REDUCTION FORMULAS FOR THE SUMMATION OF RECIPROCALS IN CERTAIN SECOND-ORDER RECURRING SEQUENCES

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1. INTRODUCTION

In [2], Brousseau considered sums of the form

$$S(k_1, k_2, \ldots, k_m) = \sum_{n=1}^{\infty} \frac{1}{F_n F_{n+k_1} F_{n+k_2} \cdots F_{n+k_m}} \tag{1.1}$$

and

$$T(k_1, k_2, \ldots, k_m) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{F_n F_{n+k_1} F_{n+k_2} \cdots F_{n+k_m}}, \tag{1.2}$$

where the k_i are positive integers with $k_1 < k_2 < \cdots < k_m$. He stated that the sums in (1.1) and (1.2) could be written as

$$S(k_1, k_2, \ldots, k_m) = r_1 + r_2 S(1, 2, \ldots, m) \tag{1.3}$$

and

$$T(k_1, k_2, \ldots, k_m) = r_3 + r_4 T(1, 2, \ldots, m), \tag{1.4}$$

where r_1, r_2, r_3, and r_4 are rational numbers that depend upon $k_1, k_2, \ldots, k_m$. He arrived at this conclusion after treating several cases involving small values of m.

Our aim in this paper is to prove Brousseau's claim by providing reduction formulas that accomplish this task. Recently, André-Jeannin [1] treated the case $m = 1$ by giving explicit expressions for the coefficients r_1, r_2, r_3, and r_4. Indeed, he worked with a generalization of the Fibonacci sequence, and we will do the same. In light of André-Jeannin's results, we consider only $m \geq 2$. We have found, for each of the sums (1.3) and (1.4), that two reduction formulas are needed for the case $m = 2$, and three are needed for $m \geq 3$. Consequently, we treat those cases separately.

Define the sequences $\{U_n\}$ and $\{W_n\}$ for all integers n by

$$\begin{cases} U_n = pU_{n-1} - qU_{n-2}, & U_0 = 0,\ U_1 = 1, \\ W_n = pW_{n-1} - qW_{n-2}, & W_0 = a,\ W_1 = b. \end{cases}$$

Here a, b, p, and q are assumed to be integers with $pq \neq 0$ and $\Delta = p^2 - 4q > 0$. Consequently, we can write down closed expressions for U_n and W_n (see [3]):

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad W_n = \frac{A\alpha^n - B\beta^n}{\alpha - \beta}, \tag{1.5}$$

where $\alpha = (p + \sqrt{\Delta})/2$, $\beta = (p - \sqrt{\Delta})/2$, $A = b - a\beta$, and $B = b - a\alpha$. Thus, $\{W_n\}$ generalizes $\{U_n\}$ which, in turn, generalizes $\{F_n\}$.

We note that

$$\alpha > 1 \text{ and } \alpha > |\beta| \text{ if } p > 0, \text{ while } \beta < -1 \text{ and } |\beta| > |\alpha| \text{ if } p < 0. \tag{1.6}$$

Consequently,

$$W_n \approx \frac{A}{\alpha-\beta}\alpha^n \text{ if } p > 0, \text{ and } W_n \approx \frac{-B}{\alpha-\beta}\beta^n \text{ if } p < 0. \tag{1.7}$$

Throughout the remainder of the paper, we take

$$S(k_1, k_2, \ldots, k_m) = \sum_{n=1}^{\infty} \frac{1}{W_n W_{n+k_1} W_{n+k_2} \cdots W_{n+k_m}} \tag{1.8}$$

and

$$T(k_1, k_2, \ldots, k_m) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{W_n W_{n+k_1} W_{n+k_2} \cdots W_{n+k_m}}, \tag{1.9}$$

where the k_i are positive integers as described earlier. From (1.6) it follows that $U_n \neq 0$ for $n \geq 1$. We shall suppose that $W_n \neq 0$ for $n \geq 1$. Then, by (1.6) and (1.7), use of the ratio test shows that the series in (1.8) and (1.9) are absolutely convergent.

We require the following identities:

$$U_m W_{n+1} - W_{n+m} = qU_{m-1}W_n, \tag{1.10}$$

$$U_{m-k+1}W_{n+k} - W_{n+m} = qU_{m-k}W_{n+k-1}, \tag{1.11}$$

$$U_m W_n + q^m U_d W_{n-m-d} = U_{m+d}W_{n-d}, \tag{1.12}$$

$$pW_{n+m} + q^2 U_{m-2}W_n = U_m W_{n+2}, \tag{1.13}$$

$$U_{m-l+1}W_{n+k} - U_{k-l+1}W_{n+m} = q^{k-l+1}U_{m-k}W_{n+l-1}. \tag{1.14}$$

Identity (1.11) follows from (1.10), which is essentially (3.14) in [3], where the initial values of $\{U_n\}$ are shifted. Identities (1.13) and (1.14) follow from (1.12), which occurs as (5.7) in [4].

2. THREE TERMS IN THE DENOMINATOR

Our results for the case in which the denominator consists of a product of three terms are contained in the following theorem.

Theorem 1: Let k_1 and k_2 be positive integers with $k_1 < k_2$. Then

$$S(k_1, k_2) = \frac{1}{qU_{k_2-k_1}}[U_{k_2-k_1+1}S(k_1-1, k_2) - S(k_1-1, k_1)] \quad \text{if } 1 < k_1, \tag{2.1}$$

$$S(1, k_2) = \frac{p}{U_{k_2}}S(1,2) + \frac{q^2U_{k_2-2}}{U_{k_2}}\left[S(1, k_2-1) - \frac{1}{W_1W_2W_{k_2}}\right] \quad \text{if } 2 < k_2, \tag{2.2}$$

$$T(k_1, k_2) = \frac{1}{qU_{k_2-k_1}}[U_{k_2-k_1+1}T(k_1-1, k_2) - T(k_1-1, k_1)] \quad \text{if } 1 < k_1, \tag{2.3}$$

$$T(1, k_2) = \frac{p}{U_{k_2}}T(1,2) + \frac{q^2U_{k_2-2}}{U_{k_2}}\left[\frac{1}{W_1W_2W_{k_2}} - T(1, k_2-1)\right] \quad \text{if } 2 < k_2. \tag{2.4}$$

Proof: With the use of (1.11), it follows that

$$\frac{qU_{k_2-k_1}}{W_nW_{n+k_1}W_{n+k_2}} = \frac{U_{k_2-k_1+1}}{W_nW_{n+k_1-1}W_{n+k_2}} - \frac{1}{W_nW_{n+k_1-1}W_{n+k_1}}, \tag{2.5}$$

and summing both sides we obtain (2.1). Likewise, to obtain (2.3), we first multiply (2.5) by $(-1)^{n-1}$ and sum both sides.

Next we have

$$\frac{U_{k_2}}{W_nW_{n+1}W_{n+k_2}} = \frac{p}{W_nW_{n+1}W_{n+2}} + \frac{q^2U_{k_2-2}}{W_{n+1}W_{n+2}W_{n+k_2}}, \tag{2.6}$$

which follows from (1.13). Now, if we sum both sides of (2.6) and note that

$$\sum_{n=1}^{\infty}\frac{1}{W_{n+1}W_{n+2}W_{n+k_2}} = S(1, k_2-1) - \frac{1}{W_1W_2W_{k_2}},$$

we obtain (2.2). Finally, to establish (2.4), we multiply (2.6) by $(-1)^{n-1}$, sum both sides, and note that

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{W_{n+1}W_{n+2}W_{n+k_2}} = \frac{1}{W_1W_2W_{k_2}} - T(1, k_2-1).$$

This proves Theorem 1. □

It is instructive to work through some examples. Taking $W_n = F_n$ and using (2.1) and (2.2) repeatedly, we find that $S(3,6) = -\frac{269}{1920} + \frac{1}{4}S(1,2)$, and this agrees with the corresponding entry in Table III of [2]. Again, with $W_n = F_n$, we have $T(3,6) = -\frac{139}{1920} + \frac{1}{4}T(1,2)$.

3. MORE THAN THREE TERMS IN THE DENOMINATOR

Let $k_1, k_2, \ldots, k_m$ be positive integers and put $P(k_1, \ldots, k_m) = W_nW_{n+k_1}\cdots W_{n+k_m}$. With this notation, the work that follows will be more succinct. The main result of this section is contained in the theorem that follows, where we give only the reduction formulas for $S(k_1, k_2, \ldots, k_m)$. After the proof, we will indicate how the corresponding reduction formulas for $T(k_1, k_2, \ldots, k_m)$ can be obtained.

Theorem 2: For $m \geq 3$, let $k_1 < k_2 < \cdots < k_m$ be positive integers and set $k_0 = 0$. Then

$$\begin{aligned} S(k_1, k_2, \ldots, k_m) = {} & \frac{U_{k_m-k_j+1}}{q^{k_{m-1}-k_j+1}U_{k_m-k_{m-1}}} S(k_1, \ldots, k_{j-1}, k_j-1, k_j, \ldots, k_{m-2}, k_m) \\ & - \frac{U_{k_{m-1}-k_j+1}}{q^{k_{m-1}-k_j+1}U_{k_m-k_{m-1}}} S(k_1, \ldots, k_{j-1}, k_j-1, k_j, \ldots, k_{m-1}) \\ & \text{if } 1 \leq j \leq m-2 \text{ and } k_{j-1} < k_j - 1; \end{aligned} \tag{3.1}$$

$$\begin{aligned} S(k_1, \ldots, k_m) = {} & \frac{U_{k_m-k_{m-1}+1}}{qU_{k_m-k_{m-1}}} S(k_1, \ldots, k_{m-2}, k_{m-1}-1, k_m) \\ & - \frac{1}{qU_{k_m-k_{m-1}}} S(k_1, \ldots, k_{m-2}, k_{m-1}-1, k_{m-1}) \text{ if } k_{m-2} < k_{m-1}-1; \end{aligned} \tag{3.2}$$

$$
\begin{aligned}
S(1,2,\ldots,m-1,k_m) &= \frac{U_m}{U_{k_m}}S(1,2,\ldots,m) \\
&\quad + \frac{q^m U_{k_m-m}}{U_{k_m}}\left[S(1,2,\ldots,m-1,k_m-1) - \frac{1}{W_1W_2\ldots W_mW_{k_m}}\right] \quad \text{if } m < k_m.
\end{aligned} \tag{3.3}
$$

Proof: With the use of (1.14), we see that

$$
\begin{aligned}
\frac{q^{k_{m-1}-k_j+1}U_{k_m-k_{m-1}}}{P(k_1,\ldots,k_m)} &= \frac{U_{k_m-k_j+1}}{P(k_1,\ldots,k_{j-1},k_j-1,k_j,\ldots,k_{m-2},k_m)} \\
&\quad - \frac{U_{k_{m-1}-k_j+1}}{P(k_1,\ldots,k_{j-1},k_j-1,k_j,\ldots,k_{m-1})},
\end{aligned}
$$

and summing both sides we obtain (3.1).

Next we have

$$
\frac{qU_{k_m-k_{m-1}}}{P(k_1,\ldots,k_m)} = \frac{U_{k_m-k_{m-1}+1}}{P(k_1,\ldots,k_{m-2},k_{m-1}-1,k_m)} - \frac{1}{P(k_1,\ldots,k_{m-2},k_{m-1}-1,k_{m-1})},
$$

which can be proved with the use of (1.11). Summing both sides, we obtain (3.2).

Finally, with the aid of (1.12) we see that

$$
\frac{U_{k_m}}{P(1,2,\ldots,m-1,k_m)} = \frac{U_m}{P(1,2,\ldots,m)} + \frac{q^m U_{k_m-m}}{W_{n+1}W_{n+2}\cdots W_{n+m}W_{n+k_m}}.
$$

The reduction formula (3.3) follows if we sum both sides and observe that

$$
\sum_{n=1}^{\infty}\frac{1}{W_{n+1}W_{n+2}\cdots W_{n+m}W_{n+k_m}} = S(1,2,\ldots,m-1,k_m-1) - \frac{1}{W_1W_2\ldots W_mW_{k_m}}.
$$

This completes the proof of Theorem 2. □

As was the case in Theorem 1, the reduction formulas for T can be obtained from those for S. In (3.1) and (3.2), we simply replace S by T. In (3.3), we *first* replace the term in square brackets by

$$
\frac{1}{W_1W_2\ldots W_mW_{k_m}} - S(1,2,\ldots,m-1,k_m-1)
$$

and *then* replace S by T.

As an application of Theorem 2 we have, with $W_n = F_n$,

$$
S(1,2,4,6,7) = -3S(1,2,3,4,7) + 2S(1,2,3,4,6) \quad \text{by (3.1)}; \tag{3.4}
$$

$$
S(1,2,3,4,7) = \frac{1}{5070} + \frac{5}{13}S(1,2,3,4,5) - \frac{1}{13}S(1,2,3,4,6) \quad \text{by (3.3)}; \tag{3.5}
$$

$$
S(1,2,3,4,6) = \frac{1}{1920} + \frac{1}{2}S(1,2,3,4,5) \quad \text{by (3.3)}. \tag{3.6}
$$

Together (3.4)-(3.6) imply that

$$
S(1,2,4,6,7) = \frac{37}{64896} - \frac{1}{26}S(1,2,3,4,5).
$$

4. CONCLUDING COMMENTS

Recently, Rabinowitz [5] considered the finite sums associated with (1.1) and (1.2). That is, he took the upper limit of summation to be N, and gave an algorithm for expressing the resulting sums in terms of

$$\sum_{n=1}^{N}\frac{1}{F_n},\ \sum_{n=1}^{N}\frac{(-1)^n}{F_n},\ \text{and}\ \sum_{n=1}^{N}\frac{1}{F_nF_{n+1}}.$$

In addition, he posed a number of interesting open questions.

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