SOME CONJECTURES CONCERNING SUMS OF ODD POWERS OF FIBONACCI AND LUCAS NUMBERS

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ABSTRACT. This paper contains observations, conjectures, and open questions concerning two finite sums that involve Fibonacci and Lucas numbers. Certain authors have become aware of the contents of this (hitherto unpublished) manuscript, and have made inroads into some of the challenges it poses. It was felt, therefore, that the contents of the original manuscript ought to be made public.

1. INTRODUCTION

Some years ago, when Professor Gerald Bergum was editor, we submitted a manuscript for possible publication in this journal. This manuscript contained observations, conjectures, and open questions concerning two finite sums. One sum involved odd powers of certain Fibonacci numbers, and the other involved odd powers of certain Lucas numbers. At that time Professor Bergum suggested (in a letter dated January 5, 1998) that, for publication, we would need to prove some of the conjectures, or answer some of the questions. We acknowledge that this was a reasonable editorial decision by Professor Bergum.

On May 15, 1998, we sent the manuscript, by facsimile, to Professor Curtis Cooper, not for publication, but in the hope that he and his coworkers could make progress by way of supplying proofs, or answering some of the questions. On August 3, 2001, we received correspondence, via e-mail, from Professor Cooper. He informed us that he and Michael Wiemann (then a graduate student) had proved a divisibility result that was inspired by one of our questions. This resulted in the publication [4]. Some years later, Ozeki [2] answered a central question that was posed in our original (unpublished) submission. Soon after, Prodinger [3] greatly extended the work in [2]. Specific details regarding the contributions of the above authors is given in Section 4.

Interestingly, in the second sentence of his introduction, Prodinger states that he had not cited our original submission. It is also not clear if Ozeki had cited our original submission. In addition, our original submission contains questions that, to the best of our knowledge, remain unanswered. For these reasons, and also because it has motivated three research papers, we feel that our original submission belongs in the public domain. Except for some streamlining suggested by the referee, what we originally presented to Professor Bergum begins in the paragraph that follows, and has the same title as above.

Clary and Hemenway [1] obtained, among other things, the charming sum

$$\sum_{k=1}^{n} F_{2k}^3 = \begin{cases} \frac{1}{4}F_n^2 L_{n+1}^2 F_{n-1} L_{n+2}, & \text{if } n \text{ is even;} \\ \frac{1}{4}L_n^2 F_{n+1}^2 L_{n-1} F_{n+2}, & \text{if } n \text{ is odd.} \end{cases} \tag{1.1}$$

They obtained this result by first showing that

$$4\sum_{k=1}^{n} F_{2k}^3 = (F_{2n+1}-1)^2 (F_{2n+1}+2). \tag{1.2}$$

This prompted us to examine the sums $\sum_{k=1}^{n} F_{2k}^{2s+1}$ for small positive integers s, and we found that our results had something in common with (1.2). In particular, we noticed the occurrence of the factor $(F_{2n+1}-1)^2$. We also explored the sums $\sum_{k=1}^{n} L_{2k}^{2s+1}$ for small values of s, and obtained the factor $(L_{2n+1}-1)$. Our observations have led us to some conjectures that we state in the next section.

2. THE CONJECTURES

Before stating our conjectures we indicate briefly, via an example, how we were led to them. We require

$$5F_m^3 = F_{3m} + 3(-1)^{m+1} F_m, \tag{2.1}$$

and

$$25F_m^5 = F_{5m} + 5(-1)^{m+1} F_{3m} + 10F_m, \tag{2.2}$$

both of which can be established with the use of the Binet forms.

From (2.2) we have, for $n \geq 1$,

$$\begin{aligned} 25\sum_{k=1}^{n} F_{2k}^5 &= \sum_{k=1}^{n} F_{10k} - 5\sum_{k=1}^{n} F_{6k} + 10\sum_{k=1}^{n} F_{2k} \\ &= \frac{F_{5(2n+1)}-5}{L_5} - 5\frac{F_{3(2n+1)}-2}{L_3} + 10(F_{2n+1}-1), \end{aligned} \tag{2.3}$$

where we have summed the relevant geometric progressions. But from (2.1) and (2.2) we have $F_{3(2n+1)} = 5F_{2n+1}^3 - 3F_{2n+1}$ and $F_{5(2n+1)} = 25F_{2n+1}^5 - 25F_{2n+1}^3 + 5F_{2n+1}$. Making these substitutions into (2.3), we obtain

$$\begin{aligned} L_1L_3L_5\sum_{k=1}^{n} F_{2k}^5 &= 4F_{2n+1}^5 - 15F_{2n+1}^3 + 25F_{2n+1} - 14 \\ &= (F_{2n+1}-1)^2 (4F_{2n+1}^3 + 8F_{2n+1}^2 - 3F_{2n+1} - 14). \end{aligned} \tag{2.4}$$

This should be compared with (1.2).

Using the same approach, we have also found

$$L_1L_3L_5L_7\sum_{k=1}^{n} F_{2k}^7 = (F_{2n+1}-1)^2 P_5(F_{2n+1}), \tag{2.5}$$

and

$$L_1L_3L_5L_7L_9\sum_{k=1}^{n} F_{2k}^9 = (F_{2n+1}-1)^2 P_7(F_{2n+1}), \tag{2.6}$$

where

$$P_5(x) = 44x^5 + 88x^4 - 92x^3 - 272x^2 + 3x + 278,$$

and

$$\begin{aligned} P_7(x) = {} & 1276x^7 + 2552x^6 - 4488x^5 - 11528x^4 + 3932x^3 \\ & + 19392x^2 + 2188x - 15016. \end{aligned}$$

With these observations we state our first conjecture.

Conjecture 2.1. *Let $m \geq 1$ be a positive integer. Then the sum*

$$L_1 L_3 L_5 \cdots L_{2m+1} \sum_{k=1}^{n} F_{2k}^{2m+1} \tag{2.7}$$

can be expressed as $(F_{2n+1}-1)^2 P_{2m-1}(F_{2n+1})$, where $P_{2m-1}(x)$ is a polynomial of degree $2m-1$ with integer coefficients.

We mention that $\sum_{k=1}^{n} F_{2k} = F_{2n+1} - 1$, which appears as (39) in [1]. For the Lucas sequence we have found

$$\sum_{k=1}^{n} L_{2k} = L_{2n+1} - 1, \tag{2.8}$$

$$L_1 L_3 \sum_{k=1}^{n} L_{2k}^3 = (L_{2n+1}-1) Q_2(L_{2n+1}), \tag{2.9}$$

$$L_1 L_3 L_5 \sum_{k=1}^{n} L_{2k}^5 = (L_{2n+1}-1) Q_4(L_{2n+1}), \tag{2.10}$$

$$L_1 L_3 L_5 L_7 \sum_{k=1}^{n} L_{2k}^7 = (L_{2n+1}-1) Q_6(L_{2n+1}), \tag{2.11}$$

$$L_1 L_3 L_5 L_7 L_9 \sum_{k=1}^{n} L_{2k}^9 = (L_{2n+1}-1) Q_8(L_{2n+1}), \tag{2.12}$$

where the polynomials Q_i are given by

$$\begin{aligned}
Q_2(x) &= x^2 + x + 16,\\
Q_4(x) &= 4x^4 + 4x^3 + 79x^2 + 79x + 704,\\
Q_6(x) &= 44x^6 + 44x^5 + 1164x^4 + 1164x^3 + 12539x^2 + 12539x + 81664,\\
Q_8(x) &= 1276x^8 + 1276x^7 + 42856x^6 + 42856x^5 + 605356x^4 + 605356x^3\\
&\quad + 4688356x^2 + 4688356x + 24825856.
\end{aligned}$$

We can now state our second conjecture.

Conjecture 2.2. *Let $m \geq 0$ be an integer. Then the sum*

$$L_1 L_3 L_5 \cdots L_{2m+1} \sum_{k=1}^{n} L_{2k}^{2m+1}$$

can be expressed as $(L_{2n+1}-1) Q_{2m}(L_{2n+1})$, where $Q_{2m}(x)$ is a polynomial of degree $2m$ with integer coefficients.

3. Some Open Questions

We have not been able to discover a general pattern associated with the polynomials $P_i(x)$ and $Q_i(x)$. Is there a recurrence which generates them? We have observed that the polynomials $P_i(x)$ and $Q_i(x)$ given in this paper are irreducible over the rational numbers. Is this true in general?

4. An Update

Our original submission finished with Section 3. In this section we indicate what has been achieved, and also what remains to be done.

To understand the contribution of Wiemann and Cooper, we need to first explain the contribution of Ozeki. Ozeki essentially expanded $L_1 L_3 \cdots L_{2m+1} \sum_{k=1}^{n} F_{2k}^{2m+1}$ as a polynomial in F_{2n+1}, giving explicit expressions for the coefficients. Much earlier, Wiemann and Cooper, with no knowledge of this expansion, proved that

$$L_1 L_3 \cdots L_{2m+1} \sum_{i=0}^{m} \binom{2m+1}{m-i} (-1)^{m-i} \frac{F_{2i+1}}{L_{2i+1}} \tag{4.1}$$

is a multiple of 5^m. In fact, the expression in (4.1) turns out to be $-5^m c_m$, where c_m is the constant in Ozeki's expansion.

In a striking coincidence, Prodinger [3] explains that he had also obtained the expansion given by Ozeki, but had to concede priority to Ozeki by two months. Prodinger then went much further, giving expansions for $\sum_{k=0}^{n} F_{2k+\delta}^{2m+\varepsilon}$, and $\sum_{k=0}^{n} L_{2k+\delta}^{2m+\varepsilon}$, in which $(\delta, \varepsilon) \in \{0,1\}$, and n and m are nonnegative integers. Of these eight sums, two are given as polynomials in F_{2n+1}, two as polynomials in $F_{2(n+1)}$, two as polynomials in L_{2n+1}, and two as polynomials in $L_{2(n+1)}$.

In Conjectures 2.1 and 2.2, there remain the unresolved issues regarding the factors $(F_{2n+1}-1)^2$ and $(L_{2n+1}-1)$, respectively. There also remain the open questions relating to the irreducibility of the polynomials specified in Section 3. In light of Prodinger's substantial contribution, the issues in our original submission that remain open are now considerably amplified. There is much for the interested reader to pursue.

Finally, we would like to acknowledge our gratitude to an anonymous referee, whose comments have enhanced our presentation.

References

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MSC2000: 11B39

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