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On Perfect Totient Numbers

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Abstract

Let $n > 2$ be a positive integer and let ϕ denote Euler's totient function. Define $\phi^1(n) = \phi(n)$ and $\phi^k(n) = \phi(\phi^{k-1}(n))$ for all integers $k \geq 2$. Define the arithmetic function S by $S(n) = \phi(n) + \phi^2(n) + \cdots + \phi^c(n) + 1$, where $\phi^c(n) = 2$. We say n is a perfect totient number if $S(n) = n$. We give a list of known perfect totient numbers, and we give sufficient conditions for the existence of further perfect totient numbers.

1 Introduction

Let $n > 2$ be a positive integer and let ϕ denote Euler's totient function. Define $\phi^1(n) = \phi(n)$ and $\phi^k(n) = \phi(\phi^{k-1}(n))$ for all integers $k \geq 2$. Shapiro [4] defines the *class number* $C(n)$ of

n by that integer c such that $\phi^c(n) = 2$. Define the arithmetic function S by

$$S(n) = \phi(n) + \phi^2(n) + \cdots + \phi^c(n) + 1,$$

where $c = C(n)$. Note that $\phi^{c+1}(n) = 1$. We say that n is a *perfect totient number* (or PTN for short) if $S(n) = n$.

Since $\phi(n)$ is even if $\phi(n) > 1$, it follows that all PTNs are odd. It is easy to show that 3^k is a PTN for all positive integers k. In Table 1 the 30 PTNs less than $5 \cdot 10^9$ which are not powers of 3 are given.

In addition to the PTNs given in Table 1, nine more were found by applying a result of Venkataraman [5]: If $p = 2^2 3^b + 1$ is prime then $3p$ is a PTN. A search of $b \leq 5000$, beyond those giving entries in Table 1, turned up the following nine values for which $p = 2^2 3^b + 1$ is prime (and therefore $3p$ is a PTN): $b = 39$, 201, 249, 885, 1005, 1254, 1635, 3306, 3522. The PTN which corresponds to $b = 3522$ has 1682 digits. Primality was verified with either UBASIC or Mathematica, by applying Lehmer's converse of Fermat's Theorem (Theorem 4.3 in Riesel [3]).

$15 = 3 \cdot 5$	$36759 = 3 \cdot 12253$
$39 = 3 \cdot 13$	$46791 = 3^3 \cdot 1733$
$111 = 3 \cdot 37$	$65535 = 3 \cdot 5 \cdot 17 \cdot 257$
$183 = 3 \cdot 61$	$140103 = 3^3 \cdot 5189$
$255 = 3 \cdot 5 \cdot 17$	$208191 = 3 \cdot 29 \cdot 239$
$327 = 3 \cdot 109$	$441027 = 3^2 \cdot 49003$
$363 = 3 \cdot 11^2$	$4190263 = 7 \cdot 11 \cdot 54419$
$471 = 3 \cdot 157$	$9056583 = 3^3 \cdot 335429$
$2199 = 3 \cdot 733$	$57395631 = 3 \cdot 19131877$
$3063 = 3 \cdot 1021$	$172186887 = 3 \cdot 57395629$
$4359 = 3 \cdot 1453$	$236923383 = 3 \cdot 1427 \cdot 55343$
$4375 = 5^4 \cdot 7$	$918330183 = 3^3 \cdot 34012229$
$5571 = 3^2 \cdot 619$	$3932935775 = 5^2 \cdot 29 \cdot 5424739$
$8751 = 3 \cdot 2917$	$4294967295 = 3 \cdot 5 \cdot 17 \cdot 255 \cdot 65537$
$15723 = 3^2 \cdot 1747$	$4764161215 = 5 \cdot 11 \cdot 86621113$

Table 1: PTNs less than $5 \cdot 10^9$ (except powers of 3).

The study of PTNs was initiated by Perez Cacho [2] when he proved that $3p$, for an odd prime p, is a PTN if and only if $p = 4n + 1$, where n is a PTN. Note that Venkataraman's result, mentioned above, follows as a corollary. Mohan and Suryanarayana [1] proved that $3p$, for an odd prime p, is not a PTN if $p \equiv 3 \pmod 4$. Thus PTNs of the form $3p$ have been completely characterized.

Applying Perez Cacho's result gives the following as the only known chains of PTNs, apart from the nine examples of length 2 mentioned earlier: $3 \to 39 \to 471$, $3^2 \to 111$, $15 \to$

$183 \to 2199$, $3^3 \to 327$, $255 \to 3063 \to 36759$, $363 \to 4359$, $3^6 \to 8751$, $3^{14} \to 57395631$, $3^{15} \to 172186887$.

The purpose of this paper is to investigate PTNs of the form $3^k p$, for $k \geq 2$ and p prime.

As an aside we note a curious result: the fact that $\phi(n) \leq n/2$ when n is even easily implies that $\phi(n) > n/2$ when n is a PTN.

2 Sufficient Conditions for PTNs

Mohan and Suryanarayana found sufficient conditions on an odd prime p for 3^2p and 3^3p to be PTNs, given in their paper as Theorem 5 and Theorem 6. In particular, let b be a nonnegative integer. Then, respectively, if $q = 2^5 3^b + 1$ and $p = 2 \cdot 3^2 q + 1$ are both prime then 3^2p is a PTN, and if $q = 2^4 3^b + 1$ and $p = 2^2 q + 1$ are both prime then 3^3p is a PTN. There is one known example of their Theorem 5, that being the PTN $15723 = 3^2 1747$ which occurs when $b = 1$. Their Theorem 6 has three known examples: the PTNs $46791 = 3^3 1733$ ($b = 3$), $140103 = 3^3 5189$ ($b = 4$), and $918330183 = 3^3 \cdot 34012229$ ($b = 12$). The values $b \leq 5000$ (for both theorems) were tested, but no further examples were found.

Let p be an odd prime. We have found four further sufficient conditions on p for 3^2p to be a PTN (three of which are given in the following Theorem), and two sufficient conditions for 3^3p to be a PTN.

Theorem 1 *Let b be a nonnegative integer. If r, q, and p, as given, are all prime then 3^2p is a PTN:*

1. $r = 2^4 3^b + 1$, $q = 2 \cdot 3r + 1$, *and* $p = 2 \cdot 3q + 1$;
2. $r = 2 \cdot 3^b + 1$, $q = 2^3 r + 1$, *and* $p = 2q + 1$;
3. $r = 2^2 3^b + 1$, $q = 2^3 3r + 1$, *and* $p = 2q + 1$.

Proof: (Part 1) We have $3^2p = 2^6 3^{b+4} + 387$ by direct substitution. Also,

$$\begin{aligned} S(3^2p) &= 2^2 3^2 q + 2^3 3^2 r + 2^7 3^{b+1} + 2^7 3^b + \cdots + 2^7 + 2^6 + \cdots + 1 \\ &= 2^7(3^{b+3} + \cdots + 3 + 1) + 451 \\ &= 2^6 3^{b+4} + 387\,. \end{aligned}$$

The proofs of Parts 2 and 3 are similar. □

In Part 1, when $b = 0$, we have $r = 17$, $q = 103$, and $p = 619$, giving the PTN 5571. There are no more examples for $b \leq 3000$. In Parts 2 and 3, no examples occur for $b \leq 3000$.

Theorem 2 *Let b be a nonnegative integer. If $q = 2^3 3^b + 1$ and $p = 2q + 1$ are both prime, then 3^2p is a PTN.*

Theorem 3 *Let b be a nonnegative integer. If $r = 2^2 3^b + 1$, $q = 2^4 r + 1$, and $p = 2^2 q + 1$ are all prime, then 3^3p is a PTN.*

THEOREM 4 *Let b be a nonnegative integer. If $s = 2^5 3^b + 1$, $r = 2 \cdot 3^2 s + 1$, $q = 2^4 3r + 1$, and $p = 2^2 q + 1$ are all prime, then $3^3 p$ is a PTN.*

Direct proofs of Theorems 2–4 may be obtained as above. In Theorem 2, examples do not occur for $b \leq 5000$, and in Theorem 3, examples do not occur for $b \leq 3000$. In Theorem 4, when $b = 1$, we have $s = 97$, $r = 1747$, $q = 83857$, and $p = 335429$, giving the PTN 9056583. There are no more examples for $b \leq 2000$.

3 PTNs of the form $3^k p$

In seeking examples of PTNs of the form $3^k p$, $k \geq 2$, we considered primes p and q such that $q = 2^a 3^b + 1$ and $p = 2^c 3^d q + 1$, where a, $c \geq 1$ and b, $d \geq 0$. Direct substitution gives

$$3^k p = 2^{a+c} 3^{b+d+k} + 2^c 3^{d+k} + 3^k .$$

On the other hand, we have

$$\begin{aligned} S(3^k p) &= 2^{c+1} 3^{d+k-1} q + 2^{a+c+1} 3^{b+d+k-2} + 2^{a+c+1} 3^{b+d+k-3} + \cdots \\ &\quad + 2^{a+c+1} + \cdots + 1 \\ &= 2^{c+1} 3^{d+k-1} + 2^{a+c+1}(3^{b+d+k-1} + \cdots + 3 + 1) + 2^{a+c} + \cdots + 1 \\ &= 2^{c+1} 3^{d+k-1} + 2^{a+c} 3^{b+d+k} + 2^{a+c} - 1. \end{aligned}$$

Assuming $3^k p$ is a PTN, we equate the above expressions for $3^k p$ and $S(3^k p)$ and simplify to obtain the diophantine equation

$$2^c (2^a - 3^{d+k-1}) = 3^k + 1. \tag{1}$$

Clearly, $a > 1$ and $c = 1$ or 2 for k even or odd, respectively.

When $k = 2$, the right-hand side of (1) is 10; thus $c = 1$ and the equation reduces to

$$2^a - 3^{d+1} = 5. \tag{2}$$

Since $2^a \equiv 2 \pmod 3$, we must have a odd. We have $a = 3$, $d = 0$ as one solution, and we have $a = 5$, $d = 2$ as another. If $a > 5$ then $3^{d+1} \equiv 123 \pmod{128}$, which implies $d + 1 \equiv 11 \pmod{32}$. This in turn implies $3^{d+1} \equiv 7 \pmod{17}$, and thus $2^a \equiv 7 + 5 \equiv 12 \pmod{17}$, which is impossible. Thus the only solutions to (2) are given by $a = 3$, $d = 0$, and by $a = 5$, $d = 2$. Since also $c = 1$, this statement includes both our Theorem 2 and Theorem 5 of Mohan and Suryanarayana [1].

When $k = 3$, (1) reduces to

$$2^a - 3^{d+2} = 7. \tag{3}$$

Clearly $a \geq 3$. Since $2^a \equiv 1 \pmod 3$ and $3^{d+2} \equiv 1 \pmod 8$, we must have a and d both even. Write $a = 2\alpha$ and $d = 2\delta$. Then (3) reduces to

$$(2^\alpha + 3^{\delta+1})(2^\alpha - 3^{\delta+1}) = 7. \tag{4}$$

Therefore $2^\alpha - 3^{\delta+1} = 1$ and $2^\alpha + 3^{\delta+1} = 7$, implying $\alpha = 2$, $\delta = 0$, which in turn implies $a = 4$, $d = 0$ as the only solution. Together with $c = 2$, this includes the statement of Theorem 6 in Mohan and Suryanarayana [1].

We show next that there are no solutions of (1) when $k \geq 4$.

Suppose first that k is even, $k \geq 4$. Then $c = 1$. Put $x = a+1$ and $y = d+k-1$ so that (1) may be given as

$$2^x - 2 \cdot 3^y = 3^k + 1, \tag{5}$$

where $x \geq 3$, $y \geq 3$. Then $2^x \equiv 1 \pmod{27}$, from which $x \equiv 0 \pmod{18}$. Since $2^{18} \equiv 1 \pmod{19}$, we then have $-2 \cdot 3^y \equiv 3^k \pmod{19}$, so that

$$\left(\frac{3}{19}\right)^y = \left(\frac{-2 \cdot 3^y}{19}\right) = \left(\frac{3^k}{19}\right) = \left(\frac{3}{19}\right)^k,$$

where $\left(\frac{\cdot}{\cdot}\right)$ is a Legendre symbol. Also, from (5), $-2 \cdot 3^y \equiv 3^k + 1 \equiv 2 \pmod{8}$, so y is odd. Then we have a contradiction since k is even, y is odd, and the Legendre symbol $(3/19) = -1$.

Suppose next that k is odd, $k \geq 5$. Then $c = 2$. Put $x = a+2$ and $y = d+k-1$ so that (1) becomes

$$2^x - 4 \cdot 3^y = 3^k + 1, \tag{6}$$

where $x \geq 4$, $y \geq 4$. There are two main cases to consider.

(a) If $k \equiv 1 \pmod{4}$, then $-4 \cdot 3^y \equiv 3^k + 1 \equiv 4 \pmod{16}$, implying that y is odd. Also, as immediately above, $x \equiv 0 \pmod{18}$. Since $2^{18} \equiv 1 \pmod{7}$, then $-4 \cdot 3^y \equiv 3^k \pmod{7}$, so $3^{y+1} \equiv 3^k \pmod{7}$. This is impossible when $y+1$ is even and k is odd.

(b) If $k \equiv 3 \pmod{4}$, then $-4 \cdot 3^y \equiv 3^k + 1 \equiv 12 \pmod{16}$, so y is even. Suppose first that $y \equiv 0 \pmod{4}$. Then $2^x \equiv 4 \cdot 3^y + 3^k + 1 \equiv 4 + 2 + 1 \equiv 2 \pmod{5}$. This implies that x is odd. But, from (6), $2^x \equiv 1 \pmod{3}$, so x is even. We have a contradiction.

The most difficult case to eliminate is when $k \equiv 3 \pmod{4}$ and $y \equiv 2 \pmod{4}$. Consideration of (6), modulo 5, implies $2^x \equiv 4 \pmod{5}$, so $x \equiv 2 \pmod{4}$. From (6), we also have $2^x \equiv 1 \pmod{27}$, so $x \equiv 0 \pmod{18}$ and then, since $x \equiv 2 \pmod{4}$, we have $x \equiv 18 \pmod{36}$. This then implies that $2^x \equiv -1 \pmod{13}$. Consideration of the nine possibilities that arise from (6), modulo 13, taking $y \equiv 2$, 6 or $10 \pmod{12}$ and $k \equiv 3$, 7 or $11 \pmod{12}$ shows that in fact $y \equiv 2 \pmod{12}$ and $k \equiv 3 \pmod{12}$. Now consider a further nine cases of (6), modulo 37, taking $y \equiv 2$, 14 or $26 \pmod{36}$ and $k \equiv 3$, 15 or $27 \pmod{36}$. The only possibility is $y \equiv 2 \pmod{36}$ and $k \equiv 27 \pmod{36}$. But in that case, since $2^{18} \equiv 3^{36} \equiv 1 \pmod{73}$, we find that $2^x - 4 \cdot 3^y \equiv 1 - 4 \cdot 9 \equiv 38 \pmod{73}$ and $3^k + 1 \equiv 27 + 1 = 28 \pmod{73}$. This contradicts (6).

We give this conclusion as:

THEOREM 5 *There are no PTNs of the form $3^k p$, $k \geq 4$, where $p = 2^c 3^d q + 1$ and $q = 2^a 3^b + 1$ are primes with a, $c \geq 1$ and b, $d \geq 0$.*

We next considered another possibility: let a, c, $e \geq 1$ and b, d, $f \geq 0$ be integers. Suppose $r = 2^a 3^b + 1$, $q = 2^c 3^d r + 1$, and $p = 2^e 3^f q + 1$ are all prime, and let $n = 3^k p$ for $k \geq 2$.

Substitution gives us

$$n = 2^{a+c+e}3^{b+d+f+k} + 2^{c+e}3^{d+f+k} + 2^e 3^{f+k} + 3^k,$$

whereas substitution and calculation gives us

$$S(n) = 2^{c+e+3}3^{d+f+k-2} + 2^{e+1}3^{f+k-1} + 2^{a+c+e}3^{b+d+f+k} + 2^{a+c+e} - 1.$$

Assuming n is a PTN, we equate the expressions for n and $S(n)$ and simplify to obtain the diophantine equation

$$2^e(2^c(2^a - 3^{d+f+k-2}) - 3^{f+k-1}) = 3^k + 1\,. \tag{7}$$

We found four solutions to (7), with $a, c, d, f \leq 20$ and $k \leq 10$. Notice that $e = 1$ or 2 if k is even or odd respectively. Our first solution is given by $a = e = 2$, $c = 4$, $d = f = 0$, and $k = 3$. Note that this is Theorem 3. The second solution is given by $a = 4$, $c = d = e = f = 1$, and $k = 2$. This is Part 1 of Theorem 1. The third solution is given by $a = e = 1$, $c = 3$, $d = f = 0$, and $k = 2$ (Part 2 of Theorem 1), and the fourth is given by $a = 2$, $c = 3$, $d = e = 1$, $f = 0$, and $k = 2$ (Part 3 of Theorem 1).

Similarly, we also considered integers $a, c, e, g \geq 1$, $b, d, f, h \geq 0$, where all of $s = 2^a3^b+1$, $r = 2^c3^ds + 1$, $q = 2^e3^fr + 1$, and $p = 2^g3^hq + 1$ are supposed prime. Then, as above, letting $n = 3^kp$ for $k \geq 3$ and supposing $S(n) = n$ implies the diophantine equation

$$2^g(2^e(2^c(2^a - 3^{d+f+h+k-3}) - 3^{f+h+k-2}) - 3^{h+k-1}) = 3^k + 1.$$

We found several solutions, but only one of them produced any PTNs: $a = 5$, $c = f = 1$, $d = g = 2$, $e = 4$, $h = 0$, and $k = 3$. This is Theorem 4, which we have already seen produces one known PTN.

The question remains open as to whether or not any PTNs exist of the form 3^kp for $k \geq 4$.

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