989) and Campbell (1993) derive and discuss special cases.
- ~~13 Constant drawdown under iid returns and an infinite horizon is a well-established result under CRRA utility. See, for example, Ramsey (1928) and Phelps and Pollack (1968).~~
- 14 This result has been widely employed in the asset pricing literature to help match up high equity premiums with relatively smooth consumption paths (see, for example, Campbell 1993).
- 15 Smith (1996) derives the feasibility and transversality condition for a related aggregator function in continuous time, but the model here is different in significant ways and Smith's result does not transfer directly.
- 16 Bhamra and Uppal (2006) set out the implicit portfolio optimality condition and the explicit optimal portfolio weights for simple examples of constant and stochastic investment opportunity sets. Explicit analytical results for portfolio choice under stochastic investment are limited to a two-state process for the risky asset.
- 17 The Wellcome Trust was founded by businessman and philanthropist Sir Henry Wellcome and supports biomedical and medical humanities research.
- 18 A consistent series of returns to hedge funds are not available prior to January 1994, so from January 1990 to December 1993 the allocations to UK, global, emerging and private equity were each increased by 0.9%, and hedge funds set to zero. Appendix B lists data sources for each returns series.
- 19 Optimal portfolio weights will satisfy a vector of moment conditions in the risk-scaled portfolio return and returns to individual assets. When returns are iid and the disbursement rate is constant, the conditions are (Bhamra and Uppal 2006, Equation 17)

$$\mathbb{E}[m^{\rho(1-\alpha)/(1-\rho)} \tilde{Z}_t^{-\alpha}(z_{it})] = \mathbb{E}[\tilde{Z}_t^{-\alpha}(z_{it})] = 0$$

This system of moment conditions and the portfolio returns data described above can be used to estimate  $\alpha$  by generalised method of moments. Estimation results are available from the authors on request.

- 20 Chen *et al* (2007) estimate parameters of the EZW utility function at the aggregate level and find that the EIS is greater than 1 and risk aversion is in the range 17–60. See also Gruber (2006) for a high estimate for the EIS. Other authors find lower values for both parameters (see, for example, Vissing-Jorgensen and Attanasio 2003; Vissing-Jorgensen 2002; Epstein and Zin 1991).
- 21 The slightly jagged shape of the surface is caused by the bootstrap process: a different set of random draws is made at each combination of  $\rho$  and  $\Delta$ . Edges of the surface are not smooth because the feasibility and boundary conditions are not met for some extreme values of  $\rho$  and  $\alpha$ .

REFERENCES

- Acharya, S., and E. Dimson, 2007, *Endowment Asset Management* (Oxford University Press).
- Albertsen, K., 1995, "The Extinction of Families", *International Statistical Review* 63(2), pp. 234–9.
- Amit, R., H. Liechtenstein, M. J. Prats, T. Millay and M. P. Pendleton, 2008, "Single Family Offices: Private Wealth Management in the Family Context", Wharton Global Family Alliance, University of Pennsylvania.
- Bakus, D. K., B. R. Routledge and S. E. Zin, 2005, "Exotic Preferences for Macroeconomists", *NBER Macroeconomics Annual 2004*, pp. 319–414 (Boston, MA: MIT Press).
- Bansal, R., and A. Yaron, 2004, "Risks for the Long Run: A Potential Resolution of Asset Pricing Puzzles", *Journal of Finance* 59, pp. 1481–509.
- Becker, G. S., 1974, "A Theory of Social Interactions", *Journal of Political Economy* 82(6), pp. 1063–93.
- Becker, G. S., 1981, "Altruism in the Family and Selfishness in the Market Place", *Economica* 48(189), pp. 1–15.
- Becker, G. S., and R. J. Barro, 1988, "A Reformulation of the Economic Theory of Fertility", *Quarterly Journal of Economics* 103(1), pp. 1–25.
- Becker, G. S., and N. Tomes, 1986, "Human Capital and the Rise and Fall of Families", *Journal of Labor Economics* 4(3, Part 2), pp. S1–39.
- Bhamra, H. S., and R. Uppal, 2006, "The Effect of Risk Aversion and Intertemporal Substitution on Dynamic Consumption and Portfolio Rules with Recursive Utility", *Journal of Economic Dynamics and Control* 30, p. 967–99.
- Brown, K. C., L. Garlappi and C. Tiu, 2010, "Asset Allocation and Portfolio Performance: Evidence from University Endowment Funds", *Journal of Financial Markets* 13(2), pp. 268–94.
- Brown, W. O., 1999, "University Endowments: Investment Strategies and Performance", *Financial Practice and Education*, Fall/Winter, pp. 61–9.
- Campbell, J. Y., 1993, "Intertemporal Asset Pricing without Consumption Data", *American Economic Review* 83, pp. 487–512.
- Campbell, J. Y., and J. Viceira, 2002, *Strategic Asset Allocation* (Oxford University Press).
- Campbell, J. Y., and T. Vuolteenaho, 2005, "Good Beta, Bad Beta", *American Economic Review* 94, pp. 1249–75.
- Canada Revenue Agency, 2007, "Completing the Registered Charity Information Return", Government of Canada, URL: <http://www.cra-arc.gc.ca/E/pub/tg/t4033a/t4033a-06e.pdf>.
- CapGemini and RBC Wealth Management, 2012, "The 16th Annual World Wealth Report 2012".
- Chen, X., J. Favilukis and S. C. Ludvigson, 2007, "An Estimation of Economic Models With Recursive Preferences", Working Paper, New York University.
- Cigno, A., 1993, "Intergenerational Transfers without Altruism: Family, Market and State", *European Journal of Political Economy* 9(4), pp. 505–18.
- Cigno, A., 2006, "A Constitutional Theory of the Family", *Journal of Population Economics* 19, pp. 259–83.
- Cigno, A., 2007, *Children and Pensions* (Cambridge, MA: MIT Press).

QUANTITATIVE APPROACHES TO HIGH NET WORTH INVESTMENT

---

**Dimmock, S. G.**, 2007, "Portfolio Choice, Background Risk and University Endowment Funds", Working Paper, Department of Finance, Michigan State University.

**Dybvig, P. H.**, 1995, "Dusenberry's Ratcheting of Consumption: Optimal Dynamic Consumption and Investment Given Intolerance for any Decline in Standard of Living", *Review of Economic Studies* 62(2), pp. 287–313.

**Dybvig, P. H.**, 1999, "Using Asset Allocation to Protect Spending", *Financial Analysts Journal* 55, pp. 49–60.

**Epstein, L. G., and S. E. Zin**, 1989, "Substitution, Risk Aversion and the Temporal Behaviour of Consumption and Asset Returns: A Theoretical Framework", *Econometrica* 57, pp. 937–69.

**Epstein, L. G., and S. E. Zin**, 1991, "Substitution, Risk Aversion and the Temporal Behavior of Consumption and Asset Returns: An Empirical Analysis", *Journal of Political Economy* 99, pp. 263–86.

**Gilding, M.**, 2005, "Families and Fortunes: Accumulation, Management Succession and Inheritance in Wealthy Families", *Journal of Sociology* 41, pp. 29–45.

**Gomes, F., and A. Michaelides**, 2005, "Optimal Life-Cycle Asset Allocation: Understanding the Empirical Evidence", *Journal of Finance* 60, pp. 869–904.

**Giovannini, A., and P. Jorion**, 1993, "Time Series Tests of a Non-Expected-Utility Model of Asset Pricing", *European Economic Review* 37, pp. 1083–100.

**Giovannini, A., and P. Weil**, 1989, "Risk Aversion and Intertemporal Substitution in the Capital Asset Pricing Model", NBER Working Paper 2824.

**Graham, R. L., D. E. Knuth and O. Patahnik**, 1994, *Concrete Mathematics: A Foundation for Computer Science* (Reading MA: Addison-Wesley).

**Gruber, J.**, 2006, "A Tax-Based Estimate of the Elasticity of Intertemporal Substitution", NBER Working Paper 11945.

**Hansen, L. P., J. Heaton and N. Li**, 2005, "Consumption Strikes Back? Measuring Long-Run Risk", Working Paper, University of Chicago.

**Huang, C.-F., and R. H. Litzenberger**, 1988, *Foundations for Financial Economics* (New York: North-Holland).

**Kingston, G.**, 1989, "Theoretical Foundations of Constant Proportion Portfolio Insurance", *Economics Letters* 29, pp. 345–47.

**Korn, R., and E. Korn**, 2001, *Option Pricing and Portfolio Optimization: Modern Methods of Financial Mathematics*, Graduate Studies in Mathematics, Volume 31 (Providence, RI: American Mathematical Society).

**Kreps, D. M., and E. L. Porteus**, 1978, "Temporal Resolution of Uncertainty and Dynamic Choice Theory", *Econometrica* 46, pp. 185–200.

**Kreps, D. M., and E. L. Porteus**, 1979, "Dynamic Choice Theory and Dynamic Programming", *Econometrica* 47, pp. 91–100.

**Lerner, J., A. Schoar and W. Wong**, 2005, "Smart Institutions, Foolish Choices? The Limited Partner Performance Puzzle", NBER Working Paper 11136.

**Litvack, J. M., B. G. Malkiel and R. E. Quandt**, 1974, "A Plan for the Definition of Endowment Income", *American Economic Review* 64, pp. 433–7.

**Manser, M., and M. Brown**, 1980, "Marriage and Household Decision-Making: A Bargaining Analysis", *International Economic Review* 21(1), pp. 31–44.

**McElroy, M. B., and M. J. Horney**, 1981, "Nash-Bargained Household Decisions: Toward a Generalization of the Theory of Demand", *International Economic Review* 22(2), pp. 333–49.

MODELLING SUSTAINABLE SPENDING PLANS FOR FAMILY OFFICES, FOUNDATIONS AND TRUSTS

**Merton, R. C.**, 1969, "Lifetime Portfolio Selection under Uncertainty: The Continuous-Time Case", *Review of Economics and Statistics* 51, pp. 247–57.

**Merton, R. C.**, 1971, "Optimum Consumption and Portfolio Rules in a Continuous-Time Model", *Journal of Economic Theory* 3, pp. 373–413.

**Merton, R. C.**, 1990, *Continuous Time Finance* (Malden: Blackwell Publishing Ltd).

**Merton, R. C.**, 2003, "Thoughts on the Future: Theory and Practice in Investment Management", *Financial Analysts Journal* 59, pp. 17–23.

**Martiros, S., and T. Millay**, 2006, "A Framework for Understanding Family Office Trends", White Paper, CCC Alliance, Boston, MA.

**Nichols, D. A.**, 1974, "The Investment Income Formula of the American Economic Association", *American Economic Review* 64, pp. 420–6.

**Office for National Statistics**, 2011, "Birth Statistics 2008", Series FM1 no. 37.

**Satchell, S. E., and S. Thorp**, 2011, "Uncertain Survival and Time Discounting: Intertemporal Consumption Plans for Family Trusts", *Journal of Population Economics* 24(1), pp. 239–66.

**Satchell, S. E., S. Thorp and O. Williams**, 2012, "Estimating Consumption Plans for Recursive Utility by Maximum Entropy Methods", QFRC Research Paper 300, University of Technology, Sydney.

**Smith, W. T.**, 1996, "Feasibility and Transversality Conditions for Models of Portfolio Choice with Non-Expected Utility in Continuous Time", *Economics Letters* 53, pp. 123–31.

**Tallarini, T. D.**, 2000, "Risk-Sensitive Real Business Cycle Models", *Journal of Monetary Economics* 45, pp. 507–32.

**Tobin, J.**, 1974, "What Is Permanent Endowment Income?", *American Economic Review* 64, pp. 427–32.

**Vissing-Jorgensen, A., and O. P. Attanasio**, 2003, "Stock Market Participation, Intertemporal Substitution and Risk Aversion", *American Economic Review Papers and Proceedings* 93, pp. 383–91.

**Vissing-Jorgensen, A.**, 2002, "Limited Asset Market Participation and the Elasticity of Intertemporal Substitution", *Journal of Political Economy* 110, pp. 825–53.

**Weil, P.**, 1990, "Unexpected Utility in Macroeconomics", *Quarterly Journal of Economics* 105, pp. 29–42.

**Wellcome Trust**, 2005, "Annual Report and Financial Statements 2005", The Wellcome Trust, London.

**Woglom, G.**, 2003, "Endowment Spending Rates, Intergenerational Equity and the Sources of Capital Gains", *Economics of Education Review* 22, pp. 591–601.

**Xu, Z.**, 2007, "A Survey on Intra-Household Models and Evidence", American Institutes for Research Working Paper, Washington, DC.