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Mechanical Model to Analyse Multilayer Geosynthetic Reinforced Granular Layer in Column Supported Embankments

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Abstract
The objective of this paper is to develop a mechanical model to predict the behaviour of a multilayer geosynthetic reinforced granular fill soft soil system improved with controlled modulus columns beneath the embankment. Deformation of geosynthetics embedded granular layer due to bending and shear is considered in this study. Therefore, geosynthetic reinforced granular fill has been idealised as a reinforced Timoshenko beam while the columns and the soft soil have been idealised as a layer of linear springs with varied stiffness. Plane strain conditions are considered for the loading and reinforced foundation soil system. Tension developed in the geosynthetics, rotation and settlements of the improved soft ground are predicted using the proposed model. This study shows the effects of multilayer geosynthetics on the settlement response of the granular layer. A notable reduction of the settlement has been observed as a result of the using multilayer weaker geosynthetic reinforcement system when compare to one stronger geosynthetics layer. It is also observed that the top reinforcement layer is subjected to maximum mobilised tension at the column edge whereas bottom reinforcement layer is more effective in controlling the deflection in the middle of two columns.

Keywords: Timoshenko beam, Geosynthetics, Soft soil, Column supported embankments

1 Introduction

Soft soils underneath the embankments are prone to excessive settlement due to the low stiffness, low bearing capacity as well as high shrink-swell potential. The controlled modulus column (CMC) is one of the ground improvement techniques to meet the higher demand for the transport infrastructure particularly near waterways comprising weak soil layers. Introducing geosynthetic reinforcement (GR) within the granular fill materials which is known as load transfer platform (LTP) results in a more

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efficient transfer of load to the columns in the form of an arching mechanism in column-supported embankments. However, the behaviour becomes much more complicated with inclusion of GR. The interaction between the reinforced granular layer, the columns, and the soft soil below the granular layer change their actual behaviour considerably. In recent years many researchers have studied the load-settlement response of such reinforced granular layer (Madhav and Poorooshasb, 1989; Shukla and Chandra, 1995; Yin, 2000).

Most of the models reported in the literature are developed for single-layer reinforced systems without considering columns. Nogami and Yong (2003) investigated the response of a multilayer geosynthetic-reinforced geomedium subjected to structural loading. The governing differential equations for the geomedium were obtained and solved by an iterative finite difference method. However, no significant difference in the response of the reinforced soil was observed for low load intensity. Chandra et al. (2005) proposed a mechanical model to predict the behaviour of a multilayer geosynthetics within the granular fill on soft soil foundation by idealising the granular fill and the soft soil as a Pasternak shear layer and a layer of non-linear springs, respectively. Effect of multilayer geosynthetics was notably observed. However the governing differential equations were solved by an iterative finite difference method. Therefore, currently no closed form solution of the load-settlement response of the reinforced granular layer-soil-column system is available in the literature which considers both the bending and the shear deformation of the multilayer geosynthetic-reinforced granular layer.

In this paper, a mechanical model is proposed for a multilayer geosynthetic reinforced granular layer on column stabilised soft soil, which incorporates the deformation of the granular fill due to both bending and shear. Using the proposed model, the load transfer mechanism in terms of deflection and rotation of the granular layer and tension mobilised in the geosynthetics with one stronger layer of geosynthetics and with two layers of weaker geosynthetics are compared.

2 Analytical Model

A granular layer with multilayer of geosynthetics on CMC improved soft soil system at the base of the embankment is shown in Figure 1a. The behaviour of such a system may be idealised in terms of the proposed mechanical model shown in Figure 1b. In this model, the geosynthetic reinforced granular fill has been idealised as a reinforced Timoshenko beam. The CMCs and the soft soil have been idealised as a layer of linear springs with different stiffness. Two geosynthetic layers are considered in the model. It is assumed that there will be no slippage between the geosynthetic reinforcement layers and granular fill materials. The assumed deformed shape of the LTP and the co-ordinate axes for a unit cell are shown in Figure 2a. The deformation of the column is assumed to remain unchanged over its width. The CMC and the soft soil are loaded with different distributed loading intensities of p_s and p_c, respectively due to soil arching. Since in the field, discrete columns are placed in a square or triangular pattern, the equivalent plane-strain material stiffness is determined by the relationship suggested by Tan et al. (2008) based on matching the column-soil composite stiffness as: $k_{c,pl}a_{r,pl} = k_{c,ax}a_{r,ax} + k_{s,ax}\left(a_{r,pl} - a_{r,ax}\right)$. Subscripts $''pl''$ and $''ax''$ denote plane-strain and axisymmetric conditions, respectively, while a_r is the area replacement ratio. Deformation of the CMC reinforced composite ground can be expressed as:

$$w = \begin{cases} w_{cz} & \text{when } \; 0 \le \xi \le d/2 \\ w_{cz} + w_{sz} & \text{when } \; 0 \le x \le s'/2 \end{cases} \tag{1}$$

where s' is the clear spacing between the CMCs, w_{sz} is the displacement of the LTP on soft soil region at a horizontal distance x, and w_{cz} is the displacement of the LTP over column region at a horizontal distance ξ.

To obtain the differential equation, governing the deflected shape of a transversely loaded LTP beam resting on elastic foundation, the LTP is divided into infinitesimal beam elements in the horizontal direction having equal thickness Δx within the soft soil region and $\Delta\xi$ within the column. Typical LTP elements over the soft soil and the columns and the stresses acting on LTP are shown in Figure 2b. The equilibrium of the vertical forces and moments of a typical element of LTP for $0 \le x \le s'/2$ under plain strain conditions yields the following equations:

$$\left.\begin{aligned} D_b \frac{d^2\theta_{sz}}{dx^2} + C_b \frac{dw_{sz}}{dx} - C_b\theta_{sz} = 0 \\ C_b \frac{d\theta_{sz}}{dx} - C_b \frac{d^2 w_{sz}}{dx^2} + q_s = p_s \end{aligned}\right\} \tag{2}$$

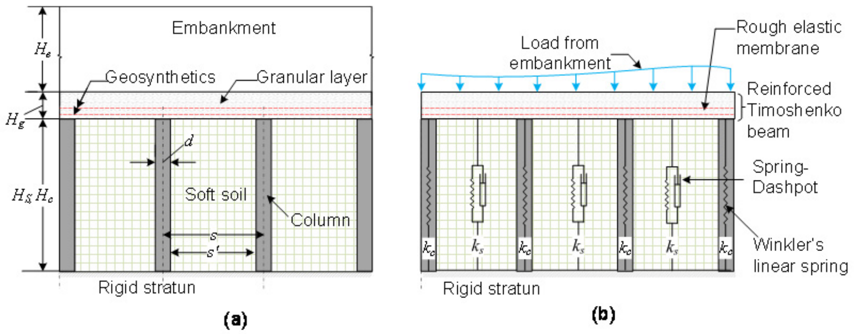

Figure 1: Illustration of (a) embankment on CMC-improved soft soil and (b) proposed foundation model

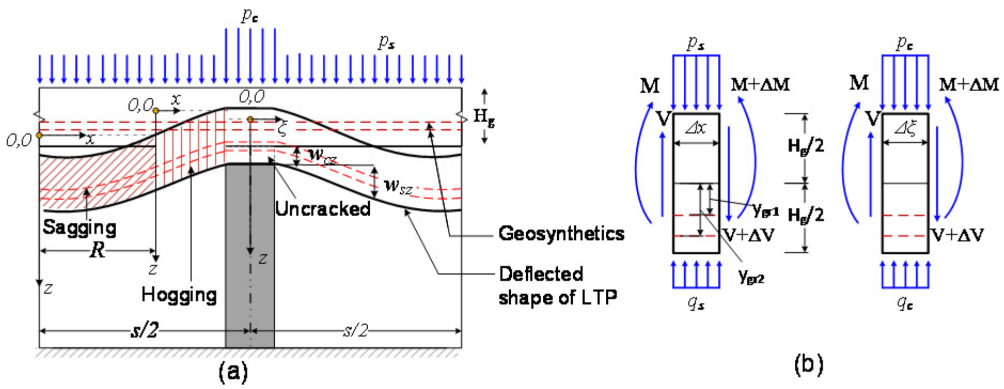

Figure 2: (a) Multilayer geosynthetic-reinforced granular fill soft soil column system and (b) stress application in different infinitesimal LTP section

Similar set of equations can be governed within the CMC region ($0 \le \xi \le d/2$) replacing θ_{sz}, w_{sz}, p_s, and q_s by θ_{cz}, w_{cz}, p_c, and q_c, respectively. In Eq. (2), w is the transverse deformation of the centroid axis of the beam, θ is the rotation angle of the cross section of beam about its neutral axis, D_b and C_b are the un-cracked bending rigidity and shear rigidity of the LTP with geosynthetics (Yin, 2000), respectively. Considering linear stress-displacement relation proposed by Winkler, 1867 and consolidation effect of the soft soil suggested by Deb et al., 2007, normal pressure at the LTP-soil (q_s) and LTP-column interfaces (q_c) can be written as:

$$q_s = \frac{k_s w_{sz}}{U} \tag{3a}$$

$$q_c = k_{c,pl} w_{cz} \tag{3b}$$

where U is the degree of consolidation of the soft soil. Although the time dependent behaviour of the soft soil is considered in the present study, it should be noted that soil cementation and creep can significantly affect the behaviour of soft soil (Azari et al., 2016; Le et al., 2015; Nguyen et al., 2014).

Combining Eq. (2) and Eq. (3) and assuming Fourier cosine series which considers the symmetric embankment loading on the LTP, the ordinary fourth-order differential equations of the deflection for the LTP for $0 \leq x \leq s'/2$ and $0 \leq \xi \leq d/2$ can be written as:

$$D_s \frac{d^4 w_1}{dx^4} - \frac{D_s k_s}{C_s U}\frac{d^2 w_1}{dx^2} + \left(\frac{k_s}{U}\right) w_1 = S_0 + \sum^{m=\infty} S_m \left[1 + \frac{D_s}{C_s}\left(\frac{2m\pi}{s'}\right)^2\right] \cos\left(\frac{2m\pi x}{s'}\right) \tag{4}$$

$$D_h \frac{d^4 w_2}{dx^4} - \frac{D_h k_s}{C_h U}\frac{d^2 w_2}{dx^2} + \left(\frac{k_s}{U}\right) w_2 = S_0 + \sum^{m=\infty} S_m \left[1 + \frac{D_h}{C_h}\left(\frac{2m\pi}{s'}\right)^2\right] \cos\left(\frac{2m\pi x}{s'}\right) \tag{5}$$

$$D_b \frac{d^4 w_{cz}}{d\xi^4} - \frac{D_b k_{c,pl}}{C_b}\frac{d^2 w_{cz}}{d\xi^2} + k_{c,pl} w_{cz} = C_0 + \sum^{m=\infty} C_m \left[1 + \frac{D_b}{C_b}\left(\frac{2m\pi}{d}\right)^2\right] \cos\left(\frac{2m\pi \xi}{d}\right) \tag{6}$$

where D_s and C_s are the bending and shear rigidity of LTP, respectively when the LTP is subjected to sagging moment; D_h and C_h are the bending and shear rigidity of LTP, respectively when the LTP is subjected to hogging moment (tension cracks appear above the neutral axis); and D_b and C_b are the bending and shear rigidity of LTP, respectively when the LTP is considered as an un-cracked on top of the column.

2.1 Analytical Solution

The solution of the fourth order nonhomogeneous differential equation Eq. (4) governing the deflected shape of a transversely loaded LTP beam resting on foundation soil which is subjected to hogging moment can be expressed as:

$$w_1 = e^{\vartheta x}(s_1 \cos\delta x + s_2 \sin\delta x) + e^{-\vartheta x}(s_3 \cos\delta x + s_4 \sin\delta x) + \frac{S_0}{k_s} + \sum_{m=1}^{m=\infty} s_s \cos\left(\frac{2m\pi x}{s'}\right) \tag{7}$$

The solution for LTP beam when it is subjected to sagging moment (Eq. (5)) is written below.

$$w_2 = e^{\alpha x}(t_1 \cos\beta x + t_2 \sin\beta x) + e^{-\alpha x}(t_3 \cos\beta x + t_4 \sin\beta x) + \frac{S_0}{k_s} + \sum_{m=1}^{m=\infty} s_h \cos\left(\frac{2m\pi x}{s'}\right) \tag{8}$$

Correspondingly, for $0 \leq \xi \leq d/2$, the analytical solution of Eq. (6) can be expressed as:

$$w_{cz} = d_1 e^{r_1 \xi} + d_2 e^{r_2 \xi} + d_3 e^{-r_1 \xi} + d_4 e^{-r_2 \xi} + \frac{C_0}{k_{c,pl}} + \sum_{m=1}^{m=\infty} c_m \cos\left(\frac{2m\pi \xi}{d}\right) \tag{9}$$

where

$$\vartheta = \left(\sqrt{\frac{k_s}{4D_s U}} + \frac{k_s}{4C_s U}\right)^{1/2};\quad \alpha = \left(\sqrt{\frac{k_s}{4D_h U}} + \frac{k_s}{4C_h U}\right)^{1/2};\quad r_1 = \frac{k_{c,pl}}{2C_b} + \sqrt{\left(\frac{k_{c,pl}}{2C_b}\right)^2 - \frac{k_{c,pl}}{D_b}}$$
$$\delta = \left(\sqrt{\frac{k_s}{4D_s U}} - \frac{k_s}{4C_s U}\right)^{1/2};\quad \beta = \left(\sqrt{\frac{k_s}{4D_h U}} - \frac{k_s}{4C_h U}\right)^{1/2};\quad r_2 = \frac{k_{c,pl}}{2C_b} - \sqrt{\left(\frac{k_{c,pl}}{2C_b}\right)^2 - \frac{k_{c,pl}}{D_b}} \tag{10}$$

$$S_0 = \frac{1}{s'}\int_{-\frac{s'}{2}}^{\frac{s'}{2}} p_s\, dx;\quad S_m = \frac{1}{s'}\int_{-\frac{s'}{2}}^{\frac{s'}{2}} p_s \cos\left(\frac{2m\pi x}{s'}\right) dx;\quad s_s = \frac{S_m\left[1+\left(\frac{2m\pi}{s'}\right)^2 \frac{D_s}{C_s}\right]}{D_s\left(\frac{2m\pi}{s'}\right)^2\left[\left(\frac{2m\pi}{s'}\right)^2 + \frac{k_s}{C_s U}\right] + \frac{k_s}{U}}$$
$$C_0 = \frac{1}{d}\int_{-\frac{d}{2}}^{\frac{d}{2}} p_c\, d\xi;\quad C_m = \frac{1}{d}\int_{-\frac{d}{2}}^{\frac{d}{2}} p_c \cos\left(\frac{2m\pi\xi}{d}\right) d\xi;\ \text{and}\quad s_h = \frac{S_m\left[1+\left(\frac{2m\pi}{s'}\right)^2 \frac{D_h}{C_h}\right]}{D_h\left(\frac{2m\pi}{s'}\right)^2\left[\left(\frac{2m\pi}{s'}\right)^2 + \frac{k_s}{C_h U}\right] + \frac{k_s}{U}} \tag{11}$$
$$c_m = \frac{C_m\left[1+\left(\frac{2m\pi}{d}\right)^2 \frac{D_b}{C_b}\right]}{D_b\left(\frac{2m\pi}{d}\right)^2\left[\left(\frac{2m\pi}{d}\right)^2 + \frac{k_{c,pl}}{C_b U}\right] + k_{c,pl}}$$

In practice, the stiffness of the LTP beam is greater than the stiffness of the soft soil and less than the stiffness of the CMC. Hence, the given solutions Eq. (7) to Eq. (9) are valid for $k_s < 4{C_{h,s}}^2U/D_{h,s}$ (for soft soil region) and $k_c > 4{C_b}^2/D_b$(for column region). It can be noted that s_1 to s_4, t_1 to t_4, and r_1 and r_2 are the constants of integration which can be determined by applying the boundary and continuity conditions. Once deflection (w) of the LTP beam is obtained, rotational angle (θ), shear force (V), bending moment (M) of the LTP, and tension (T) in the geosynthetics can be obtained using the following equations:

$$\theta = \frac{D}{C}\left(\frac{d^3w}{dx^3} + \frac{1}{C}\frac{dp}{dx} - \frac{1}{C}\frac{dq}{dx}\right) + \frac{dw}{dx};\ M = -D\frac{d\theta}{dx};\ V = C\left(\frac{dw}{dx} - \theta\right);\ T = -S_{gr}(y_n - y_r)\frac{d\theta}{dx} \tag{12}$$

where S_{gr} is the tensile stiffness of GR, y_n and y_r are the location of neutral axis and GR from the centreline of the LTP, respectively.

2.2 Boundary and Continuity Conditions

Due to symmetry, at the outside boundary of the unit cell, the shear stresses and the slope of the deflection are assumed to be zero. It is assumed that there will be no rotation and transverse deformation of the LTP at the column location. In addition, continuity conditions are satisfied at unknown distance R. For column region, at the column support, shear stress is equal to the reaction from the column support. Since the deformation of the column is constant through the width, the slope of the deflection will be zero. Boundary conditions for $0 \leq x \leq s'/2$ and $0 \leq \xi \leq d/2$ are as follows:

Within soft soil region:

$$\text{at } x = 0 \rightarrow \begin{cases} \dfrac{dw_1}{dx} = 0 \\ \tau_1 = 0 \end{cases},\ x = \frac{s'}{2} \rightarrow \begin{cases} \theta_2 = 0 \\ w_2 = 0 \end{cases} \text{and } x = R \rightarrow \begin{cases} w_1 = w_2 \\ M_1 = 0 \\ M_2 = 0 \\ V_1 = V_2 \\ \theta_1 = \theta_2 \end{cases} \tag{13a}$$

Within column region:

$$\text{at } \xi = 0 \rightarrow \frac{dw_c}{d\xi} = 0 \text{ and } \xi = \frac{d}{2} \rightarrow \tau_c = k_{c,pl}w_c \tag{13b}$$

Following the boundary and continuity conditions as stated in Eq. (13), and using Eq. (12) (for θ_s, θ_c, τ_s, and τ_c) constants of integrations and unknown R can be determined. Due to the page limitation, calculation steps are not provided in details.

3 Results and Discussions

3.1 General

Parametric study is conducted to predict the vertical deflection of LTP, bending, and rotation in the LTP and tension developed in the geosythetic reinforcement (GR) using the proposed analytical model. This study considers the typical parameters for CMC supported geosynthetic reinforced embankments which are presented in Table 1. Modulus of subgrade reaction for the soft soil (k_s) and the CMC (k_c) are estimated using the equations suggested by (Selvadurai, 1979). The stresses acting on top of the LTP layer is determined by considering the soil arching in the embankment in plane-strain condition using the expressions proposed by Low et al. (1995). In this study, two layers of geosynthetics are used for multilayer case.

Material	Parameters
Embankment fill	$H_e = 2m$ and $5m$, $\gamma_e = 18.3kN/m^3$, $\varphi_e = 30°$
Granular fill	$H_g = 0.85m$, $E_g = 35MPa$, $\nu_g = 0.3$
Soft soil	$H_s = 5m$, $E_s = 1MPa$, $\nu_s = 0.3$
CMC	$H_c = 5m$, $d = 0.45m$, $s = 2.5m$ (square arrangement) , $E_c = 10000MPa$, $\nu_c = 0.25$
Geogrids (two layers but weaker)	$S_{gr} = 1000kN/m$, $S_{gr} /T_y=10$, $\nu_{gr} = 0.3$, $y_{gr1} = 0.05m$ (location of top layer below the C.L. of LTP) $y_{gr2} = 0.30m$ (location of bottom layer below the C.L. of LTP)
Geogrids (one layer but stronger)	$S_{gr} = 2000kN/m$, $S_{gr} /T_y=10$, $\nu_{gr} = 0.3$, $y_{gr} = 0.175m$ (location of one GR layer below the C.L. of LTP)

Table 1: Material properties

Note: H: height/depth/thickness; γ: unit weight; φ: frictional angle; E: Young's modulus; d: diameter; s: spacing; ν: Poisson's ratio; S: tensile stiffness. Suffices "e", "g", "s", "c", and "gr" are used to designate embankment fill, granular fill, soft soil, CMC, and geosynthetic reinforcement (GR), respectively; T_y: yield strength of geosynthetics.

3.2 Comparison of Multilayer and One Layer of Geosynthetics

The results of multilayer weaker geosynthetics are compared with the results with the one layer of stronger geosynthetics using the proposed mechanical model. Figure 3a shows the settlement profiles of a LTP on soft soil and column region for different embankment heights. For 5m embankment height, as the number of geosynthetic layers reduces from two (weaker) to one (stronger), the maximum settlement in the middle of the two columns (Point B in Figure 3a) increases by 13%, whereas for 2m embankment this increase is 9%. Thus, for higher embankment, multiple geosynthetic reinforcements reduce the vertical settlement more effectively than for shallow embankment. This is due to the fact that the tension mobilisation of geosynthetic reinforcements increases with increasing load. However, on top of the column, since the CMC is 10000 times stiffer than the soft soil, the effect of embankment height and the geosynthetics reinforcement is very minimal on the corresponding settlement of LTP. In addition as shown in Figure 3b, adopting two layers of weaker geosynthetic reinforcements reduces the maximum rotation of the LTP by about 16% when the embankment height is 5m as shown in Figure 3b.

Figure 4 demonstrates the variation of the normalised mobilised tension T/T_y in the reinforcement (normalised against the yield strength of geosynthetics) with distance from the CMC centreline for both cases (i.e. the two layers of weaker geosynthetics and one stronger geosynthetics layer) for 5m high embankment. Results presented in Figure 3b show that the maximum negative bending moment (hogging) occurs at the column edge (Point A in Figure 4), whereas the maximum positive moment (sagging) occurs at the middle of two columns (Point B in Figure 4). Figure 4 illustrates that the top geosynthetic layer is subjected to a higher tension in the hogging region, whereas beyond this region the mobilised tension in the bottom geosynthetic layer is greater. Additionally, at the column location (Point A in Figure 4), one stronger layer of GR carries 30% less normalised tension than the top layer of GR. On the other hand, at Point B in Figure 4, it is displayed that the one layer of GR attracts 22% less normalised tension than the bottom layer of GR and 37% higher normalised tension than the top layer GR in case of two layers geosynthetic reinforcement. Therefore, one stronger layer of GR is not the equivalent of two weaker layers of geosynthetic reinforcement.

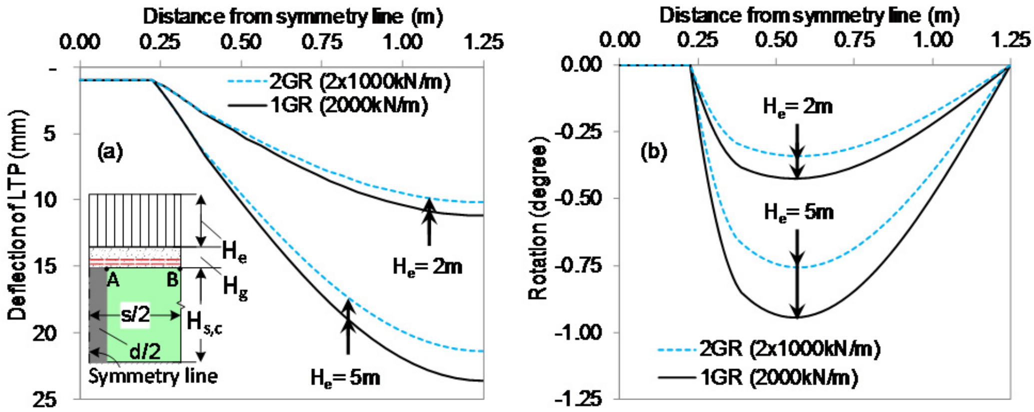

Figure 3: Effect of embankment height with multilayer geosynthetic reinforcements on (a) settlement profiles and (b) rotation of the LTP

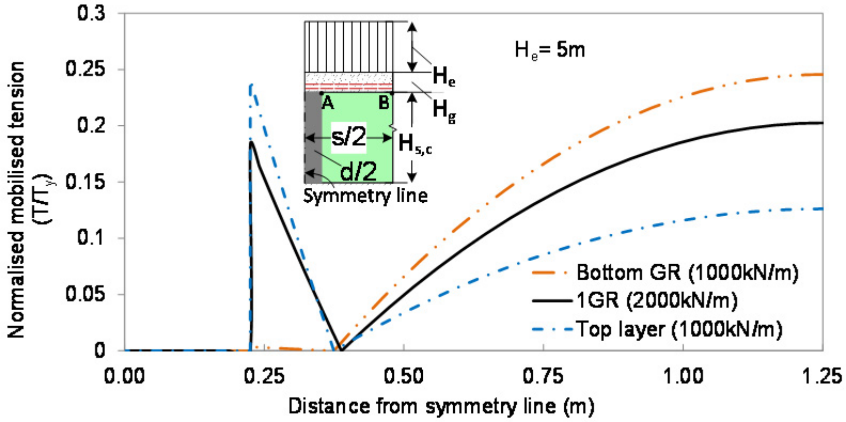

Figure 4: Typical profile of normalised mobilised tension in the geosynthetic reinforcement along the length of the LTP from the centerline of CMC

4 Conclusions

A new mechanical model with corresponding closed form analytical solutions have been proposed to study the settlement response of a granular fill layer with multilayer of geosynthetics for column stabilised soft soils. It can be concluded from this study that the use of multiple geosynthetic layers is more effective for higher embankment. Application of multiple geosynthetics reduces the maximum settlement of LTP. In case of two layers of gosynthetics, the top layer is more effective at the column location (in hogging region), whereas bottom layer is more effective in the middle of two columns (sagging region) in the column supported embankments. In addition, it is also observed that the use of one layer of stronger geosynthetics with equivalent stiffness of two layers of geosynthetics, does not results in the same settlement of LTP and tension in the GR compared to two layers of weaker GR. Therefore, it can be concluded that the load transfer mechanism in the column supported embankments with one stronger layer of geosynthetics is not equivalent to the load transfer mechanism with two weaker geosynythetics layers. Thus the practicing geotechnical engineers should be cautious in since detailed design and construction drawings arrangement of GR in the LTP can have a severe consequence on the performance of the composite system. It should be noted that although the material properties of CMC have been used in this study, the proposed model can be used to analysis the behaviour of any type of column supported embankment with load transfer platform.

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