MATHEMATICS OF COMPUTATION
Volume 78, Number 265, January 2009, Pages 421–429
S 0025-5718(08)02147-9
Article electronically published on September 5, 2008

SUPERHARMONIC NUMBERS

GRAEME L. COHEN

Abstract. Let $\tau(n)$ denote the number of positive divisors of a natural number $n > 1$ and let $\sigma(n)$ denote their sum. Then n is *superharmonic* if $\sigma(n) \mid n^k\tau(n)$ for some positive integer k. We deduce numerous properties of superharmonic numbers and show in particular that the set of all superharmonic numbers is the first nontrivial example that has been given of an infinite set that contains all perfect numbers but for which it is difficult to determine whether there is an odd member.

1. Introduction

If the harmonic mean of the positive divisors of a natural number $n > 1$ is an integer, then n is said to be *harmonic*. Equivalently, n is harmonic if $\sigma(n) \mid n\tau(n)$, where $\tau(n)$ and $\sigma(n)$ denote the number of positive divisors of n and their sum, respectively. Harmonic numbers were introduced by Ore [8], and named (some 15 years later) by Pomerance [9].

Harmonic numbers are of interest in the investigation of perfect numbers (numbers n for which $\sigma(n) = 2n$), since all perfect numbers are easily shown to be harmonic. All known harmonic numbers are even. If it could be shown that there are no odd harmonic numbers, then this would solve perhaps the most longstanding problem in mathematics: whether or not there exists an odd perfect number. Recent articles in this area include those by Cohen and Deng [3] and Goto and Shibata [5].

In [3], the following was proposed as a scheme to demonstrate the nonexistence of odd perfect numbers:

Let P_0 be the set of perfect numbers and devise a finite sequence $P_1, \ldots, P_z$ of sets of natural numbers such that

- *$P_0 \subset P_1 \subset \cdots \subset P_z$, all inclusions being strict,*
- *it seems to be difficult to find odd members of $P_1, \ldots, P_{z-1}$ as each is constructed, in turn,*
- *it can be proved that there are no odd members of P_z.*

(It would be understood that the sets $P_1, P_2, \ldots$ must not be trivially or artificially defined, for example by letting P_1 be the union of the set of all even numbers and the set of all perfect numbers.)

Let P_1 stand for the set of harmonic numbers. In [3], it was suggested that a candidate for P_2 could be the set of all numbers $n > 1$ such that $\sigma_k(n) \mid n^k\tau(n)$ where k is a positive integer and $\sigma_k(n)$ denotes the sum of the kth powers of the

Received by the editor April 12, 2007 and, in revised form, January 22, 2008.
2000 *Mathematics Subject Classification.* Primary 11A25, 11Y70.

positive divisors of n. For such a number, the harmonic mean of the kth powers of the divisors of n is an integer. Although a number of properties of this P_2 were obtained, the authors could not show that the inclusion $P_1 \subset P_2$ is strict. The article also described other examples from the literature that could be specified as P_1, such as the set of multiperfect numbers, that is, natural numbers $n > 1$ for which $\sigma(n) = ln$, for some positive integer l.

We observe at this point that there are 30 harmonic numbers up to 10^6, of which just four are perfect numbers. It is not known, however, whether the set P_1 of harmonic numbers is infinite or not. In the present paper, a different generalisation of this set P_1 is given which, in stark contrast to the set P_2 just described, is readily shown to be infinite. Up to 10^6, it contains 3453 members, all of them even. If it can be shown that this set contains no odd members, then it would follow that there are no odd perfect numbers. More data will be given below.

2. Superharmonic numbers

Our new candidate for P_2 is the set of *superharmonic numbers*: natural numbers $n > 1$ for which $\sigma(n) \mid n^k\tau(n)$, for some positive integer k. If n is superharmonic, then the smallest k such that $\sigma(n) \mid n^k\tau(n)$ is called the *index* of n, denoted by $\underline{k}(n)$. Harmonic numbers have index 1. Table 1 gives the number of superharmonic numbers in three intervals of length $2 \cdot 10^8$, counted according to their index; all 141,336 superharmonic numbers in these intervals are even. Table 2 gives the smallest superharmonic numbers with index 1, 2, ..., 24.

Regarding notation, roman letters always denote integers, with p and q reserved for primes. We write $v_p(m)$ for the exponent on p in m, that is, the integer $a \geq 0$ such that $p^a \mid m$ and $p^{a+1} \nmid m$. It is often convenient to write $p^a \parallel m$ when $v_p(m) = a$.

We recall that, if $n = \prod p^a$ represents the canonical decomposition of n into primes, then

$$\tau(n) = \prod (a+1), \qquad \sigma(n) = \prod (1 + p + p^2 + \cdots + p^a) = \prod \frac{p^{a+1} - 1}{p - 1}.$$

The functions τ and σ are multiplicative.

The definition of a superharmonic number does not provide an effective search technique when n is very large, as in later examples (see Table 3) where we have n around 10^{1500}, with $\underline{k}(n)$ exceeding 490. Our first result is far more useful in the search for numerical information.

Theorem 1. *Let n be a superharmonic number. Then, for each prime divisor p of $\sigma(n)$, $p \mid n\tau(n)$. Further, if $v_p(\sigma(n)) > v_p(\tau(n))$, then $p \mid n$. The index of n is given by*

$$\underline{k}(n) = \max_{\substack{p \mid \sigma(n) \\ v_p(n) \neq 0}} \left\lceil \frac{v_p(\sigma(n)) - v_p(\tau(n))}{v_p(n)} \right\rceil.$$

TABLE 1. The number of superharmonic numbers, counted according to their index $\underline{k}$, in each of the three given intervals.

$\underline{k}$	$[2, 2 \cdot 10^8]$	$[10^9 + 1, 10^9 + 2 \cdot 10^8]$	$[10^{10} + 1, 10^{10} + 2 \cdot 10^8]$
1	81	3	0
2	6317	1062	292
3	14689	3708	1255
4	16998	4868	1773
5	15804	4997	2032
6	13524	4315	1802
7	10379	3667	1691
8	7325	3243	1297
9	4658	1940	1067
10	2857	1253	692
11	1683	880	508
12	994	690	326
13	572	268	159
14	344	183	130
15	172	152	114
16	101	105	72
17	54	28	29
18	42	23	14
19	13	21	15
20	11	8	10
21	3	0	4
22	4	3	3
23	0	7	2
Totals	96625	31424	13287

Proof. The first two statements follow from the definition of a superharmonic number, since an integer k exists such that $p^a \mid n^k\tau(n)$ for each p, where $a = v_p(\sigma(n))$. Then also

$$kv_p(n) + v_p(\tau(n)) \geq v_p(\sigma(n)).$$

Let $k'(p)$ be the smallest k for which this holds. We set $k'(p) = 0$ if $v_p(n) = 0$ and observe that $v_p(n) > 0$ for at least one p, since $\sigma(n) > \tau(n)$. Then $\sigma(n) \mid n^k\tau(n)$ provided $k \geq k'(p)$ for all p. The result follows. □

It is easy to see, as Ore [8] showed for harmonic numbers, that a single prime power cannot be superharmonic. The proof by Callan [2] (see also Pomerance [9]) that the only harmonic numbers with two distinct prime factors are the even perfect numbers does not carry through to superharmonic numbers although it would seem to be true.

The next theorem quickly implies that there are infinitely many superharmonic numbers.

Theorem 2. *Let q_j denote the jth prime ($q_1 = 2$, $q_2 = 3$, ...) and put $M_j = q_1q_2\cdots q_j$. Then M_j is superharmonic for all $j \geq 2$.*

TABLE 2. The smallest superharmonic number n with index $\underline{k}(n)$. If $\underline{k}(n) \geq 25$, then $n > 10^9$.

$\underline{k}(n)$	n	$\underline{k}(n)$	n
1	$6 = 2 \cdot 3$	13	$540606 = 2 \cdot 3 \cdot 11 \cdot 8191$
2	$30 = 2 \cdot 3 \cdot 5$	14	$344022 = 2 \cdot 3 \cdot 7 \cdot 8191$
3	$102 = 2 \cdot 3 \cdot 17$	15	$2309862 = 2 \cdot 3 \cdot 47 \cdot 8191$
4	$186 = 2 \cdot 3 \cdot 31$	16	$786426 = 2 \cdot 3 \cdot 131071$
5	$1146 = 2 \cdot 3 \cdot 191$	17	$4718586 = 2 \cdot 3 \cdot 786431$
6	$762 = 2 \cdot 3 \cdot 127$	18	$3145722 = 2 \cdot 3 \cdot 524287$
7	$8022 = 2 \cdot 3 \cdot 7 \cdot 191$	19	$33030102 = 2 \cdot 3 \cdot 7 \cdot 786431$
8	$5334 = 2 \cdot 3 \cdot 7 \cdot 127$	20	$22020054 = 2 \cdot 3 \cdot 7 \cdot 524287$
9	$35526 = 2 \cdot 3 \cdot 31 \cdot 191$	21	$146276166 = 2 \cdot 3 \cdot 31 \cdot 786431$
10	$23622 = 2 \cdot 3 \cdot 31 \cdot 127$	22	$97517382 = 2 \cdot 3 \cdot 31 \cdot 524287$
11	$145542 = 2 \cdot 3 \cdot 127 \cdot 191$	23	$599260422 = 2 \cdot 3 \cdot 127 \cdot 786431$
12	$49146 = 2 \cdot 3 \cdot 8191$	24	$399506694 = 2 \cdot 3 \cdot 127 \cdot 524287$

Proof. The number 2 is not superharmonic and the number $2 \cdot 3$ is (see Table 2). For $j \geq 3$, consider the quotient

$$\theta = \frac{M_j^k \tau(M_j)}{\sigma(M_j)} = \frac{(2 \cdot 3 q_3 q_4 \cdots q_j)^k \cdot 2^j}{\sigma(2 \cdot 3 q_3 q_4 \cdots q_j)} = \frac{(2 \cdot 3 q_3 q_4 \cdots q_j)^k}{3 \cdot \frac{3+1}{2^2} \frac{q_3+1}{2} \frac{q_4+1}{2} \cdots \frac{q_j+1}{2}},$$

where we have used the multiplicativity of τ and σ. Every prime divisor of the denominator is less than q_j and hence the denominator divides M_j^k for suitably large k. For such k, θ is an integer, so M_j is superharmonic. □

Corollary 3. *There are infinitely many superharmonic numbers. Furthermore, the set $\{\underline{k}(n) : n$ superharmonic$\}$ of all possible indices is infinite.*

Proof. There are infinitely many superharmonic numbers because there are infinitely many prime numbers. Moreover, there are infinitely many primes congruent to 3 (mod 4), so $v_2(\sigma(M_j))$ increases indefinitely with j. Then k must be increasingly large, as j increases, to allow θ to be an integer. This implies the second statement of the corollary. □

To illustrate the growth of $\underline{k}(M_j)$ with j, its values are given in Table 3 for $2 \leq j \leq 40$ and $493 \leq j \leq 500$. The sequence $\{\underline{k}(M_j)\}_{j \geq 2}$ is necessarily increasing, but Table 3 shows that it is not strictly increasing.

The following result shows that every natural number divides a superharmonic number. This provides a second proof that there are infinitely many such numbers.

Theorem 4. *For any natural number N, there exists a superharmonic number which is a multiple of N.*

Proof. Set $N_0 = N$ if N is even and $N_0 = 2N$ if N is odd. Suppose N_0 is not superharmonic. Let p_1 be the smallest prime such that $p_1 \mid \sigma(N_0)$ and $p_1 \nmid N_0$; put $N_1 = N_0 p_1$. Let p_2 be the smallest prime such that $p_2 \mid \sigma(N_1)$ and $p_2 \nmid N_1$; put

TABLE 3. The index $\underline{k}(M_j)$ of M_j, the product of the first j primes, for $2 \le j \le 40$ and $493 \le j \le 500$.

j	$\underline{k}(M_j)$	$\log_{10}(M_j)$	j	$\underline{k}(M_j)$	$\log_{10}(M_j)$	j	$\underline{k}(M_j)$	$\log_{10}(M_j)$
2	1	0.778	18	15	23.069	34	33	55.001
3	2	1.477	19	16	24.895	35	33	57.174
4	2	2.322	20	18	26.747	36	35	59.353
5	3	3.364	21	18	28.610	37	35	61.549
6	3	4.478	22	21	30.508	38	36	63.761
7	5	5.708	23	22	32.427	39	38	65.984
8	5	6.987	24	22	34.376	40	38	68.222
9	6	8.348	25	22	36.363	...		
10	7	9.811	26	22	38.367	493	486	1494.895
11	10	11.302	27	24	40.380	494	486	1498.443
12	10	12.870	28	25	42.409	495	487	1501.992
13	10	14.483	29	25	44.447	496	487	1505.541
14	11	16.117	30	25	46.500	497	488	1509.091
15	14	17.789	31	31	48.604	498	488	1512.642
16	14	19.513	32	32	50.721	499	490	1516.194
17	15	21.284	33	32	52.858	500	491	1519.746

$N_2 = N_1 p_2$. Let p_3 be the smallest prime such that $p_3 \mid \sigma(N_2)$ and $p_3 \nmid N_2$, and continue in this fashion until the process ends with a prime p_j such that $p_j \mid \sigma(N_{j-1})$ and $p_j \nmid N_{j-1}$. This end will occur when all prime divisors of $\sigma(p_j) = p_j + 1$, being less than p_j, occur among $p_1, \ldots, p_{j-1}$ and the prime divisors of N_0. Note that $2 \nmid p_1 p_2 \cdots p_j$. Put $N_j = N_{j-1} p_j$ and consider the quotient

$$\theta = \frac{N_j^k \tau(N_j)}{\sigma(N_j)} = \frac{(N_0 p_1 p_2 \cdots p_j)^k \cdot 2^j \tau(N_0)}{\sigma(N_0 p_1 p_2 \cdots p_j)} = \frac{(N_0 p_1 p_2 \cdots p_j)^k \tau(N_0)}{\sigma(N_0) \dfrac{p_1+1}{2} \dfrac{p_2+1}{2} \cdots \dfrac{p_j+1}{2}}.$$

We have again used the multiplicativity of τ and σ. By construction, every prime factor of the denominator divides $N_0 p_1 p_2 \cdots p_j$ so that, for suitably large k, the numerator of θ is divisible by its denominator. For such k, θ is an integer, so N_j is superharmonic. □

3. ODD SUPERHARMONIC NUMBERS

There are infinitely many superharmonic numbers, and all of those known to date are even. Furthermore, there are infinitely many superharmonic numbers that are squarefree, that is, not divisible by p^2 for any prime p, as is clear from Theorem 2. But, as we show below, no odd superharmonic number, if there is one, can be squarefree. Ore [8] proved the corresponding result for harmonic numbers long ago. Another long-established result for harmonic numbers, due to Garcia [4], is that if n is odd and harmonic and $p^a \parallel n$, then $p^a \equiv 1 \pmod 4$. These properties for superharmonic numbers are given in the following theorem.

Theorem 5. *Let N be an odd superharmonic number. Then* (i) *N cannot be squarefree,* (ii) *if $p^a \parallel n$, then $p^a \equiv 1 \pmod 4$.*

Proof. (i) Suppose $N = p_1 p_2 \cdots p_j$ for odd primes $p_1 < p_2 < \cdots < p_j$. Consider the quotient

$$\theta = \frac{N^k \tau(N)}{\sigma(N)} = \frac{(p_1 \cdots p_j)^k \cdot 2^j}{\sigma(p_1 \cdots p_j)} = \frac{(p_1 \cdots p_j)^k}{\dfrac{p_1+1}{2} \cdots \dfrac{p_j+1}{2}}.$$

The prime factors of $(p_1+1)/2$ are less than p_1, so the denominator cannot be a factor of the numerator, for any k. Then θ cannot be an integer, so N is not superharmonic.

(ii) Suppose $p^a \not\equiv 1 \pmod 4$, so that $p \equiv 3 \pmod 4$ and a is odd. Write $a = 2^j b - 1$, where $b \geq 1$ is odd and $j \geq 1$, and consider the factorisation

$$\begin{aligned}\sigma(p^a) &= \frac{p^{a+1}-1}{p-1} = \frac{(p^b)^{2^j}-1}{p-1} \\ &= \frac{p^b-1}{p-1}(p^b+1)(p^{2b}+1)(p^{2^2 b}+1)\cdots(p^{2^{j-1}b}+1).\end{aligned}$$

Since $p^b \equiv 3 \pmod 4$, this implies that $v_2(\sigma(p^a)) \geq j+1$, whereas $v_2(\tau(p^a)) = j$. On the other hand, if $p^a \equiv 1 \pmod 4$, then $v_2(\sigma(p^a)) = v_2(\tau(p^a))$. So $v_2(\sigma(N)) > v_2(\tau(N))$ if $p^a \equiv 3 \pmod 4$ for at least one prime factor p of N. In that case, $n^k\tau(N)/\sigma(N)$ cannot be an integer, for any k, so N is not superharmonic. The result follows. □

Ore's proof of the harmonic version of (i) was different from the above. By his approach, he was able to show further that the only squarefree harmonic number, odd or even, is 6. Of course, that does not extend to superharmonic numbers. The proof given here of (ii) is essentially Garcia's.

As one consequence of Theorem 5, not previously noted for harmonic numbers, we have the following. The proof is omitted.

Corollary 6. *Suppose $p^2 M$ is an odd superharmonic number, where M is squarefree and not divisible by the prime p. Then $p \equiv \pm 1 \pmod{12}$.*

4. Conclusion. Thoughts on odd perfect numbers

To return to the scheme outlined in the Introduction, let P_1 be the set of harmonic numbers and P_2 the set of superharmonic numbers. The earlier attempt, in [3], to find a nontrivial set that extends P_1, yet for which there seem to be no odd members, was not satisfactory for the more general purpose, nor are such extensions as the set of natural numbers $n > 1$ for which $\sigma(n) \mid n\tau^2(n)$, since 3 is a member of this set. Superharmonic numbers were arrived at by wondering first about numbers $n > 1$ for which $\sigma(n) \mid n^2\tau(n)$.

The set P_0 of perfect numbers may well be infinite. Realistically, this depends on showing that the set of Mersenne primes (primes of the form $2^p - 1$) is infinite. It is unlikely that any other approach could be used to show that P_1 is infinite, also. So the fact that P_2 is provably infinite is a bonus.

What about the density of P_2? The density of a set P is

$$\delta_P = \lim_{x \to \infty} \frac{N_P(x)}{x},$$

if the limit exists, where x is a real variable and $N_P(x)$ is the number of elements of P not exceeding x. It was shown by Kanold [6] that P_1 has zero density, but a similar argument does not carry through to P_2 (nor to the set of superharmonic numbers of index k, for any given $k > 1$), but see the Appendix. The preferred next step in this scheme is to seek P_3, such that $P_2 \subset P_3$, P_3 has no apparent odd members, and $\delta_{P_2} < \delta_{P_3}$.

Compared to superharmonic numbers, the tighter definition of odd perfect numbers allows more to be determined regarding their prime factor decomposition, if there is such a number. In place of Theorem 5, it has long been known that an odd perfect number must have the form $p_0^{b_0} p_1^{2b_1} p_2^{2b_2} \cdots p_t^{2b_t}$, where $p_0 \equiv b_0 \equiv 1 \pmod 4$. Furthermore, Nielsen [7] has recently proved that $t \geq 8$, the first improvement on that bound for more than 25 years, and since 1991 it has been known that there are no odd perfect numbers below 10^{300} (Brent et al. [1]). Anything at all close to such results for odd superharmonic numbers, or odd harmonic numbers, would be very difficult.

Searches for odd perfect numbers, and more generally for a multiperfect number n (even or odd), must make use of the fact that the set of prime factors of $\sigma(n)$ must almost coincide with the set Q of prime factors of n itself; in particular, if $p^a \parallel n$, then the prime factors of $\sigma(p^a)$ mostly must remain within Q. The proofs of most of the theorems in this paper depend essentially on the same requirement. That is, the likelihood of finding an odd superharmonic number would seem to be about the same as that of finding an odd perfect number.

5. Appendix

I am grateful to the referee of this paper for the following theorem.

Theorem 7. *The set P_2 of superharmonic numbers has density zero.*

Proof. We must show that $\delta_{P_2} = 0$ or, equivalently, $N_{P_2}(x) = o(x)$ as $x \to \infty$. For any set P, write $\#\{n \leq x : n \in P\}$ for $N_P(x)$, with variations on this notation that will be obvious. Let $P^+(n)$ denote the largest prime factor of an integer $n \geq 2$, and put $P^+(1) = 1$.

Define

$$\Psi(x, y) = \#\{n \leq x : P^+(n) \leq y\}, \quad x \geq y \geq 2.$$

It is shown in Tenenbaum [10] that, for large x,

$$\Psi(x, y) \ll x e^{-u/2}, \tag{1}$$

where $u = (\log x)/(\log y)$, uniformly in y. Setting $y = \exp((\log x)^{2/3})$, it follows that $\Psi(x, y) = o(x)$, so to prove that $\delta_{P_2} = 0$ it suffices to show that

$$\#\{n \leq x : n \in P_2,\ P^+(n) > y\} = o(x).$$

Moreover, since

$$\#\{n \leq x : q = P^+(n) > y,\ q^2 \mid n\} \leq \sum_{y < q \leq x} \sum_{\substack{n \leq x \\ q^2 \mid n}} 1 \ll \sum_{q > y} \frac{x}{q^2} \ll \frac{x}{y},$$

it suffices to show that

$$\#\{n \leq x : n \in P_2,\ q = P^+(n) > y,\ q \parallel n\} = o(x).$$

In fact, it it is enough to show that

$$\#\{n \le x : n \in P_2,\ q = P^+(n) > y,\ q \parallel n,\ P^+(q+1) > z\} = o(x), \tag{2}$$

where $z = \exp((\log x)^{1/3})$. To see why, observe that

$$\#\{n \le x : q = P^+(n) > y,\ q \parallel n,\ P^+(q+1) \le z\} \le \sum_{\substack{y<q\le x \\ P^+(q+1)\le z}} \sum_{\substack{n\le x \\ q\parallel n}} 1$$

$$\le x \sum_{\substack{y<q\le x \\ P^+(q+1)\le z}} \frac{1}{q} \ll x \sum_{\substack{y<n\le x \\ P^+(n)\le z}} \frac{1}{n}\,.$$

By partial summation, the last sum is

$$\int_y^x \frac{d\Psi(u,z)}{u} = \frac{\Psi(x,z)}{x} - \frac{\Psi(y,z)}{y} + \int_y^x \frac{\Psi(u,z)}{u^2}\,du,$$

and, using the bound in (1), each term on the right is easily seen to be $o(1)$ as $x \to \infty$. Thus, we need only prove (2).

Let n lie in the subset of P_2 indicated in (2). Write $n = qm$, where $q = P^+(n)$, so $q \nmid m$, and put $r = P^+(q+1)$. Then, we have $\sigma(n) \mid n^k\tau(n)$ for some k if and only if

$$(q+1)\sigma(m) \mid 2(qm)^k\tau(m).$$

Since q and $q+1$ are coprime, it follows that $r \mid 2m^k\tau(m)$. Suppose $r \mid \tau(m)$. Then $p^{r-1} \mid n$ for some prime p, so $x \ge n > 2^{r-1} > 2^{z-1}$, but this is not possible for $z = \exp((\log x)^{1/3})$. Therefore, $r \nmid \tau(m)$, and $r \ne 2$, so $r \mid m$. That is, $qP^+(q+1) \mid n$. But the number of such integers n up to x does not exceed

$$\sum_{\substack{y<q\le x \\ P^+(q+1)>z}} \frac{x}{qP^+(q+1)} < \frac{x}{z} \sum_{\substack{y<q\le x \\ P^+(q+1)>z}} \frac{1}{q} < \frac{x}{z} \sum_{n\le x} \frac{1}{n} \ll \frac{x\log x}{z} = o(x),$$

which proves (2). □

References

[1] R. P. Brent, G. L. Cohen and H. J. J. te Riele, "Improved techniques for lower bounds for odd perfect numbers", *Math. Comp.*, **57** (1991), 857–868. MR1094940 (92c:11004)

[2] D. Callan, Problems and Solutions: Solutions: 6616, *Amer. Math. Monthly*, **99** (1992), 783–789. MR1542194

[3] G. L. Cohen and Moujie Deng, "On a generalisation of Ore's harmonic numbers", *Nieuw Archief voor Wiskunde*, **16** (1998), 161–172. MR1680101 (2000k:11008)

[4] M. Garcia, "On numbers with integral harmonic mean", *Amer. Math. Monthly*, **61** (1954), 89–96. MR0059291 (15:506d)

[5] T. Goto and S. Shibata, "All numbers whose positive divisors have integral harmonic mean up to 300", *Math. Comp.*, **73** (2004), 475–491. MR2034133 (2004j:11005)

[6] H.-J. Kanold, "Über das harmonische Mittel der Teiler einer natürlichen Zahl", *Math. Ann.*, **133** (1957), 371–374. MR0089219 (19:635f)

[7] P. P. Nielsen, "Odd perfect numbers have at least nine distinct prime factors", *Math. Comp.*, **76** (2007), 2109–2126. MR2336286

[8] O. Ore, "On the averages of the divisors of a number", *Amer. Math. Monthly*, **55** (1948), 614–619. MR0027292 (10:284a)

[9] C. Pomerance, *On a problem of Ore: Harmonic numbers*, unpublished manuscript (1973).
[10] G. Tenenbaum, *Introduction to Analytic and Probabilistic Number Theory*, Cambridge University Press, Cambridge (1995). MR1342300 (97e:11005b)

Department of Mathematical Sciences, University of Technology, Sydney, Broadway, NSW 2007, Australia

Current address: 1201/95 Brompton Road, Kensington, NSW 2033, Australia

E-mail address: `g.cohen@bigpond.net.au`