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An α-fuzzy goal approximate algorithm for Fuzzy multiple objective linear programming problems

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Abstract. Many business decision problems involve multiple objectives and can be described by multiple objective linear programming (MOLP) models. Referring to the imprecision or fuzziness inherent in human judgments, two types of inaccuracies should be incorporated in MOLP problems. One is the experts' ambiguous understanding of the nature of the parameters in the problem formulation process, and the other is the fuzzy goals of the decision maker (DM) for each of the objective functions. Consider these two fuzziness features, this paper proposes an α-fuzzy goal approximate (α-FGA) algorithm for achieving the fuzzy goals for the objective functions specified by the DM in dealing with fuzzy multiple objective linear programming (FMOLP) problems with fuzzy parameters under the different satisfactory degree α. The detail description and analysis to the algorithm are supplied.

Keywords: Optimization; Fuzzy multiple objective linear programming; Approximate algorithm; Satisfactory degree α.

1. Introduction

Multiple objective linear programming (MOLP) is an efficient way for modeling and solving many realistic optimization problems. Such problems often include several conflicting and incommensurable objectives, which are to be optimized, and are subject to certain specified constraints [1].

In modeling a real MOLP problem, the possible values of parameters in the objective functions and constraints may be assigned in an experimental or subjective manner through the experts' understanding of the nature for the parameters. With this observation, the possible values of these parameters are often imprecisely or ambiguously known to the experts. In this case, it may be more appropriate to interpret the experts' understanding of the parameters as fuzzy numerical data that can be represented by fuzzy numbers [5]. The fuzzy multiple objective linear programming (FMOLP) problems involving fuzzy parameters would be more adequate to describe reality [4].

Under some circumstances for the FMOLP problems, the decision maker (DM) has some goals for the objective functions which need to be achieved. In fact, to ask the DM what attainments are desired for each objective function is a difficult job. Considering the imprecise nature of the DM's judgment, it is nature to assume that the DM may have imprecise or fuzzy goals for each of the objective functions in the FMOLP problem. Also, using fuzzy numbers to represent such goals would be more appropriate than the crisp numbers [2] [4].

This paper proposes an α-fuzzy goal approximate (α-FGA) algorithm for achieving the fuzzy goals for the objective functions specified by the DM in dealing with FMOLP problems with fuzzy parameters under the different satisfactory degree α.

2. Related Definition and Theorems

In this paper, we consider the situation that all coefficients of the objective functions and constraints are represented by fuzzy numbers in any form of membership function. Here, we define the ranking way between two n-dimensional fuzzy numbers under a certain satisfactory degree α as follows.

Definition 2.1 Let $\tilde{\alpha}, \tilde{\beta}$ be two n-dimensional fuzzy numbers, we define

1) $\tilde{\alpha} \underset{=\alpha}{\succ} \tilde{\beta}$ iff $\alpha_{i\lambda}^{L} \geqq \beta_{i\lambda}^{L}$ and $\alpha_{i\lambda}^{R} \geqq \beta_{i\lambda}^{R}, i=1,\ldots,n, \forall \lambda \in [\alpha,1]$,

2) $\tilde{\alpha} \succeq_{\alpha} \tilde{\beta}$ iff $\alpha_{i\lambda}^{L} \geq \beta_{i\lambda}^{L}$ and $\alpha_{i\lambda}^{R} \geq \beta_{i\lambda}^{R}, i=1,\ldots,n, \forall \lambda \in [\alpha,1]$,

3) $\tilde{\alpha} \succ_{\alpha} \tilde{\beta}$ iff $\alpha_{i\lambda}^{L} > \beta_{i\lambda}^{L}$ and $\alpha_{i\lambda}^{R} > \beta_{i\lambda}^{R}$, $i=1,\ldots,n$, $\forall \lambda \in [\alpha,1]$.

With a certain satisfactory degree α, the FMOLP problems can be formulated as follows:

$$(\text{FMOLP}_\alpha) \begin{cases} \text{Max}_\alpha \quad \tilde{f}(x) = \tilde{C}x \\ \text{s.t.} \quad x \in X = \{x \in R^n \mid \tilde{A}x \underset{=\alpha}{\prec} \tilde{b}, x \geq 0\} \end{cases} \tag{1}$$

where $\mu_{\tilde{c}_{ij}}(x)\geq\alpha$, $\mu_{\tilde{a}_{ij}}(x)\geq\alpha$, and $\mu_{\tilde{b}_l}(x)\geq\alpha$ with $i=1,\ldots,k$, $j=1,\ldots,n$, and $l=1,\ldots,m$.

Based on Definition 2.1, we propose the following definitions about the optimal solutions for FMOLP$_\alpha$ problems with fuzzy parameters.

Definition 2.1 x^* is said to be a complete optimal solution to the FMOLP$_\alpha$ problems, if and only if there exists $x^*\in X$ such that $\tilde{f}_i(x^*)\succeq_\alpha \tilde{f}_i(x)$, $i=1,\ldots,k$, for all $x\in X$.

Definition 2.3 x^* is said to be a Pareto optimal solution to the FMOLP$_\alpha$ problems, if and only if there does not exists another $x\in X$ such that $\tilde{f}_i(x)\succeq_\alpha \tilde{f}_i(x^*)$, for all i.

Definition 2.4 x^* is said to be a weak Pareto optimal solution to the FMOLP$_\alpha$ problems, if and only if there does not exists another $x\in X$ such that $\tilde{f}_i(x)\succ_\alpha \tilde{f}_i(x^*)$, for all i.

Associated with the FMOLP$_\alpha$ problems, let's consider the following crisp multiple objective linear programming MOLP$_{\alpha\lambda}$ problems:

$$(\text{MOLP}_{\alpha\lambda})\begin{cases}\text{Max } \left(\langle C_\lambda^L,x\rangle,\langle C_\lambda^R,x\rangle\right)^T, \forall\lambda\in[\alpha,1]\\ \text{s.t. } x\in X=\left\{x\in R^n \mid A_\lambda^L x\leq b_\lambda^L, A_\lambda^R x\leq b_\lambda^R, x\geq 0, \forall\lambda\in[\alpha,1]\right\}\end{cases}\tag{2}$$

where

$$C_\lambda^L=\begin{bmatrix}c_{11\lambda}^L & c_{12\lambda}^L & \cdots & c_{1n\lambda}^L\\ c_{21\lambda}^L & c_{22\lambda}^L & \cdots & c_{2n\lambda}^L\\ \vdots & \vdots & \ddots & \vdots\\ c_{k1\lambda}^L & c_{k2\lambda}^L & \cdots & c_{kn\lambda}^L\end{bmatrix},\quad C_\lambda^R=\begin{bmatrix}c_{11\lambda}^R & c_{12\lambda}^R & \cdots & c_{1n\lambda}^R\\ c_{21\lambda}^R & c_{22\lambda}^R & \cdots & c_{2n\lambda}^R\\ \vdots & \vdots & \ddots & \vdots\\ c_{k1\lambda}^R & c_{k2\lambda}^R & \cdots & c_{kn\lambda}^R\end{bmatrix},\quad A_\lambda^L=\begin{bmatrix}a_{11\lambda}^L & a_{12\lambda}^L & \cdots & a_{1n\lambda}^L\\ a_{21\lambda}^L & a_{22\lambda}^L & \cdots & a_{2n\lambda}^L\\ \vdots & \vdots & \ddots & \vdots\\ a_{m1\lambda}^L & a_{m2\lambda}^L & \cdots & a_{mn\lambda}^L\end{bmatrix},\quad A_\lambda^R=\begin{bmatrix}a_{11\lambda}^R & a_{12\lambda}^R & \cdots & a_{1n\lambda}^R\\ a_{21\lambda}^R & a_{22\lambda}^R & \cdots & a_{2n\lambda}^R\\ \vdots & \vdots & \ddots & \vdots\\ a_{m1\lambda}^R & a_{m2\lambda}^R & \cdots & a_{mn\lambda}^R\end{bmatrix},$$

$$b_\lambda^L=\left[b_{1\lambda}^L,b_{2\lambda}^L,\cdots,b_{m\lambda}^L\right]^T,\qquad b_\lambda^R=\left[b_{1\lambda}^R,b_{2\lambda}^R,\cdots,b_{m\lambda}^R\right]^T.$$

For the crisp MOLP$_{\alpha\lambda}$ problems, we also have the following definitions.

Definition 2.5 [3] x^* is said to be a complete optimal solution, if and only if there exists $x^*\in X$ such that $f_i(x^*)\geq f_i(x)$, i = 1, ..., k, for all $x\in X$.

Definition 2.6 [3] x^* is said to be a Pareto optimal solution, if and only if there does not exists another $x\in X$ such that $f_i(x)\geq f_i(x^*)$, for all i and $f_j(x)\neq f_j(x^*)$ for at least one j.

Definition 2.7 [3] x^* is said to be a weak Pareto optimal solution, if and only if there does not exists another $x\in X$ such that $f_i(x)>f_i(x^*)$, i = 1, ... , k.

The following theorem shows the relationships between FMOLP$_\alpha$ problem and the MOLP$_{\alpha\lambda}$ problem.

Theorem 2.1 Let $x^*\in X$ be a solution to the MOLP$_{\alpha\lambda}$ problem. Then x^* is also a solution to the FMOLP problem.

4. An α-Fuzzy Goal Approximated (α-FGA) Algorithm for FMOLP Problem

Under some circumstances, the DM needs to specify fuzzy goals for the objective functions in dealing with FMOLP problems. The key idea behind goal programming is to minimize the deviations from goals set by the DM. Therefore in most cases, goal programming seems to yield a satisfying solutions rather than optimal one. Considering the FMOLP problem, for each of the fuzzy multiple objective functions $\tilde{f}(x)=\left(\tilde{f}_1(x),\tilde{f}_2(x),\ldots,\tilde{f}_k(x)\right)^T$, the DM can specify some fuzzy goals $\tilde{g}=(\tilde{g}_1,\tilde{g}_2,\ldots,\tilde{g}_k)^T$ which reflect the desired values of the objective functions of the DM. Also, the DM can change the fuzzy goal interactively due to learning and improved understanding during the interactive solution process.

From the definition of MOLP$_{\alpha\lambda}$ problem in Section 3, when the DM sets up some fuzzy goals $\tilde{g}=(\tilde{g}_1,\tilde{g}_2,\ldots,\tilde{g}_k)^T$ under the satisfactory degree α, which are represented by fuzzy numbers with any form of membership functions, the corresponding Pareto optimal solution, which is the nearest to the fuzzy goals or better than that, is obtained by solving the following minimax problem:

$$(\text{MOLP}_{\alpha\lambda m})\begin{cases}\text{Min Max } \left(\langle C_\lambda^L,x\rangle-g_\lambda^L,\langle C_\lambda^R,x\rangle-g_\lambda^R\right)^T, \forall\lambda\in[\alpha,1]\\ \text{s.t. } x\in X=\left\{x\in R^n \mid A_\lambda^L x\leq b_\lambda^L, A_\lambda^R x\leq b_\lambda^R, x\geq 0, \forall\lambda\in[\alpha,1]\right\}\end{cases}\tag{3}$$

where $g_\lambda^L=\left[g_{1\lambda}^L,g_{2\lambda}^L,\cdots,g_{k\lambda}^L\right]^T$, $g_\lambda^R=\left[g_{1\lambda}^R,g_{2\lambda}^R,\cdots,g_{k\lambda}^R\right]^T$.

Refer to the description of MOLP$_{\alpha\lambda m}$ problem in (3), there are an infinite number of objective functions and an infinite number of constraints included in MOLP$_{\lambda m}$ problem. In order to fix the problem, the α-FGA algorithm is proposed as follows for solving MOLP$_{\alpha\lambda m}$ problem.

For the simplicity in presentation, we define

$$X_\lambda=\left\{x\in R^n \mid A_\lambda^L x\leq b_\lambda^L, A_\lambda^R x\leq b_\lambda^R, x\geq 0\right\}\ \lambda\in[\alpha,1]\tag{4}$$

The main steps of the α-FGA algorithm are described as follows:

Let the interval [α,1] be decomposed into l mean sub-intervals with $(l+1)$ nodes λ_i $(i=0,\cdots,l)$ which are arranged in the order of $\alpha=\lambda_0<\lambda_1<\cdots<\lambda_l=1$. Based on the current decomposing, we define the constraint as $X^l=\bigcap_i^l X_{\lambda_i}$, and denote:

$$(\mathrm{MOLP}_{\alpha\lambda m})_l \begin{cases}\text{Min Max } \left(\langle C^L_{\lambda_i},x\rangle-g^L_{\lambda_i},\langle C^R_{\lambda_i},x\rangle-g^R_{\lambda_i}\right)^T, \alpha=\lambda_0\ <\ldots<\lambda_l=1\\ \text{s.t.}\quad x\in X^l\end{cases} \tag{5}$$

Step 1: Set $l=1$, then solve $(\mathrm{MOLP}_{\alpha\lambda m})_l$ with solution $(x)_l$, and the solution $(x)_l$ is subject to the constraint $x\in X^l$.

Step 2: Solve $(\mathrm{MOLP}_{\alpha\lambda m})_{2l}$ with solution $(x)_{2l}$, and the solution obtained is subject to the constraint $x\in X^{2l}$.

Step 3: If $\|(x)_{2l}-(x)_l\|<\varepsilon$, then the final solution x^* of $\mathrm{MOLP}_{\alpha\lambda m}$ problem is $(x)_{2l}$. Otherwise, update l to $2l$ and go back to Step 2.

5. An illustrative example

To illustrate the IFGDM method developed, let us consider the following FMOLP problem with two fuzzy objective functions and three fuzzy constraints:

$$\text{Max } \tilde{f}(x)=\text{Max}\begin{pmatrix}\tilde{f}_1(x)\\ \tilde{f}_2(x)\end{pmatrix}=\text{Max}\begin{pmatrix}\tilde{c}_{11}x_1+\tilde{c}_{12}x_2\\ \tilde{c}_{21}x_1+\tilde{c}_{22}x_2\end{pmatrix}=\text{Max}\begin{pmatrix}\tilde{4}x_1+\tilde{2}x_2\\ \text{-}\,\tilde{2}x_1+\tilde{4}x_2\end{pmatrix} \tag{6}$$

$$\text{s.t.}\quad\begin{cases}\tilde{a}_{11}x_1+\tilde{a}_{12}x_2=\text{-}\,\tilde{1}x_1+\tilde{3}x_2\underset{=}{\prec}\tilde{b}_1=\tilde{21}\\ \tilde{a}_{21}x_1+\tilde{a}_{22}x_2=\ \tilde{1}x_1+\tilde{3}x_2\underset{=}{\prec}\tilde{b}_2=\tilde{27}\\ \tilde{a}_{31}x_1+\tilde{a}_{32}x_2=\ \tilde{4}x_1+\tilde{3}x_2\underset{=}{\prec}\tilde{b}_3=\tilde{45}\\ \tilde{a}_{41}x_1+\tilde{a}_{42}x_2=\ \tilde{3}x_1+\tilde{1}x_2\underset{=}{\prec}\tilde{b}_4=\tilde{30}\\ x_1\le 0;\quad x_2\le 0\end{cases} \tag{7}$$

And the membership functions of the coefficients of the objective functions and constraints are set up as follows:

$$\mu_{\tilde{c}_{11}}(x)=\begin{cases}0 & x<3\text{ or }6<x\\ (x^2-9)/9 & 3\le x<4\\ 1 & x=4\\ (36-x^2)/20 & 4<x\le 6\end{cases}\qquad \mu_{\tilde{c}_{12}}(x)=\begin{cases}0 & x<1\text{ or }4<x\\ (x^2-1)/3 & 1\le x<2\\ 1 & x=2\\ (16-x^2)/12 & 2<x\le 4\end{cases}$$

$$\mu_{\tilde{c}_{21}}(x)=\begin{cases}0 & x<-2.5\text{ or }-1<x\\ (6.25-x^2)/2.25 & -2.5\le x<-2\\ 1 & x=-2\\ (x^2-1)/3 & -2<x\le -1\end{cases}\qquad \mu_{\tilde{c}_{22}}(x)=\begin{cases}0 & x<3\text{ or }6<x\\ (x^2-9)/9 & 3\le x<4\\ 1 & x=4\\ (36-x^2)/20 & 4<x\le 6\end{cases}$$

$$\mu_{\tilde{a}_{11}}(x)=\begin{cases}0 & x<-2\text{ or }-0.5<x\\ (4-x^2)/3 & -2\le x<-1\\ 1 & x=-1\\ (x^2-0.25)/0.75 & -1<x\le -0.5\end{cases}\qquad \mu_{\tilde{a}_{12}}(x)=\begin{cases}0 & x<2\text{ or }5<x\\ (x^2-4)/5 & 2\le x<3\\ 1 & x=3\\ (25-x^2)/16 & 3<x\le 5\end{cases}$$

$$\mu_{\tilde{a}_{21}}(x)=\begin{cases}0 & x<0.5\text{ or }2<x\\ (x^2-0.25)/0.75 & 0.5\le x<1\\ 1 & x=1\\ (4-x^2)/3 & 1<x\le 2\end{cases}\qquad \mu_{\tilde{a}_{22}}(x)=\begin{cases}0 & x<2\text{ or }5<x\\ (x^2-4)/5 & 2\le x<3\\ 1 & x=3\\ (25-x^2)/16 & 3<x\le 5\end{cases}$$

$$\mu_{\tilde{c}_{31}}(x)=\begin{cases}0 & x<3\text{ or }6<x\\ (x^2-9)/9 & 3\le x<4\\ 1 & x=4\\ (36-x^2)/20 & 4<x\le 6\end{cases}\qquad \mu_{\tilde{a}_{32}}(x)=\begin{cases}0 & x<2\text{ or }5<x\\ (x^2-4)/5 & 2\le x<3\\ 1 & x=3\\ (25-x^2)/16 & 3<x\le 5\end{cases}$$

$$\mu_{\tilde{a}_{41}}(x)=\begin{cases}0 & x<2 \text{ or } 5<x\\ (x^2-4)/5 & 2\le x<3\\ 1 & x=3\\ (25-x^2)/16 & 3<x\le 5\end{cases}\qquad \mu_{\tilde{a}_{42}}(x)=\begin{cases}0 & x<0.5 \text{ or } 2<x\\ (x^2-0.25)/0.75 & 0.5\le x<1\\ 1 & x=1\\ (4-x^2)/3 & 1<x\le 2\end{cases}$$

$$\mu_{\tilde{b}_1}(x)=\begin{cases}0 & x<20 \text{ or } 23<x\\ (x^2-400)/41 & 20\le x<21\\ 1 & x=21\\ (529-x^3)/88 & 21<x\le 23\end{cases}\qquad \mu_{\tilde{b}_2}(x)=\begin{cases}0 & x<26 \text{ or } 29<x\\ (x^2-676)/53 & 26\le x<27\\ 1 & x=27\\ (841-x^2)/112 & 27<x\le 29\end{cases}$$

$$\mu_{\tilde{b}_3}(x)=\begin{cases}0 & x<44 \text{ or } 47<x\\ (x^2-1936)/89 & 44\le x<45\\ 1 & x=.45\\ (2209-x^2)/184 & 45<x\le 47\end{cases}\qquad \mu_{\tilde{b}_4}(x)=\begin{cases}0 & x<29 \text{ or } 32<x\\ (x^2-841)/59 & 29\le x<30\\ 1 & x=.30\\ (1024-x^2)/124 & 30<x\le 32\end{cases}$$

Suppose the DM wants to set up the fuzzy goals whose membership functions are listed as follows

$$g_1(x)=\begin{cases}0 & x<14 \text{ or } 37<x\\ (x^2-196)/245 & 14\le x<21\\ 1 & x=21\\ (1369-x^2)/928 & 21<x\le 37\end{cases} \tag{8}$$

$$g_2(x)=\begin{cases}0 & x<6.5 \text{ or } 25<x\\ (x^2-42.25)/114 & 6.5\le x<12.5\\ 1 & x=12.5\\ (625-x^2)/468.75 & 12.5<x\le 25\end{cases} \tag{9}$$

And the initial value of satisfactory degree α is set to 0.2.

Based on the new fuzzy goals $(\tilde{g}_1,\tilde{g}_2)$ specified in (8) and (9) and degree α = 0.2, By the α-FGA algorithm, the running result is that the decision variables are $x_1^*=2.8992$ and $x_2^*=4.9829$, and two fuzzy objective functions are $\tilde{f}_1^*(x_1^*,x_2^*)=\tilde{f}_1^*(2.8992,4.9829)=2.8992\tilde{c}_{11}+4.9829\tilde{c}_{12}$ and $\tilde{f}_2^*(x_1^*,x_2^*)=\tilde{f}_2^*(2.8992,4.9829)=2.8992\tilde{c}_{21}+4.9829\tilde{c}_{22}$, which are shown as in Figure 1

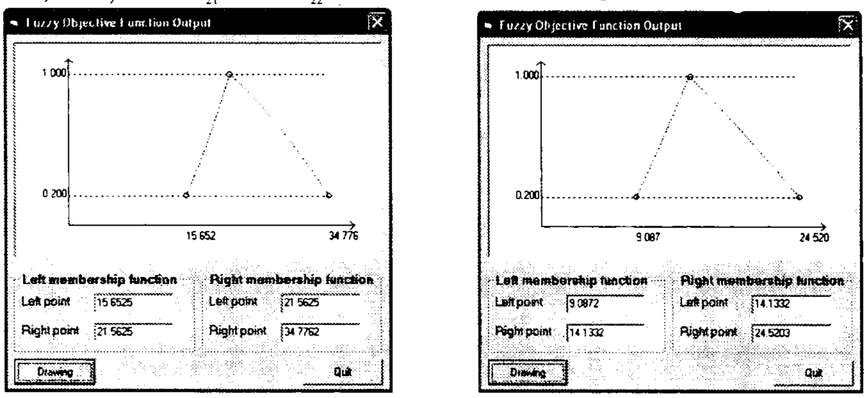

Figure 1: Running result in fuzzy objection functions by α-FGA algorithm

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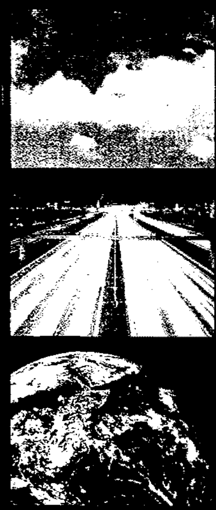

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