Deterministic distributed dense coding with stabilizer states

Guoming Wang[1, *] and Mingsheng Ying[1, †]

[1]*State Key Laboratory of Intelligent Technology and Systems, Department of Computer Science and Technology, Tsinghua University, Beijing, China, 100084*

(Dated: February 17, 2013)

We consider the possibility of using stabilizer states to perform deterministic dense coding among multiple senders and a single receiver. In the model we studied, the utilized stabilizer state is partitioned into several subsystems and then each subsystem is held by a distinct party. We present a sufficient condition for a stabilizer state to be useful for deterministic distributed dense coding with respect to a given partition plan. The corresponding protocol is also constructed. Furthermore, we propose a method to partially solve a more general problem of finding the set of achievable alphabet sizes for an arbitrary stabilizer state with respect to an arbitrary partition plan. Finally, our work provides a new perspective from the stabilizer formalism to view the standard dense coding protocol and also unifies several previous results in a single framework.

PACS numbers: 03.67.Mn, 03.67.Hk

I. INTRODUCTION

Since its proposal by Bennett and Wiesner in 1992 [1], dense coding has become one of the most important constituents in quantum information science. This communication protocol enables enhancement of the classical capacity of a noiseless quantum channel by using previously shared entanglement between the sender and the receiver. Up to now, researchers are still trying to thoroughly understand the power of a general bipartite entangled state in this task [2, 3, 4, 5, 6, 7, 8, 9, 10]. Typically there are two classes of dense coding schemes considered. One is called deterministic dense coding, which requires the protocol to succeed all the time; while the other one, performing unambiguous discrimination [11, 12, 13] on the final state, allows the protocol to succeed in a probabilistic manner.

Recently several authors have begun to consider the possibility of using a multipartite entangled state to perform dense coding among multiple parties [14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24]. In the multipartite case, many senders may *simultaneously* transmit classical information to a single receiver with the aid of *a priori* multipartite entanglement. Since each sender can only encode on his own subsystem, this scheme is called 'distributed dense coding'[17, 18]. Specifically, our model of deterministic distributed dense coding is as follows. Suppose ρ is an n-qudit state. Divide its n qudits into m groups $T_1, T_2, \ldots, T_m$ for some $2 \leq m \leq n$ and then distribute the subsystem T_i to the i-th party A_i, for $i = 1, 2, \ldots, m$. Now assume that A_i performs one out of b_i different quantum operations on the subsystem T_i, for $i = 1, 2, \ldots, m-1$. Then $A_1, A_2, \ldots, A_{m-1}$ send all their subsystems to A_m. If A_m can perfectly distinguish among all possible states, then this procedure actually accomplishes transmission of $log_2 b_i$ bits of classical information from A_i to A_m, for $i = 1, 2, \ldots, m-1$. In this case, we say that $(b_1, b_2, \ldots, b_{m-1})$ is an achievable alphabet size for ρ with respect to the grouping plan $T_1, T_2, \ldots, T_m$. Then for a given state ρ, any grouping strategy will define a region of achievable alphabet sizes. The most general question would be to determine such a region for all possible partition plans. For a more practical concern, we want to know whether the utilization of ρ really improves the classical capacity of the senders. So only when there exists an achievable alphabet size $(b_1, b_2, \ldots, b_{m-1})$ with $b_i > d^{|T_i|}$ for at least one $1 \leq i \leq m-1$ and $b_j \geq d^{|T_j|}$ for other $j \neq i$ (where $|T_i|$ denotes the number of qudits in T_i), we say that ρ is useful for deterministic distributed dense coding with respect to $T_1, T_2, \ldots, T_m$.

The purpose of this paper is to investigate the usefulness of stabilizer states for deterministic distributed dense coding. Stabilizer states have played an important role in quantum information theory, especially in the field of quantum error correction [25, 26] and cluster state quantum computation [27]. They can be described in an elegant and compact form named the stabilizer formalism [28, 29], which has also lead to novel perspectives to many phenomena in quantum information science and quantum mechanics [30, 31, 32]. We present a sufficient condition for a stabilizer state to be useful for deterministic distributed dense coding with respect to a given partition plan. The corresponding protocol is also constructed. Furthermore, we propose a method to partially solve the general problem of finding the region of achievable alphabet sizes for an arbitrary stabilizer state with respect to an arbitrary partition plan. Finally, our work provides a new perspective from the stabilizer formalism to view the standard dense coding protocol and also unifies several previous results in a single framework.

This paper is organized as follows. In Sec. II we briefly recall some fundamental facts about the stabilizer formalism. In Sec. III, we study the power of stabilizer states in deterministic distributed dense coding and also construct

*Electronic address: wgm00@mails.tsinghua.edu.cn
†Electronic address: yingmsh@tsinghua.edu.cn

the corresponding protocol. In Sec. IV we analyze several concrete examples by using our theorems. Finally, Sec. V summarizes our results.

II. PRELIMINARY

In this section, we review some fundamental facts about stabilizer state and its corresponding stabilizer formalism. Although in most literatures the notion of stabilizer state was put forward in the context of multiqubit systems, it can actually be generalized without essential difficulty to arbitrary higher-dimensional systems as well. Similar topics have also been explored in Refs. [33, 34, 35, 36]. So here we directly start with the general higher-dimensional case.

Consider a d-dimensional Hilbert space. Define

$$\begin{aligned} X^{(d)} &= \sum_{j=0}^{d-1} |j \oplus 1\rangle\langle j|, \\ Z^{(d)} &= \sum_{j=0}^{d-1} \omega^j |j\rangle\langle j|, \end{aligned} \tag{1}$$

where $\omega = e^{i\frac{2\pi}{d}}$ is the d-th root of unity over the complex field and the '$\oplus$' sign denotes addition modulo d. In what follows, without causing ambiguity, we will omit the superscript '(d)' in $X^{(d)}$ and $Z^{(d)}$. The matrices $\{\sigma_{a,b} = X^a Z^b : a, b = 0, 1, \ldots, d-1\}$ are considered as the generalized Pauli matrices over d-dimensional space. The commutation relations among them are given by

$$\sigma_{a,b}\sigma_{j,k} = \omega^{bj-ak}\sigma_{j,k}\sigma_{a,b}. \tag{2}$$

It can be checked that if d is even and ab is odd, the eigenvalues of $\sigma_{a,b}$ are $\omega^{1/2}, \omega^{c+1/2}, \omega^{2c+1/2}, \ldots, \omega^{d-c+1/2}$ for some factor c of d; otherwise, the eigenvalues of $\sigma_{a,b}$ are $1, \omega^c, \omega^{2c}, \ldots, \omega^{d-c}$ for some factor c of d.

The generalized Pauli group on n qudits $G_n^{(d)}$ is defined to consist all n-fold tensor products of generalized Pauli matrices over d-dimensional space, allowing overall phase factor γ^a, where $\gamma = \sqrt{\omega}$ and $0 \le a \le 2d-1$, i.e.

$$\begin{aligned} G_n^{(d)} &= \{\gamma^a \sigma_{i_1,j_1} \otimes \sigma_{i_2,j_2} \otimes \cdots \otimes \sigma_{i_n,j_n} : 0 \le a \le 2d-1, \\ &\quad 0 \le i_1, j_1, i_2, j_2, \ldots, i_n, j_n \le d-1\}. \end{aligned} \tag{3}$$

Actually, when d is odd, the introduction of γ is unnecessary and it can be replaced by ω. For a detailed discussion about this, one can see Ref. [34].

Define the map $\chi : G_n^{(d)} \rightarrow \mathbb{Z}_d^{2n}$ as follows: for $g = \gamma^c \sigma_{a_1,b_1} \otimes \sigma_{a_2,b_2} \otimes \cdots \otimes \sigma_{a_n,b_n}$, $\chi(g) = (a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n)$. From now on, all additions and multiplications of $\chi(g)$ will be taken over $\mathbb{Z}_d$. By Eq.(2), for any $g = \gamma^c \sigma_{a_1,b_1} \otimes \sigma_{a_2,b_2} \otimes \cdots \otimes \sigma_{a_n,b_n}$, $h = \gamma^{c'} \sigma_{a'_1,b'_1} \otimes \sigma_{a'_2,b'_2} \otimes \cdots \otimes \sigma_{a'_n,b'_n} \in G_n^{(d)}$, their commutation relation is

$$gh = \omega^{\sum_{i=1}^n (b_i a'_i - a_i b'_i)} hg = \omega^{\chi(g)\Lambda_n \chi(h)^T} hg, \tag{4}$$

where Λ_n is a $2n \times 2n$ matrix given by

$$\Lambda_n = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} \tag{5}$$

and I_n is the $n \times n$ identity matrix. So we have

$$ghg^\dagger = \omega^{\chi(g)\Lambda_n \chi(h)^T} h. \tag{6}$$

In particular, g and h commute if and only if

$$\chi(g)\Lambda_n \chi(h)^T = 0. \tag{7}$$

For a set of commuting operators $g_1, g_2, \ldots, g_k \in G_n^{(d)}$, we say that they are independent if $\forall i = 1, 2, \ldots, k$,

$$\langle g_1, g_2, \ldots, g_k \rangle \neq \langle g_1, g_2, \ldots, g_{i-1}, g_{i+1}, \ldots, g_k \rangle. \tag{8}$$

Define $G_n^{\prime(d)}$ to be the subset of $G_n^{(d)}$ composed of all the operators whose eigenvalues are of the form $1, \omega^c, \omega^{2c}, \ldots, \omega^{d-c}$ for some factor c of d. Now suppose $g_1, g_2, \ldots, g_n$ are independent commuting operators in $G_n^{\prime(d)}$. Let

$$S = \langle g_1, g_2, \ldots, g_n \rangle \tag{9}$$

be the Abelian subgroup generated by them. If there exists a unique state $|\psi_S\rangle$ (up to an overall phase) such that

$$g_i |\psi_S\rangle = |\psi_S\rangle, \ \forall i = 1, 2, \ldots, n, \tag{10}$$

we say that S is a *complete* stabilizer and $|\psi_S\rangle$ is stabilized by S. In this case, with the fact $\sum_{j=0}^{d-1} \omega^{j\lambda} = 0$, $\forall \lambda = 1, 2, \ldots, d-1$, one can verify

$$\rho_S \equiv |\psi_S\rangle\langle\psi_S| = \frac{1}{d^n} \prod_{i=1}^n (\sum_{j=0}^{d-1} g_i^j). \tag{11}$$

Suppose $S = \langle g_1, g_2, \ldots, g_k \rangle$, where $g_1, g_2, \ldots, g_k$ are independent commuting operations in $G_n^{\prime(d)}$. There is an extremely useful way of presenting the generators $g_1, g_2, \ldots, g_k$ using the check matrix M. This matrix is of size $k \times 2n$ and its i-th row is simply the representation row of the i-th generator $\chi(g_i)$, $\forall i = 1, 2, \ldots, k$. Since $g_1, g_2, \ldots, g_k$ mutually commute, the check matrix M satisfies

$$M\Lambda_n M^\dagger = 0. \tag{12}$$

For example, consider a four-qutrit system, i.e. $d = 3$, $n = 4$.

$$\begin{aligned} g_1 &= \sigma_{1,0} \otimes \sigma_{0,1} \otimes \sigma_{1,1} \otimes \sigma_{0,1}, \\ g_2 &= \sigma_{0,2} \otimes \sigma_{2,0} \otimes \sigma_{0,1} \otimes \sigma_{1,1}, \\ g_3 &= \sigma_{1,1} \otimes \sigma_{0,1} \otimes \sigma_{0,2} \otimes \sigma_{0,0} \end{aligned} \tag{13}$$

are three independent commuting operators from $G_4^{\prime(3)}$. Then the corresponding check matrix is

$$M = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 2 & 0 & 1 & 2 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 2 & 0 \end{pmatrix}. \tag{14}$$

III. DETERMINISTIC DENSE CODING WITH STABILIZER STATES

In this section we investigate the usefulness of stabilizer states for deterministic distributed dense coding.

At first, we need to introduce two groups of definitions and notations. The first group is about sets of integers. We use $[1,n]$ to denote the set of integers $\{1,2,\ldots,n\}$. If $T_1,T_2,\ldots,T_k$ are disjoint proper subsets of $[1,n]$ and they satisfy $\cup_{i=1}^{k}T_i=[1,n]$, then we say $(T_1,T_2,\ldots,T_k)$ a partition of $[1,n]$. We also use $|T|$ to denote the number of elements in a set T. The second group is about vectors in $\mathbb{Z}_d^n$. Note that all additions and multiplications of vectors in $\mathbb{Z}_d^n$ are taken over $\mathbb{Z}_d$. For any $\overrightarrow{\alpha}_1,\overrightarrow{\alpha}_2,\ldots,\overrightarrow{\alpha}_k\in\mathbb{Z}_d^n$, their linear span is defined as

$$span\{\overrightarrow{\alpha}_1,\overrightarrow{\alpha}_2,\ldots,\overrightarrow{\alpha}_k\}=\{\sum_{i=1}^{k}\lambda_i\overrightarrow{\alpha}_i:\lambda_1,\lambda_2,\ldots,\lambda_k\in\mathbb{Z}_d\}. \tag{15}$$

$\overrightarrow{\alpha}_1,\overrightarrow{\alpha}_2,\ldots,\overrightarrow{\alpha}_k$ are said to be linearly independent if for any $a_1,a_2,\ldots,a_k\in\mathbb{Z}_d$, $\sum_{i=1}^{k}a_i\overrightarrow{\alpha}_i=\overrightarrow{0}$ if and only if $a_1=a_2=\cdots=a_k=0$. In particular, any n linearly independent vectors $\overrightarrow{\alpha}_1,\overrightarrow{\alpha}_2,\ldots,\overrightarrow{\alpha}_n$ in $\mathbb{Z}_d^n$ are called a basis of $\mathbb{Z}_d^n$.

Now let us reformulate our problem precisely. Suppose $g_1,g_2,\ldots,g_n$ are independent commuting operators in $G_n'^{(d)}$ and $S=\langle g_1,g_2,\ldots,g_n\rangle$ is a complete stabilizer. $|\psi_S\rangle$ is the state stabilized by S. Assume that $(T_1,T_2,\ldots,T_{m+1})$ is a partition of $[1,n]$. $A_1,A_2,\ldots,A_{m+1}$ are distant parties and A_i holds the subsystem T_i of $|\psi_S\rangle$, for $i=1,2,\ldots,m+1$. Now suppose A_i performs one out of b_i different quantum operations on the subsystem T_i, for $i=1,2,\ldots,m$. Then $A_1,A_2,\ldots,A_m$ send all their subsystems to A_{m+1}. If A_{m+1} can perfectly discriminate among all possible states, then $(b_1,b_2,\ldots,b_m)$ is said to be an achievable alphabet size for $|\psi\rangle$ with respect to $(T_1,T_2,\ldots,T_{m+1})$. Our primary goal is to determine whether there exists an achievable alphabet size $(b_1,b_2,\ldots,b_m)$ such that $b_i>d^{|T_i|}$ for at least one $1\leq i\leq m$ and $b_j\geq d^{|T_j|}$ for other $j\neq i$. If so, $|\psi_S\rangle$ is useful for deterministic distributed dense coding with respect to $(T_1,T_2,\ldots,T_{m+1})$. Our ultimate goal is to completely determine the set of achievable alphabet sizes for $|\psi_S\rangle$ with respect to an arbitrary partition plan.

Now suppose a deterministic distributed dense coding protocol achieves the alphabet size $(b_1,b_2,\ldots,b_m)$ for $|\psi_S\rangle$ with respect to $(T_1,T_2,\ldots,T_{m+1})$ by setting A_i's encoding operations to be unitary operations $\{U_{ij}:j=1,2,\ldots,b_i\}$, $\forall i=1,2,\ldots,m$. Then there are totally $\prod_{i=1}^{m}b_i$ possible encoded states which are given by

$$|\psi(\overrightarrow{j})\rangle=(U_{1j_1}\otimes U_{2j_2}\otimes\cdots\otimes U_{mj_m}\otimes I)|\psi_S\rangle, \tag{16}$$

where $\overrightarrow{j}=(j_1,j_2,\ldots,j_m)$ with $1\leq j_i\leq b_i$, $\forall i=1,2,\ldots,m$. These states can be perfectly discriminated by the receiver A_{m+1} if and only if they are mutually orthogonal, i.e. $\langle\psi(\overrightarrow{j})|\psi(\overrightarrow{j}')\rangle=0$, $\forall\overrightarrow{j}\neq\overrightarrow{j}'$. Furthermore, we can prove that $(b_1,b_2,\ldots,b_m)$ satisfies two constraints. The first one is

$$\prod_{i=1}^{m}b_i\leq d^n, \tag{17}$$

since there could be at most d^n mutually orthogonal n-qudit states. Apparently the protocol reaches the best efficiency if and only if $\prod_{i=1}^{m}b_i=d^n$. In this case, we say that $|\psi_S\rangle$ is *optimally* useful for deterministic distributed dense coding with respect to $(T_1,T_2,\ldots,T_{m+1})$. The second constraint is

$$b_i\leq d^{2|T_i|},\ \ \forall i=1,2,\ldots,m. \tag{18}$$

To see this, one needs to realize that the states

$$\begin{aligned}|\phi(j)\rangle&=(U_{1j}\otimes U_{21}\otimes U_{31}\otimes\cdots\otimes U_{m1}\otimes I)|\psi_S\rangle\\&=(U_{1j}\otimes I)|\psi_S'\rangle\end{aligned} \tag{19}$$

for $j=1,2,\ldots,b_1$ are mutually orthogonal, where

$$|\psi_S'\rangle=(I\otimes U_{21}\otimes U_{31}\otimes\cdots\otimes U_{m1}\otimes I)|\psi_S\rangle. \tag{20}$$

This means that b_1 is an achievable alphabet size for $|\psi_S'\rangle$ with respect to the bipartition $(T_1,\bigcup_{i=2}^{m+1}T_i)$. Since in a bipartite dense coding scheme with an arbitrary $d_1\times d_2$ state (where the d_1-dimensional subsystem is held by the sender) the alphabet size that cannot exceed d_1^2 [17], we obtain $b_1\leq d^{2|T_1|}$. Similarly, $b_i\leq d^{2|T_i|}$, $\forall i=2,3,\ldots,m$. The second constraint tells us no matter how we group the n qudits and how we encode, eventually every sender can acquire at most twice the classical information capacity of the original noiseless quantum channel.

From now on we will focus on deterministic distributed dense coding schemes whose encoding operations are chosen from the generalized Pauli group on multiple qudits. Let us first look at the effect of this kind of operations on the state $|\psi_S\rangle$. From the fact that $|\psi_S\rangle$ is stabilized by $S=\langle g_1,g_2,\ldots,g_n\rangle$, we know for any $g\in G_n$, $g|\psi_S\rangle$ is the state stabilized by

$$\begin{aligned}gSg^\dagger&=\langle gg_1g^\dagger,gg_2g^\dagger,\ldots,gg_ng^\dagger\rangle\\&=\langle\omega^{\chi(g)\Lambda\chi(g_1)}g_1,\omega^{\chi(g)\Lambda\chi(g_2)}g_2,\ldots,\omega^{\chi(g)\Lambda\chi(g_n)}g_n\rangle,\end{aligned} \tag{21}$$

where the second equality comes from Eq.(6). In other words, $g|\psi_S\rangle$ becomes the simultaneous eigenstate of $g_1,g_2,\ldots,g_n$ with the eigenvalues $\omega^{-\chi(g)\Lambda\chi(g_1)},\omega^{-\chi(g)\Lambda\chi(g_2)},\ldots,\omega^{-\chi(g)\Lambda\chi(g_n)}$ respectively. Now we introduce a map Γ as follows: if $|\psi\rangle$ is the simultaneous eigenstate of $g_1,g_2,\ldots,g_n$ corresponding to the eigenvalues $\omega^{x_1},\omega^{x_2},\ldots,\omega^{x_n}$ for some $x_1,x_2,\ldots,x_n\in\mathbb{Z}_d$, then $\Gamma(|\psi\rangle)=(x_1,x_2,\ldots,x_n)^T$; otherwise, $\Gamma(|\psi\rangle)$ is not defined. So for any $g\in G_n^{(d)}$, we have

$$\begin{aligned}
&\Gamma(g|\psi_S\rangle)\\
&=(-\chi(g)\Lambda\chi(g_1),-\chi(g)\Lambda\chi(g_2),\ldots,-\chi(g)\Lambda\chi(g_n))\\
&=(\sum_{j=1}^{n}a_jb_{1j}-\sum_{j=1}^{n}b_ja_{1j},\sum_{j=1}^{n}a_jb_{2j}-\sum_{j=1}^{n}b_ja_{2j},\\
&\ldots,\sum_{j=1}^{n}a_jb_{nj}-\sum_{j=1}^{n}b_ja_{nj})^T\\
&=\sum_{j=1}^{n}a_j(b_{1j},b_{2j},\ldots,b_{nj})^T-\sum_{j=1}^{n}b_j(a_{1j},a_{2j},\ldots,a_{nj})^T\\
&=\sum_{j=1}^{n}a_j\overrightarrow{\beta_j}-\sum_{j=1}^{n}b_j\overrightarrow{\alpha_j}
\end{aligned} \tag{22}$$

where we suppose $\chi(g)=(a_1,a_2,\ldots,a_n,b_1,b_2,\ldots,b_n)$, $\chi(g_i)=(a_{i1},a_{i2},\ldots,a_{in},b_{i1},b_{i2},\ldots,b_{in})$, $\forall i=1,2,\ldots,n$, and

$$\begin{aligned}\overrightarrow{\alpha_j}=(a_{1j},a_{2j},\ldots,a_{nj})^T,\\ \overrightarrow{\beta_j}=(b_{1j},b_{2j},\ldots,b_{nj})^T,\end{aligned} \tag{23}$$

$\forall j=1,2,\ldots,n$. Note that $\overrightarrow{\alpha_j}$ and $\overrightarrow{\beta_j}$ are exactly the j-th and $(j+n)$-th columns of the check matrix M for $g_1,g_2,\ldots,g_n$ respectively, $\forall j=1,2,\ldots,n$.

Now suppose we have a valid deterministic dense coding protocol in which A_i's encoding operations are

$$U_{ij}=\bigotimes_{l\in T_i}\sigma_{a_{ijl},b_{ijl}} \tag{24}$$

for $j=1,2,\ldots,b_i$, where $a_{ijl},b_{ijl}\in\mathbb{Z}_d$, $\forall i=1,2,\ldots,m$. Then by Eqs.(16) and (22),

$$\Gamma(|\psi(\overrightarrow{j})\rangle)=\sum_{i=1}^{m}\sum_{l\in T_i}(a_{ij_il}\overrightarrow{\beta}_l-b_{ij_il}\overrightarrow{\alpha}_l)=\sum_{i=1}^{m}\overrightarrow{\gamma}_{ij_i}, \tag{25}$$

where

$$\begin{aligned}\overrightarrow{\gamma}_{ij_i}&=\sum_{l\in T_i}a_{ij_il}\overrightarrow{\beta}_l-b_{ij_il}\overrightarrow{\alpha}_l\\ &\in S_i\equiv span\{\overrightarrow{\alpha}_l,\overrightarrow{\beta}_l:l\in T_i\}.\end{aligned} \tag{26}$$

So for any $\overrightarrow{j}\neq\overrightarrow{j'}$, $|\psi(\overrightarrow{j})\rangle$ and $|\psi(\overrightarrow{j}')\rangle$ are orthogonal if and only if $\Gamma(|\psi(\overrightarrow{j})\rangle)\neq\Gamma(|\psi(\overrightarrow{j}')\rangle)$. Therefore we have

$$\sum_{i=1}^{m}\overrightarrow{\gamma}_{ij_i}\neq\sum_{i=1}^{m}\overrightarrow{\gamma}_{ij_i'},\ \forall\overrightarrow{j}\neq\overrightarrow{j}'. \tag{27}$$

Conversely, suppose we are given a set of vectors $\{\overrightarrow{\gamma}_{ij}\in S_i:i=1,2,\ldots,m,j=1,2,\ldots,b_i\}$ which satisfy inequality (27). Since $S_i=span\{\overrightarrow{\alpha}_l,\overrightarrow{\beta}_l:l\in T_i\}$, $\overrightarrow{\gamma}_{ij}$ can be written as

$$\overrightarrow{\gamma}_{ij}=\sum_{l\in T_i}a_{ijl}\overrightarrow{\beta}_l-b_{ijl}\overrightarrow{\alpha}_l \tag{28}$$

for some $a_{ijl},b_{ijl}\in\mathbb{Z}_d$. Then consider the protocol in which A_i uses the encoding operations $\{U_{ij}=\bigotimes_{l\in T_i}\sigma_{a_{ijl},b_{ijl}}:j=1,2,\ldots,b_i\}$, $\forall i=1,2,\ldots,m$. One can easily see that it is also a valid deterministic dense coding protocol.

Summarizing the argument in the above two paragraphs, we know there exists a protocol which achieves the alphabet size $(b_1,b_2,\ldots,b_m)$ by using generalized Pauli group elements to encode if and only if there exist vectors $\{\overrightarrow{\gamma}_{ij}\in S_i:i=1,2,\ldots,m,j=1,2,\ldots,b_i\}$ which satisfy inequality (27). Thus our problem can be rephrased as follows: given the subspaces $S_1,S_2,\ldots,S_m$ of $\mathbb{Z}_d^n$, for any $(b_1,b_2,\ldots,b_m)\in\mathbb{Z}^m$, do there exist vectors $\{\overrightarrow{\gamma}_{ij}\in S_i:i=1,2,\ldots,m,j=1,2,\ldots,b_i\}$ satisfying inequality (27)?

One can easily see that a necessary condition for $\sum_{i=1}^{m}\overrightarrow{\gamma}_{ij_i}=\sum_{i=1}^{m}\overrightarrow{\gamma}_{ij_i'}$ is the vectors $\{\overrightarrow{\gamma}_{ij_i}-\overrightarrow{\gamma}_{ij_i'}:i=1,2,\ldots,m\}$ are linearly dependent. With this observation, we obtain a sufficient condition for $|\psi_S\rangle$ to be useful for deterministic distributed dense coding with respect to $(T_1,T_2,\ldots,T_{m+1})$, as the following theorem states:

Theorem 1 *If there exist $R_i,Q_i\subset T_i$, $\forall i=1,2,\ldots,m$, such that: (1) $|R_i|+|Q_i|>|T_i|$ for at least one $1\leq i\leq m$, while $|R_j|+|Q_j|\geq|T_j|$ for other $j\neq i$; (2) the vectors $\{\overrightarrow{\alpha}_k:k\in\cup_{i=1}^mR_i\}\cup\{\overrightarrow{\beta}_l:l\in\cup_{i=1}^mQ_i\}$ are linearly independent, then $|\psi_S\rangle$ is useful for deterministic distributed dense coding with respect to $(T_1,T_2,\ldots,T_{m+1})$.*

Proof: Consider the following protocol: $\forall i=1,2,\ldots,m$, A_i's encoding operations are

$$U_i(\{a_{ik},b_{ik}\}_{k\in T_i})=\bigotimes_{k\in T_i}\sigma_{a_{ik},b_{ik}}, \tag{29}$$

where $a_{ik}=0,1,\ldots,d-1$, for $k\in Q_i$; $a_{ik}=0$, for $k\in T_i-Q_i$; $b_{ik}=0,1,\ldots,d-1$, for $k\in R_i$; $b_{ik}=0$, for $k\in T_i-R_i$. There are totally $d^{|R_i|+|Q_i|}$ different choices of $\{a_{ik},b_{ik}\}_{k\in T_i}$. In other words, the alphabet size of A_i is $d^{|R_i|+|Q_i|}$. If we prove that this protocol is valid, then by condition (1), $|\psi_S\rangle$ is useful for deterministic distributed dense coding with respect to $(T_1,T_2,\ldots,T_{m+1})$.

By Eq.(26), the vector corresponding to $U_i(\{a_{ik},b_{ik}\}_{k\in T_i})$ is

$$\begin{aligned}\overrightarrow{\gamma}_i(\{a_{ik},b_{ik}\}_{k\in T_i})&=\sum_{k\in T_i}a_{ik}\overrightarrow{\beta}_k-\sum_{k\in T_i}b_{ik}\overrightarrow{\alpha}_k\\ &=\sum_{k\in Q_i}a_{ik}\overrightarrow{\beta}_k-\sum_{k\in R_i}b_{ik}\overrightarrow{\alpha}_k.\end{aligned} \tag{30}$$

Now suppose for some $\{a_{ik},b_{ik}\}$, $\{a_{ik}',b_{ik}'\}$, we have

$$\sum_{i=1}^{m}\overrightarrow{\gamma}_i(\{a_{ik},b_{ik}\}_{k\in T_i})=\sum_{i=1}^{m}\overrightarrow{\gamma}_i(\{a_{ik}',b_{ik}'\}_{k\in T_i}). \tag{31}$$

Then by Eq.(30), we obtain

$$\sum_{i=1}^{m}(\sum_{k\in Q_i}(a_{ik}-a_{ik}')\overrightarrow{\beta}_k-\sum_{k\in R_i}(b_{ik}-b_{ik}')\overrightarrow{\alpha}_k)=\overrightarrow{0}. \tag{32}$$

Since $\{\overrightarrow{\alpha}_k:k\in\cup_{i=1}^mR_i\}\cup\{\overrightarrow{\beta}_k:k\in\cup_{i=1}^mQ_i\}$ are linearly independent, this equation implies $a_{ik}=a_{ik}'$,

$\forall k \in Q_i$ and $b_{ik} = b'_{ik}$, $\forall k \in R_i$, $\forall i = 1, 2, \ldots, m$. So the vectors $\{\overrightarrow{\gamma}_i(\{a_{ik}, b_{ik}\}_{k\in T_i})\}$ satisfy inequality (27). In other words, this protocol is valid. This ends the proof ■

With the help of this theorem, we find that when d is prime, the power of n-qudit stabilizer states in deterministic distributed dense coding is strong, as the following corollary states:

Corollary 1 *If d is prime, then any genuinely entangled n-qudit stabilizer state is optimally useful for deterministic distributed dense coding with respect to at least one partition of $[1, n]$.*

Proof: In Ref.[37], the authors present a procedure which can transform any $k \times 2n$ check matrix for $S = \langle g_1, g_2, \ldots, g_k\rangle$ (where $d = 2$) into the following standard form

$$M = \begin{pmatrix} I_r & A_1 & A_2 & | & B & 0 & C \\ 0 & 0 & 0 & | & D & I_{k-r} & E \end{pmatrix}, \tag{33}$$

(re-labelling the original n qudits and re-selecting stabilizer generators if necessary), where A_1, A_2, B, C, D, E are matrices of size $r \times (k - r)$, $r \times (n - k)$, $r \times r$, $r \times (n-k)$, $(k-r)\times r$, $(k-r)\times(n-k)$ respectively, for some $r \le k$. Their procedure includes three basic kinds of operations about the original matrix: swapping rows, swapping columns and adding one row to another. We realize that their conclusion can be readily extended to arbitrary prime dimensions, since the essential prerequisite of their method is that $\mathbb{Z}_d$ needs to be a field.

Now $|\psi_S\rangle$ is stabilized by a complete stabilizer $S = \langle g_1, g_2, \ldots, g_n\rangle$. In this case, the above standard form reduces into

$$M = \begin{pmatrix} I_r & A_1 & | & B & 0 \\ 0 & 0 & | & D & I_{n-r} \end{pmatrix}, \tag{34}$$

where A_1, B, D are matrices of size $r \times (n - r)$, $r \times r$, $(n - r) \times r$ respectively, for some $r \le n$. Then we have

$$\begin{aligned} 0 &= M\Lambda_n M^\dagger \\ &= \begin{pmatrix} B - B^T & -A_1 - D^T \\ A_1^T + D & 0 \end{pmatrix}, \end{aligned} \tag{35}$$

which yields $B = B^T$ and $A_1 + D^T = 0$.

Now we prove $D \neq 0$ by contradiction. Assume $D = 0$. Define

$$M_1 = \begin{pmatrix} I_r & | & B \\ 0 & | & D \end{pmatrix}. \tag{36}$$

Then we have

$$M_1\Lambda_r M_1^\dagger = \begin{pmatrix} B - B^T & -D^T \\ D & 0 \end{pmatrix} = 0. \tag{37}$$

For any $T \subset [1, n]$ and any $g = \gamma^c \sigma_{a_1,b_1} \otimes \sigma_{a_2,b_2} \otimes \cdots \otimes \sigma_{a_n,b_n} \in G_n^{(d)}$, define the restriction of g on T as

$$g^{(T)} = \bigotimes_{k\in T} \sigma_{a_k,b_k}. \tag{38}$$

Then one can see that $M_1\Lambda_r M_1^\dagger = 0$ implies $g_1'^{([1,r])}, g_2'^{([1,r])}, \ldots, g_n'^{([1,r])}$ mutually commute, where g_i' is the stabilizer generator corresponding to the i-th row of M in Eq.(34), for $i = 1, 2, \ldots, n$. Thus by lemma 1 of Ref. [32], $|\psi_S\rangle$ should be separable with respect to the bipartition $([1, r], [r + 1, n])$. This contradicts with the given fact that $|\psi_S\rangle$ is genuinely entangled. So $D \neq 0$.

Now suppose the entry on the k-th row and l-th column of D is nonzero. Assume the l-th column of B is $(b_1, b_2, \ldots, b_r)^T$ and the l-th column of D is $(d_1, d_2, \ldots, d_{n-r})^T$ with $d_k \neq 0$. Then the $(n + l)$-th column of M is $\overrightarrow{\beta}_l = (b_1, b_2, \ldots, b_r, d_1, d_2, \ldots, d_{n-r})^T$. Note that the i-th column of M is $\overrightarrow{\alpha}_i = (0, \ldots, 0, 1, 0, \ldots, 0)^T$ where 1 is the i-th element, for $i = 1, 2, \ldots, r$. Also, the $(n+i)$-th column of M is $\overrightarrow{\beta}_i = (0, \ldots, 0, 1, 0, \ldots, 0)^T$ where 1 is the i-th element, for $i = r+1, r+2, \ldots, n$.

Consider the partition $(T_1, T_2, \ldots, T_n)$ with $T_i = \{i\}$, $\forall i = 1, 2, \ldots, r + k - 1$; $T_i = \{i + 1\}$, $\forall i = r + k, r + k + 1, \ldots, n - 1$; $T_n = \{r + k\}$. In other words, the receiver holds the $(r + k)$-th qudit and $n - 1$ senders each hold one of the other $n - 1$ qudits. Consider the vectors $\overrightarrow{\alpha}_1, \overrightarrow{\alpha}_2, \ldots, \overrightarrow{\alpha}_r$, $\overrightarrow{\beta}_l$, $\overrightarrow{\beta}_{r+1}, \overrightarrow{\beta}_{r+2}, \ldots, \overrightarrow{\beta}_{r+k-1}, \overrightarrow{\beta}_{r+k+1}, \ldots, \overrightarrow{\beta}_n$. They are linearly independent. Actually, suppose for $c_1, c_2, \ldots, c_{r+k-1}, c_{r+k+1}, \ldots, c_n, \lambda \in \mathbb{Z}_d$, we have

$$\begin{aligned} \overrightarrow{0} &= \sum_{i=1}^{r} c_i\overrightarrow{\alpha}_i + \sum_{i=r+1}^{r+k-1} c_i\overrightarrow{\beta}_i + \sum_{i=r+k+1}^{n} c_i\overrightarrow{\beta}_i + \lambda\overrightarrow{\beta}_l \\ &= (c_1 + b_1\lambda, c_2 + b_2\lambda, \ldots, c_r + b_r\lambda, \\ &\quad c_{r+1} + d_1\lambda, c_{r+2} + d_2\lambda, \ldots, c_{r+k-1} + d_{k-1}\lambda, \\ &\quad d_k\lambda, c_{r+k-1} + d_{k+1}\lambda, \ldots, c_n + d_{n-r}\lambda)^T. \end{aligned} \tag{39}$$

Since $d_k \neq 0$, the entry $d_k\lambda = 0$ implies $\lambda = 0$. Taking this back to the above equation, we obtain $c_1 = c_2 = \cdots = c_{r+k-1} = c_{r+k+1} = \cdots = c_n = 0$. Now define $R_i = T_i$, $\forall i = 1, 2, \ldots, r$; $R_i = \emptyset$, $\forall i = r + 1, r + 2, \ldots, n - 1$; $Q_i = \emptyset$, $\forall i = 1, 2, \ldots, l-1, l+1, l+2, \ldots, r$; $Q_i = T_i$, $\forall i = l, r+1, r+2, \ldots, n-1$. Then $\{R_i, Q_i : i = 1, 2, \ldots, n-1]\}$ satisfy the conditions of theorem 1. Furthermore, note that $\sum_{i=1}^{n-1}(|R_i| + |Q_i|) = n$. So by the proof of theorem 1, we know $|\psi_S\rangle$ is optimally useful for deterministic distributed dense coding with respect to $(T_1, T_2, \ldots, T_n)$. ■

Remark. From theorem 1 and corollary 1, we see that the linear independency among the columns of check matrix can affect the dense coding power of $|\psi\rangle$. The more linearly independent they are, the more powerful $|\psi_S\rangle$ is for dense coding.

The dense coding protocol given by the proof of theorem 1 always has alphabet size of the form $(d^{a_1}, d^{a_2}, \ldots, d^{a_m})$ for some integers $a_1, a_2, \ldots, a_m$. One may wonder whether a wider class of alphabet sizes can be reached. Indeed this is true. In what follows, we will propose a method to partially solve the general problem of determining the whole set of achievable alphabet sizes.

Now suppose $\{\overrightarrow{x}_1, \overrightarrow{x}_2, \ldots, \overrightarrow{x}_n\}$ is an arbitrary basis of $\mathbb{Z}_d^n$. Let $\overrightarrow{X}$ denote this basis. For $j = 1, 2, \ldots, n$, define

$$W_j = span\{\overrightarrow{x}_j, \overrightarrow{x}_{j+1}, \ldots, \overrightarrow{x}_n\}. \tag{40}$$

Also define $W_{n+1} = \emptyset$. For any $\overrightarrow{z} = \sum_{i=1}^n \lambda_i \overrightarrow{x}_i \in \mathbb{Z}_d^n$, define $C_j(\overrightarrow{z}; \overrightarrow{X}) = \lambda_j$, $\forall j = 1, 2, \ldots, n$. For $j = 1, 2, \ldots, n$, define

$$P_j = \{1 \le i \le m : S_i \cap (W_j - W_{j+1}) \ne \emptyset\}. \tag{41}$$

Then for all $i \in P_j$, choose $\overrightarrow{z}_{ij} \in S_i \cap (W_j - W_{j+1})$. Let

$$c_{ij} = C_j(\overrightarrow{z}_{ij}; \overrightarrow{X}). \tag{42}$$

Note that by the definition of $\overrightarrow{z}_{ij}$, it satisfies

$$C_k(\overrightarrow{z}_{ij}; \overrightarrow{X}) = 0, \ \forall k < j. \tag{43}$$

For any $t \ge 1$, define

$$\begin{aligned} A(P_j; t) \ &= \{(a_1, a_2, \ldots, a_m) \in \mathbb{Z}^m : \forall i \notin P_j, a_i = 1; \\ &\prod_{i=1}^m a_i \le t\}. \end{aligned} \tag{44}$$

For $j = 1, 2, \ldots, n$, choose $\overrightarrow{a}_j = (a_{j1}, a_{j2}, \ldots, a_{jm}) \in A(P_j; d)$. Then define

$$Q_j = \{i \in P_j : \exists \eta \in \mathbb{Z}_d, \ s.t. \ \eta c_{ij} \equiv \prod_{t=1}^{i-1} a_{jt} (mod \ \ d)\}, \tag{45}$$

and furthermore,

$$\overrightarrow{b}_j = F(\overrightarrow{a}_j; Q_j) = (b_{j1}, b_{j2}, \ldots, b_{jm}), \tag{46}$$

where $b_{ji} = a_{ji}$, $\forall i \in Q_j$; $b_{ji} = 1$, $\forall i \notin Q_j$. Let $B(\overrightarrow{X})$ denote the set of $(\overrightarrow{b}_1, \overrightarrow{b}_2, \ldots, \overrightarrow{b}_n)$ that can be obtained by this procedure.

With these definitions and notations introduced above, we have the following theorem:

Theorem 2 *For any* $(\overrightarrow{b}_1, \overrightarrow{b}_2, \ldots, \overrightarrow{b}_n) \in B(\overrightarrow{X})$, $(\prod_{j=1}^n b_{j1}, \prod_{j=1}^n b_{j2}, \ldots, \prod_{j=1}^n b_{jm})$ *is an achievable alphabet size for* $|\psi_S\rangle$ *with respect to* $(T_1, T_2, \ldots, T_{m+1})$.

Proof: In what follows, if not explicitly pointed out, all computations will be taken over $\mathbb{Z}_d$. By definition, $\forall j = 1, 2, \ldots, n$, $\forall i \in Q_j$, there exists $\eta_{ij} \in \mathbb{Z}_d$ such that

$$\eta_{ij} c_{ij} = \prod_{t=1}^{i-1} a_{jt}. \tag{47}$$

Define

$$\overrightarrow{y}_{ij} = \eta_{ij} \overrightarrow{z}_{ij} \in S_i. \tag{48}$$

Then we have

$$C_j(\overrightarrow{y}_{ij}; \overrightarrow{X}) = \prod_{t=1}^{i-1} a_{jt}. \tag{49}$$

Moreover, by Eq.(43),

$$C_k(\overrightarrow{y}_{ij}; \overrightarrow{X}) = 0, \ \forall k < j. \tag{50}$$

For all $i \notin Q_j$, define

$$\overrightarrow{y}_{ij} = \overrightarrow{0}. \tag{51}$$

Now for $i = 1, 2, \ldots, m$, we define

$$\overrightarrow{\gamma}(i; \overrightarrow{\lambda}_i) = \sum_{j=1}^n \lambda_{ij} \overrightarrow{y}_{ij} \in S_i, \tag{52}$$

where $\overrightarrow{\lambda}_i = (\lambda_{i1}, \lambda_{i2}, \ldots, \lambda_{in})$ with $1 \le \lambda_{ij} \le b_{ji}$, $\forall j = 1, 2, \ldots, n$. We will prove the vectors $\{\overrightarrow{\gamma}(i; \overrightarrow{\lambda}_i)\}$ satisfy inequality (27).

Suppose for some $\{\overrightarrow{\lambda}_i\}$, $\{\overrightarrow{\mu}_i\}$,

$$\sum_{i=1}^m \overrightarrow{\gamma}(i; \overrightarrow{\lambda}_i) = \sum_{i=1}^m \overrightarrow{\gamma}(i; \overrightarrow{\mu}_i), \tag{53}$$

or equivalently,

$$\sum_{i=1}^m \sum_{j=1}^n (\lambda_{ij} - \mu_{ij}) \overrightarrow{y}_{ij} = 0. \tag{54}$$

Note that for $j = 1, 2, \ldots, n$, $\forall i \notin Q_j$, $\lambda_{ij} = \mu_{ij} = 1$ because by definition $1 \le \lambda_{ij}, \mu_{ij} \le b_{ji} = 1$. So Eq.(54) reduces into

$$\sum_{j=1}^n \sum_{i \in Q_j} (\lambda_{ij} - \mu_{ij}) \overrightarrow{y}_{ij} = 0. \tag{55}$$

If we write the left-hand side of Eq.(55) as linear combination of the basis $\overrightarrow{x}_1, \overrightarrow{x}_2, \ldots, \overrightarrow{x}_n$, then the coefficient corresponding to $\overrightarrow{x}_1$ should be zero, i.e.

$$\begin{aligned} 0 &= \sum_{j=1}^n \sum_{i \in Q_j} (\lambda_{ij} - \mu_{ij}) C_1(\overrightarrow{y}_{ij}; \overrightarrow{X}) \\ &= \sum_{i \in Q_1} (\lambda_{i1} - \mu_{i1}) \prod_{t=1}^{i-1} a_{1t}, \end{aligned} \tag{56}$$

where the second equality comes from Eqs.(49) and (50).

Now define

$$R_1 = \{i \in Q_1 : a_{1i} \ne 1\}. \tag{57}$$

Then $\forall i \in Q_1 - R_1$, $b_{1i} = a_{1i} = 1$, and consequently $\lambda_{i1} = \mu_{i1} = 1$ since by definition $1 \le \lambda_{i1}, \mu_{i1} \le b_{1i}$. So

$$0 = \sum_{i \in Q_1} (\lambda_{i1} - \mu_{i1}) \prod_{t=1}^{i-1} a_{1t} = \sum_{i \in R_1} (\lambda_{i1} - \mu_{i1}) \prod_{t=1}^{i-1} a_{1t}. \tag{58}$$

Note that the above additions and multiplications are taken over $\mathbb{Z}_d$. Eq.(58) actually means

$$\sum_{i \in R_1} (\lambda_{i1} - \mu_{i1}) \prod_{t=1}^{i-1} a_{1t} \equiv 0 (mod \ \ d). \tag{59}$$

Now we turn back to normal computation over $\mathbb{Z}$. We actually can prove

$$\sum_{i\in R_1}(\lambda_{i1}-\mu_{i1})\prod_{t=1}^{i-1}a_{1t}=0. \tag{60}$$

To see this, one only needs to realize

$$\begin{aligned}&|\sum_{i\in R_1}(\lambda_{i1}-\mu_{i1})\prod_{t=1}^{i-1}a_{1t}|\\ &\le\sum_{i\in R_1}(b_{1i}-1)\prod_{t=1}^{i-1}a_{1t}\\ &=\sum_{i\in R_1}(\prod_{t=1}^{i}a_{1t}-\prod_{t=1}^{i-1}a_{1t})\\ &\le\sum_{i=1}^{m}(\prod_{t=1}^{i}a_{1t}-\prod_{t=1}^{i-1}a_{1t})\\ &=\prod_{t=1}^{m}a_{1t}-1\\ &\le d-1,\end{aligned} \tag{61}$$

where the first inequality comes from $1\le\lambda_{i1},\mu_{i1}\le b_{1i}$, the second equality comes from $b_{1i}=a_{1i}$, $\forall i\in R_1\subset Q_1$, and the last inequality comes from $(a_{11},a_{12},\ldots,a_{1m})\in A(P_1;d)$.

Suppose i_1, i_2 is the smallest and second smallest number in R_1. Then for $\forall i\ge i_2\in R_1$, $\prod_{t=1}^{i-1}a_{1t}$ is a multiple of $\prod_{t=1}^{i_2-1}a_{1t}$. Consequently, from $\sum_{i\in R_1}(\lambda_{i1}-\mu_{i1})\prod_{t=1}^{i-1}a_{1t}=0$ we get that $(\lambda_{i_11}-\mu_{i_11})\prod_{t=1}^{i_1-1}a_{1t}$ is a multiple of $\prod_{t=1}^{i_2-1}a_{1t}$. But on the other hand,

$$\begin{aligned}&|(\lambda_{i_11}-\mu_{i_11})\prod_{t=1}^{i_1-1}a_{1t}|\le(b_{1i_1}-1)\prod_{t=1}^{i_1-1}a_{1t}\\ &=(a_{1i_1}-1)\prod_{t=1}^{i_1-1}a_{1t}<\prod_{t=1}^{i_1}a_{1t}\le\prod_{t=1}^{i_2-1}a_{1t}.\end{aligned} \tag{62}$$

So we must have $\lambda_{i_11}=\mu_{i_11}$, which furthermore implies

$$0=\sum_{i\in R_1-\{i_1\}}(\lambda_{i1}-\mu_{i1})\prod_{t=1}^{i-1}a_{1t}. \tag{63}$$

Repeating the above argument for i_2 and the third smallest number i_3 in R_1, one can get $\lambda_{i_21}=\mu_{i_21}$. So by iterating this procedure one can eventually get $\lambda_{i1}=\mu_{i1}$, $\forall i\in R_1$.

Summarizing the above argument, we obtain $\lambda_{i1}=\mu_{i1}$, $\forall i=1,2,\ldots,m$. Taking this back to Eq.(55), we get

$$\sum_{j=2}^{n}\sum_{i\in Q_j}(\lambda_{ij}-\mu_{ij})\overrightarrow{y}_{ij}=0. \tag{64}$$

Again, by taking a similar analysis for $\overrightarrow{x}_2$, we can obtain $\lambda_{i2}=\mu_{i2}$, $\forall i=1,2,\ldots,m$.

Repeat this procedure, and eventually we prove $\lambda_{ij}=\mu_{ij}$, $\forall i=1,2,\ldots,m$, $\forall j=1,2,\ldots,n$. Therefore $\{\overrightarrow{\gamma}(i;\overrightarrow{\lambda}_i)\in S_i:1\le i\le m\}$ satisfy inequality (27), where $\overrightarrow{\lambda}_i=(\lambda_{i1},\lambda_{i2},\ldots,\lambda_{in})$ with $1\le\lambda_{ij}\le b_{ji}$, $\forall j=1,2,\ldots,n$. So $(\prod_{j=1}^{n}b_{j1},\prod_{j=1}^{n}b_{j2},\ldots,\prod_{j=1}^{n}b_{jm})$ is an achievable alphabet size for $|\psi_S\rangle$ with respect to $(T_1,T_2,\ldots,T_{m+1})$. ■

Remark 1. One can see that several ingredients of theorem 2 can be chosen freely. These ingredients include the basis $\overrightarrow{x}_i$, the vectors $\overrightarrow{y}_{ij}\in S_i$ and $(a_{j1},a_{j2},\ldots,a_{jm})\in A(P_j;d)$. Every possible selection of these variables can lead to an achievable alphabet size by applying theorem 2.

Remark 2. One can see that the overall alphabet size of all senders is

$$\begin{aligned}&\prod_{i=1}^{m}\prod_{j=1}^{n}b_{ji}=\prod_{j=1}^{n}\prod_{i\in Q_j}a_{ji}\\ &\le\prod_{j=1}^{n}\prod_{i\in P_j}a_{ji}=\prod_{j=1}^{n}\prod_{i=1}^{m}a_{ji}\\ &\le d^n,\end{aligned} \tag{65}$$

where the first equality comes from the definition of b_{ji}, the second inequality comes from $Q_j\subset P_j$ and $a_{ji}\ge 1$, the third inequality comes from $a_{ji}=1$, $\forall i\notin P_j$, and the last inequality comes from $(a_{j1},a_{j2},\ldots,a_{jm})\in A(P_j;d)$. So as long as $Q_j=P_j$ and $\prod_{i=1}^{m}a_{ji}=d$, $\forall j=1,2,\ldots,n$, the alphabet size obtained by theorem 2 is optimal.

IV. ILLUSTRATIONS

In this section we will analyze several states by applying our theorems. In each example, the matrices X, Z are $X^{(d)}$, $Z^{(d)}$ defined by Eq.(1) with the corresponding dimension d, and similarly for $\sigma_{i,j}$. We will also use the notation X_j to denote the operation X acting on the jth qudit and similarly for Z_j. Moreover, all the entries of check matrices range over $\mathbb{Z}_d$. So we can use $-c$ to equivalently denote $d-c$, $\forall c=1,2,\ldots,d-1$.

We will consider four examples. The first two examples are re-examinations of old results from our perspective. The third and fourth examples are detailed illustrations of how to utilize theorem 1 and theorem 2 respectively.

Example 1 *Let us begin with the standard bipartite dense coding protocol. Let*

$$|\Phi^+\rangle=\frac{1}{\sqrt{d}}\sum_{i=0}^{d-1}|ii\rangle \tag{66}$$

be the maximally entangled state in the $d\times d$ system. It is a stabilizer state and its stabilizer is $S=\langle g_1,g_2\rangle$, where

$$\begin{aligned}g_1&=X_1X_2,\\ g_2&=Z_1Z_2^{-1}.\end{aligned} \tag{67}$$

The check matrix of g_1,g_2 is

$$M=\begin{pmatrix}1&1&0&0\\0&0&1&-1\end{pmatrix} \tag{68}$$

Consider the partition $(\{1\},\{2\})$. *The first and third columns of* M *are* $\overrightarrow{\alpha}_1 = (1,0)^T, \overrightarrow{\beta}_1 = (0,1)^T$. *Let* $R_1 = Q_1 = \{1\}$. *Then by theorem 1 and its proof,* d^2 *is an achievable alphabet size for* $|\psi_S\rangle$ *with respect to the bipartition* $(\{1\},\{2\})$, *and the corresponding encoding operations are* $\{\sigma_{i,j} : i,j = 0,1,\ldots,d-1\}$.

Example 2 *Now consider the generalization of GHZ state to arbitrary n-qudit system*

$$|GHZ_{d,n}\rangle = \frac{1}{\sqrt{d}}\sum_{i=0}^{d-1}|ii\ldots i\rangle. \tag{69}$$

Its distributed dense coding capability has been investigated by Refs. [14, 15, 16]. One can see $|GHZ_{d,n}\rangle$ *is a stabilizer state and its stabilizer is* $S = \langle g_1, g_2, \ldots, g_n\rangle$, *where*

$$\begin{aligned} &g_1 = X_1X_2\ldots X_n, \\ &g_j = Z_{j-1}Z_j^{-1}, \qquad \forall j = 2,3,\ldots,n. \end{aligned} \tag{70}$$

The check matrix for $g_1, g_2, \ldots, g_n$ *is defined as follows:* $\forall i = 1,2,\ldots,n$, *the i-th column is*

$$\overrightarrow{\alpha}_i = (1,0,0,\ldots,0)^T; \tag{71}$$

the $(n+1)$*-th column is*

$$\overrightarrow{\beta}_1 = (0,1,0,\ldots,0)^T; \tag{72}$$

$\forall i = 2,3,\ldots,n-1$, *the* $(n+i)$*-th column is*

$$\overrightarrow{\beta}_i = (0,\ldots,0,-1,1,0,\ldots,0)^T, \tag{73}$$

where -1 *and* 1 *are the i-th and* $(i+1)$*-th entries of* $\overrightarrow{\beta}_i$ *respectively; the* $2n$*-th column is*

$$\overrightarrow{\beta}_n = (0,0,\ldots,0,-1)^T. \tag{74}$$

For example, when $n = 4$, *we have* $g_1 = X_1X_2X_3X_4$, $g_2 = Z_1Z_2^{-1}$, $g_3 = Z_2Z_3^{-1}$, $g_4 = Z_3Z_4^{-1}$. *The corresponding check matrix is*

$$M = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix} \tag{75}$$

Now consider the partition $(\{1\},\{2\},\ldots,\{n-1\},\{n\})$. *Define*

$$S_i = span\{\overrightarrow{\alpha}_i, \overrightarrow{\beta}_i\}, \tag{76}$$

$\forall i = 1,2,\ldots,n-1$.

Choose $\overrightarrow{X} = (\overrightarrow{\alpha}_1, \overrightarrow{\beta}_1, \overrightarrow{\beta}_2, \ldots, \overrightarrow{\beta}_{n-1})$ *as the basis of* $\mathbb{Z}_d^n$. *Then we obtain* $P_1 = \{1,2,\ldots,n-1\}$, $P_j = \{j-1\}$, $\forall j = 2,3,\ldots,n$. *For* $i = 1,2,\ldots,n-1$, $j \in P_i$, *choose* $\overrightarrow{z}_{ij}$ *as follows:*

$$\begin{aligned} &\overrightarrow{z}_{i1} = \overrightarrow{\alpha}_i, \\ &\overrightarrow{z}_{i(i+1)} = \overrightarrow{\beta}_i, \end{aligned} \tag{77}$$

where $i = 1,2,\ldots,n-1$. *Then* $c_{i1} = c_{i(i+1)} = 1$, $i = 1,2,\ldots,n-1$.

Choose arbitrary

$$\overrightarrow{a}_1 = (\lambda_1, \lambda_2, \ldots, \lambda_{n-1}) \in A(P_1; d), \tag{78}$$

i.e. $\prod_{i=1}^{n-1}\lambda_i \le d$. *Also,* $\forall j = 2,3,\ldots,n$, *choose*

$$\overrightarrow{a}_j = (1,\ldots,1,d,1,\ldots,1) \in A(P_j; d), \tag{79}$$

where d *is the* $(j-1)$*-th entry of* $\overrightarrow{a}_j$. *Since* $\forall j = 1,2,\ldots,n$, $\forall i \in P_j$, $c_{ij} = 1$, *we have* $Q_j = P_j$. *Consequently,* $\forall j = 1,2,\ldots,n-1$,

$$\overrightarrow{b}_j = F(\overrightarrow{a}_j; Q_j) = \overrightarrow{a}_j. \tag{80}$$

By theorem 2, we have $(\lambda_1 d, \lambda_2 d, \ldots, \lambda_{n-1} d)$ *is an achievable alphabet size for* $|\psi_S\rangle$ *with respect to* $(\{1\},\{2\},\ldots,\{n-1\},\{n\})$. *We realize that similar results were also obtained by Refs. [14, 15].*

Example 3 *Now consider a* $5\times5\times5\times5\times5$ *system, i.e.* $d = 5$, $n = 5$. *Define*

$$\begin{aligned} &g_1 = \sigma_{1,1}\otimes\sigma_{1,2}\otimes\sigma_{2,1}\otimes\sigma_{0,1}\otimes\sigma_{3,0}, \\ &g_2 = \sigma_{2,4}\otimes\sigma_{1,1}\otimes\sigma_{0,2}\otimes\sigma_{1,2}\otimes\sigma_{2,2}, \\ &g_3 = \sigma_{2,2}\otimes\sigma_{4,1}\otimes\sigma_{1,1}\otimes\sigma_{0,1}\otimes\sigma_{3,2}, \\ &g_4 = \sigma_{3,0}\otimes\sigma_{0,1}\otimes\sigma_{4,2}\otimes\sigma_{1,2}\otimes\sigma_{4,1}, \\ &g_5 = \sigma_{4,3}\otimes\sigma_{1,4}\otimes\sigma_{2,3}\otimes\sigma_{4,2}\otimes\sigma_{2,3}. \end{aligned} \tag{81}$$

They are five independent commuting operators in $G_5^{\prime(5)}$. *Let* $S = \langle g_1, g_2, g_3, g_4, g_5\rangle$. *The density matrix of the state stabilized by* S *is given by*

$$\rho_S = \frac{1}{5^5}\prod_{i=1}^{5}(\sum_{j=0}^{4} g_i^j). \tag{82}$$

The check matrix for g_1, g_2, g_3, g_4, g_5 *is*

$$M = \begin{pmatrix} 1 & 1 & 2 & 0 & 3 & 1 & 2 & 1 & 1 & 0 \\ 2 & 1 & 0 & 1 & 2 & 4 & 1 & 2 & 2 & 2 \\ 2 & 4 & 1 & 0 & 3 & 2 & 1 & 1 & 1 & 2 \\ 3 & 0 & 4 & 1 & 4 & 0 & 1 & 2 & 2 & 1 \\ 4 & 1 & 2 & 4 & 2 & 3 & 4 & 3 & 2 & 3 \end{pmatrix}. \tag{83}$$

Consider the partition $(T_1, T_2, T_3) = (\{1,2\},\{3\},\{4,5\})$. *Suppose* A_1 *holds the first and second qudits,* A_2 *holds the third qudit, and* A_3 *holds the fourth and fifth qudits. Let* $\overrightarrow{\alpha}_i$, $\overrightarrow{\beta}_i$ *be the i-th and* $(i+5)$*-th columns of* M, $\forall i = 1,2,3$. *Define* $R_1 = \{1,2\}$, $Q_1 = \{2\}$, $R_2 = \{3\}$, $Q_2 = \{3\}$. *One can check that* $\overrightarrow{\alpha}_1, \overrightarrow{\alpha}_2, \overrightarrow{\alpha}_3, \overrightarrow{\beta}_2, \overrightarrow{\beta}_3$ *are linearly independent. Therefore, by theorem 1 and its proof,* ρ_S *is optimally useful for deterministic distributed dense coding with respect to* $(\{1,2\},\{3\},\{4,5\})$. *It can achieve the alphabet size* $(125, 25)$ *with the following protocol:* A_1*'s encoding operations are* $\{\sigma_{0,b_1}\otimes\sigma_{a_2,b_2} : a_2, b_1, b_2 = 0,1,2,3,4\}$; A_2*'s encoding operations are* $\{\sigma_{a_3,b_3} : a_3, b_3 = 0,1,2,3,4\}$.

Example 4 *Now consider a* $7 \times 7 \times 7 \times 7$ *system, i.e.* $d = 7$, $n = 4$. *Define*

$$\begin{aligned} g_1 &= \sigma_{1,3} \otimes \sigma_{2,2} \otimes \sigma_{2,0} \otimes \sigma_{1,1}, \\ g_2 &= \sigma_{1,5} \otimes \sigma_{1,1} \otimes \sigma_{3,2} \otimes \sigma_{2,3}, \\ g_3 &= \sigma_{2,3} \otimes \sigma_{1,0} \otimes \sigma_{4,5} \otimes \sigma_{3,5}, \\ g_4 &= \sigma_{3,1} \otimes \sigma_{0,0} \otimes \sigma_{6,6} \otimes \sigma_{5,1}. \end{aligned} \tag{84}$$

They are four independent commuting operators in $G_7^{\prime(4)}$. *Let* $S = \langle g_1, g_2, g_3, g_4 \rangle$. *The density matrix of the state stabilized by* S *is given by*

$$\rho_S = \frac{1}{7^4} \prod_{i=1}^{4} (\sum_{j=0}^{6} g_i^j). \tag{85}$$

The check matrix for g_1, g_2, g_3, g_4 *is*

$$M = \begin{pmatrix} 1 & 2 & 2 & 1 & 3 & 2 & 0 & 1 \\ 1 & 1 & 3 & 2 & 5 & 1 & 2 & 3 \\ 2 & 1 & 4 & 3 & 3 & 0 & 5 & 5 \\ 3 & 0 & 6 & 5 & 1 & 0 & 6 & 1 \end{pmatrix}. \tag{86}$$

Consider the partition $(\{1\}, \{2\}, \{3\}, \{4\})$. *Suppose* A_1, A_2, A_3, A_4 *hold the first, second, third and fourth qudits respectively.*

Let $\overrightarrow{\alpha}_i$, $\overrightarrow{\beta}_i$ *denote the* i*-th and* $(i+4)$*-th columns of* M, $\forall i = 1, 2, 3$. *Define*

$$S_i = span\{\overrightarrow{\alpha}_i, \overrightarrow{\beta}_i\}, \tag{87}$$

$\forall i = 1, 2, 3$.

Choose the basis $\overrightarrow{X} = (\overrightarrow{x}_1, \overrightarrow{x}_2, \overrightarrow{x}_3, \overrightarrow{x}_4)$ *of* $\mathbb{Z}_7^4$ *as follows:*

$$\begin{aligned} \overrightarrow{x}_1 &= (0, 0, 0, 1)^T, \\ \overrightarrow{x}_2 &= (0, 0, 1, 0)^T, \\ \overrightarrow{x}_3 &= (0, 1, 0, 0)^T, \\ \overrightarrow{x}_4 &= (1, 0, 0, 0)^T. \end{aligned} \tag{88}$$

Then we have $P_1 = \{1, 3\}$, $P_2 = \{2, 3\}$, $P_3 = \{2\}$, $P_4 = \{1\}$.

For $j = 1, 2, 3, 4$, $i \in P_j$, *choose* $\overrightarrow{z}_{ij}$ *as follows:*

$$\begin{aligned} \overrightarrow{z}_{11} &= \overrightarrow{\beta}_1 = (3, 5, 3, 1)^T, \\ \overrightarrow{z}_{14} &= 3\overrightarrow{\beta}_1 - \overrightarrow{\alpha}_1 = (1, 0, 0, 0)^T, \\ \overrightarrow{z}_{22} &= \overrightarrow{\alpha}_2 = (2, 1, 1, 0)^T, \\ \overrightarrow{z}_{23} &= \overrightarrow{\beta}_2 = (2, 1, 0, 0)^T, \\ \overrightarrow{z}_{31} &= -\overrightarrow{\beta}_3 = (0, 5, 2, 1)^T, \\ \overrightarrow{z}_{32} &= \overrightarrow{\beta}_3 - \overrightarrow{\alpha}_3 = (5, 6, 1, 0)^T. \end{aligned} \tag{89}$$

Consequently, $c_{11} = c_{14} = c_{22} = c_{23} = c_{31} = c_{32} = 1$.

Now choose

$$\begin{aligned} (a_{11}, a_{12}, a_{13}) &= (2, 1, 3) \in A(P_1; 7), \\ (a_{21}, a_{22}, a_{23}) &= (1, 2, 3) \in A(P_2; 7), \\ (a_{31}, a_{32}, a_{33}) &= (1, 7, 1) \in A(P_3; 7), \\ (a_{41}, a_{42}, a_{43}) &= (7, 1, 1) \in A(P_4; 7). \end{aligned} \tag{90}$$

Since $\forall j = 1, 2, 3, 4$, $\forall i \in P_j$, $c_{ij} = 1$, *we get* $Q_j = P_j$. *Thus* $b_{ji} = a_{ji}$, $\forall i = 1, 2, 3$, $\forall j = 1, 2, 3, 4$. *By theorem 2,* $(2 \times 7, 2 \times 7, 3 \times 3) = (14, 14, 9)$ *is an achievable alphabet size for* ρ_S *with respect to* $(\{1\}, \{2\}, \{3\}, \{4\})$. *So* ρ_S *is useful for deterministic distributed dense coding with respect to this partition. The corresponding dense coding protocol is built as follows.* $\forall j = 1, 2, 3, 4$, $\forall i \in P_j$, *since* $c_{ij} = 1$, *we define*

$$\overrightarrow{y}_{ij} = \prod_{t=1}^{i-1} a_{jt} \overrightarrow{z}_{ij}. \tag{91}$$

For all $i \notin P_j$, *define* $\overrightarrow{y}_{ij} = 0$. *Then by Eq.(52),*

$$\begin{aligned} &\overrightarrow{\gamma}(1; \overrightarrow{\lambda}_1) = \lambda_{11} \overrightarrow{y}_{11} + \lambda_{14} \overrightarrow{y}_{14} \\ &= (\lambda_{11} + 3\lambda_{14}) \overrightarrow{\beta}_1 - \lambda_{14} \overrightarrow{\alpha}_1, \end{aligned} \tag{92}$$

where $\lambda_{11} = 1, 2$, $\lambda_{14} = 1, 2, \ldots, 7$;

$$\begin{aligned} &\overrightarrow{\gamma}(2; \overrightarrow{\lambda}_2) = \lambda_{22} \overrightarrow{y}_{22} + \lambda_{23} \overrightarrow{y}_{23} \\ &= \lambda_{23} \overrightarrow{\beta}_2 + \lambda_{22} \overrightarrow{\alpha}_2, \end{aligned} \tag{93}$$

where $\lambda_{22} = 1, 2$, $\lambda_{23} = 1, 2, \ldots, 7$;

$$\begin{aligned} &\overrightarrow{\gamma}(3; \overrightarrow{\lambda}_3) = \lambda_{31} \overrightarrow{y}_{31} + \lambda_{32} \overrightarrow{y}_{32} \\ &= 2(-\lambda_{31} + \lambda_{32}) \overrightarrow{\beta}_3 - 2\lambda_{32} \overrightarrow{\alpha}_3, \end{aligned} \tag{94}$$

where $\lambda_{31}, \lambda_{32} = 1, 2, 3$. *So* A_1*'s encoding operations are* $\{\sigma_{\lambda_{11}+3\lambda_{14}, \lambda_{14}} : \lambda_{11} = 1, 2, \lambda_{14} = 1, 2, \ldots, 7\}$; A_2*'s encoding operations are* $\{\sigma_{\lambda_{23}, -\lambda_{22}} : \lambda_{22} = 1, 2, \lambda_{23} = 1, 2, \ldots, 7\}$; A_3*'s encoding operations are* $\{\sigma_{2(-\lambda_{31}+\lambda_{32}), 2\lambda_{32}} : \lambda_{31}, \lambda_{32} = 1, 2, 3\}$.

V. CONCLUSION

In sum, we have investigated the possibility of performing deterministic distributed dense coding with the aid of a previously shared stabilizer state. We present a sufficient condition for a stabilizer state to be useful for deterministic distributed dense coding with respect to a given partition plan. The corresponding protocol is also constructed. Then a method is proposed to partially solve the general problem of finding the set of achievable alphabet sizes for an arbitrary stabilizer state with respect to an arbitrary partition plan. Finally, our work provides a new perspective from the stabilizer formalism to view the standard dense coding protocol and also unifies several previous results in a single framework.

We would like to point out several open questions that deserve further research. The first question is whether one can achieve the optimal alphabet sizes for any stabilizer state by using only generalized Pauli group elements to encode. If so, can all the optimal protocols be generated by our theorem 1 and theorem 2? The second problem would be to consider deterministic distributed dense coding with multiple copies of a stabilizer state. We do not know whether the dense coding capacity of a stabilizer state can be improved asymptotically. Finally, to our knowledge, there are almost no results about *deterministic* distributed dense coding with a general multipartite entangled state. We hope our results can shed light on the power of general multipartite entanglement in this task.

Acknowledgement

This work was partly supported by the Natural Science Foundation of China (Grant Nos. 60621062 and 60503001) and the Hi-Tech Research and Development Program of China (863 project) (Grant No. 2006AA01Z102).

[1] C. H. Bennett and S. J. Wiesner, Phys. Rev. Lett. **69**, 2881 (1992).
[2] S. Mozes, J. Oppenheim and B. Reznik, Phys. Rev. A **71**, 012311 (2005).
[3] A. K. Pati, P. Parashar and P. Agrawal, Phys. Rev. A **72**, 012329 (2005).
[4] Z. Ji, Y. Feng, R. Duan and M. Ying, Phys. Rev. A **73**, 034307 (2006).
[5] S. Wu, S. M. Cohen, Y. Sun and R. B. Griffiths, Phys. Rev. A **73**, 042311 (2006).
[6] Y. Feng, R. Duan and Z. Ji, Phys. Rev. A **74**, 012310 (2006).
[7] Q. B. Fan and S. Zhang, Phys. Lett. A **348** 160 (2006).
[8] P. S. Bourdon, E. Gerjuoy, J. P. McDonald and H. T. Williams, e-print quant-ph/0709.0563.
[9] N. D. Mermin, Phys. Rev. A **66**, 032308 (2002).
[10] G. Bowen, Phys. Rev. A **63**, 022302 (2001).
[11] I. D. Ivanovic, Phys. Lett. A **123,** 257 (1987).
[12] D. Dieks, Phys. Lett. A **126,** 303 (1988).
[13] A. Peres, Phys. Lett. A **128,** 19 (1988).
[14] X. S. Liu, G. L. Long, D. M. Tong and L. Feng, Phys. Rev. A **65**, 022304 (2002).
[15] A. Grudka and A. Wo'jcik, Phys. Rev. A **66**, 014301 (2002).
[16] H. J. Lee, D. Ahn and S. W. Hwang, Phys. Rev. A **66**, 024304 (2002).
[17] D. Bruß, G. M. D'Ariano, M. Lewenstein, C. Macchiavello, A. Sen(De) and U. Sen, Phys. Rev. Lett. **93**, 210501 (2004).
[18] D. Bruß, M. Lewenstein, A. Sen(De), U. Sen, G. M. D'Ariano and C. Macchiavello, International Journal of Quantum Information **4**, 415 (2006).
[19] Y. Yeo and W. K. Chua, Phys. Rev. Lett. **96**, 060502 (2006).
[20] P. Agrawal and A. Pati, Phys. Rev. A **74**, 062320 (2006).
[21] M. Horodecki and M. Piani, e-print quant-ph/0701134.
[22] G. Rigolin, e-print quant-ph/0407193.
[23] S. Muralidharan and P. K. Panigrahi, e-print quant-ph/0708.3785.
[24] B. Pradhan, P. Agrawal and A. K. Pati, e-print quant-ph/0705.1917.
[25] P.W. Shor, Phys. Rev. A **52**, R2493 (1995).
[26] A. M. Steane, Phys. Rev. A **54**, 4741 (1996).
[27] R. Raussendorf and H. J. Briegel, Phys. Rev. Lett. **86**, 5188 (2001).
[28] D. Gottesman, Phys. Rev. A **54**, 1862 (1996).
[29] D. Gottesman, Ph.D. thesis, California Institute of Technology, Pasadena, CA, 1997.
[30] M. Van den Nest, W. Dür and H. J. Briegel, Phys. Rev. Lett. **98**, 117207 (2007).
[31] G. Tóth and O. Gühne, Phys. Rev. A **72**, 022340 (2005).
[32] G. Wang, M. Ying, Phys. Rev. A **75**, 052332 (2007).
[33] D. Gottesman, in Quantum Computing and Quantum Communications: First NASA International Conference, edited by C. P. Williams Springer-Verlag, Berlin, 1999.
[34] E. Hostens, J. Dehaene and B. De Moor, Phys. Rev. A **71**, 042315 (2005).
[35] A. Y. Vlasov, e-print quant-ph/0210049.
[36] M. A. Nielsen, M. J. Bremner, J. L. Dodd, A. M. Childs, and C. M. Dawson, Phys. Rev. A **66**, 022317 (2002).
[37] M. A. Nielsen, and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, 2000), p.470.