Perfect many-to-one teleportation with stabilizer states

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We study the possibility of performing perfect teleportation of unknown quantum states from multiple senders to a single receiver with a previously shared stabilizer state. In the model we considered, the utilized stabilizer state is partitioned into several subsystems and then each subsystem is distributed to a distinct party. We present two sufficient conditions for a stabilizer state to achieve a given nonzero teleportation capacity with respect to a given partition plan. The corresponding teleportation protocols are also explicitly given. Interestingly, we find that even mixed stabilizer states are also useful for perfect many-to-one teleportation. Finally, our work provides a new perspective from stabilizer formalism to view the standard teleportation protocol and also suggests a new technique for analyzing teleportation capability of multipartite entangled states.

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I. INTRODUCTION

Entanglement is an intrigue feature of quantum mechanics. It has been exploited as a resource to carry out various amazing tasks which are impossible in classical physics. A remarkable example is quantum teleportation [1] which allows indirect transmission of quantum information between distant parties by using previously shared entanglement and classical communication between them. Indeed, teleportation has become a basic building block of many quantum communication and quantum computation protocols nowadays.

It is widely acknowledged that a thorough understanding of the power of entanglement in information procession is one of the major goals of quantum information theory. One of the key steps toward this goal is to give a complete characterization of teleportation capability of quantum entanglement. However, up to now, most progress in this direction is restricted to the simple case of bipartite entangled states [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18]. Results about the power of general multipartite entangled states in teleportation are still scarce. This is partially due to the exponentially growing complexity of multipartite states. To avoid such an unmanageable complexity, some authors chose to consider special multipartite states that own certain symmetry. For examples, one can see Refs. [19, 20, 21, 22, 23, 24, 25, 26, 27].

In this paper we study the usefulness of stabilizer states for perfect teleportation. Stabilizer states have played an important role in quantum information theory, especially in the field of quantum error correction [28, 29] and cluster state quantum computation [30]. They can be described in an elegant and compact form named the stabilizer formalism [31, 32], which has also lead to novel perspectives to many phenomena in quantum information science and quantum mechanics [33, 34, 35].

Let us first fix the model of teleportation in the multipartite case. Suppose ρ is an n-qudit state. Divide its n qudits into m groups $T_1, T_2, \ldots, T_m$ for some $2 \leq m \leq n$ and distribute the subsystem T_i to the i-th party A_i, for $i = 1, 2, \ldots, m$. Now assume that A_i has an unknown a_i-qudit state σ_i, for $i = 1, 2, \ldots, m-1$. We want to know whether $A_1, A_2, \ldots, A_{m-1}$ can *simultaneously faithfully* teleport the states $\sigma_1, \sigma_2, \ldots, \sigma_{m-1}$ to A_m by performing local operations on the particles they have and classical communications among them(LOCC). If this is possible, then $(a_1, a_2, \ldots, a_{m-1})$ is called an achievable teleportation capacity for ρ with respect to the grouping plan $T_1, T_2, \ldots, T_m$. For a given state ρ, each grouping strategy will define a region of achievable teleportation capacities. Our question is exactly to determine such a region for all possible grouping strategies.

Our main results are two sufficient conditions for a stabilizer state to achieve a given nonzero teleportation capacity with respect to a given partition plan. While the first condition is only suitable for bipartitions, the second can be applied to general partitions. The corresponding teleportation protocols are also explicitly given. Interestingly, we find that even mixed stabilizer states are also useful for perfect many-to-one teleportation. Finally, our work provides a new perspective from the stabilizer formalism to view the standard teleportation protocol and also suggests a new technique for analyzing teleportation capability of multipartite entangled states.

This paper is organized as follows. In Sec. II we briefly recall some basic facts about the stabilizer formalism. In Sec. III, we study the usefulness of stabilizer states for perfect teleportation and also construct our teleportation protocols. In Sec. IV, we analyze several concrete examples by applying our theorems. Finally, Sec. V summarizes our results.

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II. PRELIMINARY

In this section, we review some fundamental facts about stabilizer state and its corresponding stabilizer formalism. Although in most literatures the notion of stabilizer state was put forward in the context of multiqubit systems, it can actually be generalized without essential difficulty to arbitrary higher-dimensional systems as well. Similar topics have also been explored in Refs. [36, 37, 38, 39]. So here we directly start with the general higher-dimensional case.

Consider a d-dimensional Hilbert space. Define $X_{(d)}$ and $Z_{(d)}$ as follows:

$$\begin{aligned} X_{(d)}|j\rangle &= |j \oplus 1\rangle, \\ Z_{(d)}|j\rangle &= \omega^j |j\rangle, \end{aligned} \tag{1}$$

where $j \in \mathbb{Z}_d$, $\omega = e^{i\frac{2\pi}{d}}$ is the d-th root of unity over the complex field and the '$\oplus$' sign denotes addition modulo d. Then the matrices $\{X_{(d)}^a Z_{(d)}^b : a, b = 0, 1, \ldots, d-1\}$ are considered as the generalized Pauli matrices over the d-dimensional space. In what follows, without causing ambiguity, we will omit the subscript '(d)' and use X, Z to denote $X_{(d)}, Z_{(d)}$. The commutation relations among the generalized Pauli matrices over the d-dimensional space are given by

$$(X^a Z^b)(X^j Z^k) = \omega^{bj-ak}(X^j Z^k)(X^a Z^b). \tag{2}$$

It can be checked that if d is even and ab is odd, the eigenvalues of $X^a Z^b$ are $\omega^{1/2}, \omega^{c+1/2}, \omega^{2c+1/2}, \ldots, \omega^{d-c+1/2}$ for some factor c of d; otherwise, the eigenvalues of $X^a Z^b$ are $1, \omega^c, \omega^{2c}, \ldots, \omega^{d-c}$ for some factor c of d.

Let X_i, Z_i denote the operations of X, Z on the i-th qudit respectively. The generalized Pauli group on n qudits $G_n^{(d)}$ is generated under multiplication by the Pauli matrices acting on each qudit, together with the phase factor $\gamma = \sqrt{\omega}$, i.e.

$$G_n^{(d)} = \langle \gamma, X_1, Z_1, X_2, Z_2, \ldots, X_n, Z_n \rangle. \tag{3}$$

By Eq.(2), for any $g = \gamma^c X^{a_1} Z^{b_1} \otimes X^{a_2} Z^{b_2} \otimes \cdots \otimes X^{a_n} Z^{b_n}$, $h = \gamma^{c'} X^{a_1'} Z^{b_1'} \otimes X^{a_2'} Z^{b_2'} \otimes \cdots \otimes X^{a_n'} Z^{b_n'} \in G_n^{(d)}$, their commutation relation is given by

$$gh = \omega^{\sum_{i=1}^{n}(b_i a_i' - a_i b_i')} hg. \tag{4}$$

In particular, g and h commute if and only if $\sum_{i=1}^{n}(b_i a_i' - a_i b_i')$ is a multiple of d.

For a set of commuting operators $g_1, g_2, \ldots, g_k \in G_n^{(d)}$, we say that they are independent if $\forall i = 1, 2, \ldots, k$,

$$\langle g_1, g_2, \ldots, g_k \rangle \neq \langle g_1, g_2, \ldots, g_{i-1}, g_{i+1}, \ldots, g_k \rangle. \tag{5}$$

Define $G_n'^{(d)}$ to be the subset of $G_n^{(d)}$ composed of all the operators whose eigenvalues are of the form $1, \omega^c, \omega^{2c}, \ldots, \omega^{d-c}$ for some factor c of d. Now suppose $g_1, g_2, \ldots, g_k$ are independent commuting operators in $G_n'^{(d)}$. Let

$$S = \langle g_1, g_2, \ldots, g_k \rangle \tag{6}$$

be the Abelian subgroup generated by them. A state $|\psi\rangle$ is said to be stabilized by S, or S is the stabilizer of $|\psi\rangle$, if

$$g_i|\psi\rangle = |\psi\rangle, \ \forall i = 1, 2, \ldots, k. \tag{7}$$

All the states stabilized by S constitute a subspace denoted by V_S. With the fact $\sum_{j=0}^{d-1} \omega^{j\lambda} = 0, \forall \lambda = 1, 2, \ldots, d-1$, one can verify that the projection operator onto V_S is

$$P_S = \frac{1}{d^k} \prod_{i=1}^{k} (\sum_{j=0}^{d-1} g_i^j). \tag{8}$$

Then the maximally mixed state over V_S is

$$\rho_S = P_S / tr(P_S). \tag{9}$$

In particular, if there is a unique pure state (up to an overall phase) stabilized by S, then $g_1, g_2, \ldots, g_k$ are called a *complete* set of stabilizer generators and S is called a *complete* stabilizer.

In practice we are often interested in the stabilized subspace V_S, which is the simultaneous eigenspace of the operators $g_1, g_2, \ldots, g_k$ corresponding to the eigenvalues $1, 1, \ldots, 1$. But in general we can also consider the simultaneous eigenspace of $g_1, g_2, \ldots, g_k$ corresponding to their other eigenvalues. In what follows, we will use $P(g_1, g_2, \ldots, g_k; \overrightarrow{x})$ to denote the projection operator onto the simultaneous eigenspace of $g_1, g_2, \ldots, g_k$ corresponding to the eigenvalues $\omega^{x_1}, \omega^{x_2}, \ldots, \omega^{x_k}$, where $\overrightarrow{x} = (x_1, x_2, \ldots, x_k) \in \mathbb{Z}_d^k$. With the fact $\sum_{j=0}^{d-1} \omega^{j\lambda} = 0, \forall \lambda = 1, 2, \ldots, d-1$, one can see

$$P(g_1, g_2, \ldots, g_k; \overrightarrow{x}) = \frac{1}{d^k} \prod_{i=1}^{k} (\sum_{j=0}^{d-1} \omega^{-jx_i} g_i^j), \tag{10}$$

In particular, $P_S = P(g_1, g_2, \ldots, g_k; \overrightarrow{0})$.

For any two subgroups H_1, H_2 of $G_n^{(d)}$, if there exists a bijective map $N : H_1 \to H_2$ such that for any $h_1, h_2 \in H_1$, $N(h_1 h_2) = N(h_1)N(h_2)$, then we say H_1 and H_2 are isomorphic. We will denote this isomorphism by $H_1 \cong H_2$. Given several operators $g_1, g_2, \ldots, g_k \in G_n^{(d)}$, we are usually interested in the commutation relations among them. In this situation, we may write, e.g. $g_1 = \overline{Z}_1$, $g_2 = \overline{X}_1$, $g_3 = \overline{Z}_2$, $g_4 = \overline{Z}_3$. The intention of this writing is to indicate that $\langle g_1, g_2, g_3, g_4 \rangle \cong \langle Z_1, X_1, Z_2, Z_3 \rangle$ and the isomorphism between them is induced by $N(g_1) = Z_1$, $N(g_2) = X_1$, $N(g_3) = Z_2$, $N(g_4) = Z_3$. Note that g_1 may not actually be the action of Z on the first qudit, and similarly for g_2, g_3, g_4.

III. PERFECT TELEPORTATION WITH STABILIZER STATES

In this section we study the usefulness of the state ρ_S given by Eqs.(8) and (9) for perfect teleportation with multiple senders and one receiver. Note that only when S is a complete stabilizer, ρ_S is a pure state. In other cases, ρ_S is a mixed state. But our discussion below does not need to discriminate between the two cases because it essentially does not depend on the purity of ρ_S.

At first, we need to introduce several definitions and notations. We will use $[1,n]$ to denote the set of integers $\{1,2,\ldots,n\}$. If $T_1,T_2,\ldots,T_l$ are disjoint proper subsets of $[1,n]$ and they satisfy $\cup_{i=1}^{l}T_i=[1,n]$, then we say $\{T_1,T_2,\ldots,T_l\}$ is a partition of $[1,n]$. For any $T\subset[1,n]$, we use $|T|$ to denote the number of elements in T and also use T^C to denote the complement of T in $[1,n]$. For any $T\subset[1,n]$ and $g=\gamma^c X^{a_1}Z^{b_1}\otimes X^{a_2}Z^{b_2}\otimes\cdots\otimes X^{a_n}Z^{b_n}\in G_n^{(d)}$, define the restriction of g on T to be

$$g^{(T)}=\bigotimes_{i\in T}X^{a_i}Z^{b_i}. \tag{11}$$

Furthermore, for $S=\langle g_1,g_2,\ldots,g_k\rangle$, define the restriction of S on T to be

$$S^{(T)}=\langle\gamma,g_1^{(T)},g_2^{(T)},\ldots,g_k^{(T)}\rangle. \tag{12}$$

One can easily see that the choice of stabilizer generators $g_1,g_2,\ldots,g_k$ does not affect the result $S^{(T)}$. So it is well-defined. In addition, in what follows, we will use the subset $T=\{i_1,i_2,\ldots,i_t\}\subset[1,n]$ to represent the subsystem of ρ_S composed of the i_1-th, i_2-th, ..., i_t-th qudits. We also use $\rho_S^{(T)}$ to denote the reduced density matrix of ρ_S on this subsystem. Finally, for several subgroups $P,P_1,P_2,\ldots,P_k$ of $G_n^{(d)}$, if we write

$$P=\prod_{i=1}^{k}P_i=P_1P_2\ldots P_k \tag{13}$$

we mean that each element of P_i commutes with each element of P_j, $\forall 1\le i\ne j\le k$, and

$$P=\{g_1g_2\ldots g_k:\forall i=1,2,\ldots,k,g_i\in P_i\}. \tag{14}$$

Now let us reformulate our problem precisely. Suppose $g_1,g_2,\ldots,g_k$ are independent commuting operators in $G_n'^{(d)}$. The state ρ_S given by Eqs.(8) and (9) is the maximally mixed state over the subspace stabilized by $S=\langle g_1,g_2,\ldots,g_k\rangle$. Assume that $\{T_1,T_2,\ldots,T_{m+1}\}$ is a partition of $[1,n]$. $A_1,A_2,\ldots,A_{m+1}$ are distant parties and A_i holds the subsystem T_i of ρ_S, for $i=1,2,\ldots,m+1$. Now suppose A_i has an unknown a_i-qudit state σ_i, for $i=1,2,\ldots,m$. If $A_1,A_2,\ldots,A_m$ can *simultaneously faithfully* teleport the states $\sigma_1,\sigma_2,\ldots,\sigma_m$ to A_{m+1} by performing LOCC operations on the particles they have, the $(a_1,a_2,\ldots,a_m)$ is said to be an achievable teleportation capacity for ρ_S with respect to $\{T_1,T_2,\ldots,T_{m+1}\}$. Our goal is to determine the region of achievable teleportation capacities for ρ_S with respect to an arbitrary partition plan.

Before presenting our main theorems, it is necessary to prove a lemma at first.

In Ref.[40] the authors found an interesting theorem which states that for any two isomorphic subgroups G and H of the Pauli group on n qubits, there exists a unitary operation U such that for any $g\in G$, there exists $h\in H$ such that $g=UhU^\dagger$ up to an overall phase. Here our lemma can be viewed as a partial extension of this theorem to the higher dimensional case.

Lemma 1 *If a subgroup H of $G_n^{(d)}$ is isomorphic to $G=\langle\gamma^c,Z_1^{a_1},Z_2^{a_2},\ldots,Z_s^{a_s},X_1^{b_1},X_2^{b_2},\ldots,X_t^{b_t}\rangle$ for some $t\le s\le n$, $c\in\mathbb{Z}_{2d}$ and $a_1,a_2,\ldots,a_s,b_1,b_2,\ldots,b_t\in\mathbb{Z}_d$, then there exists a unitary operation U such that for any $h\in H$, there exists $g\in G$ such that $h=UgU^\dagger$.*

Proof: By the definition of isomorphism we can write H as

$$H=\langle\gamma^c,\overline{Z}_1^{a_1},\overline{Z}_2^{a_2},\ldots,\overline{Z}_s^{a_s},\overline{X}_1^{b_1},\overline{X}_2^{b_2},\ldots,\overline{X}_t^{b_t}\rangle \tag{15}$$

for some $\overline{Z}_1,\overline{Z}_2,\ldots,\overline{Z}_s,\overline{X}_1,\overline{X}_2,\ldots,\overline{X}_t\in G_n^{(d)}$.

Note that $\overline{Z}_1,\overline{Z}_2,\ldots,\overline{Z}_s$ mutually commute and their simultaneous eigenspace corresponding to the eigenvalues $\omega^{x_1},\omega^{x_2},\ldots,\omega^{x_s}$ is d^{n-s}-dimensional, $\forall x_1,x_2,\ldots,x_s\in\mathbb{Z}_d$. For any $\overrightarrow{x}=(x_1,x_2,\ldots,x_s)\in\mathbb{Z}_d^s$, define $\overrightarrow{x}|_{[t+1,s]}=(x_{t+1},x_{t+2},\ldots,x_s)$. Suppose $\{|\overline{\psi}(\overrightarrow{x}|_{[t+1,s]};\alpha)\rangle\}_{\alpha=1}^{d^{n-s}}$ is an arbitrary orthonormal basis of the simultaneous eigenspace of $\overline{Z}_1,\overline{Z}_2,\ldots,\overline{Z}_t,\overline{Z}_{t+1},\ldots,\overline{Z}_s$ corresponding to the eigenvalues $1,1,\ldots,1,\omega^{x_{t+1}},\omega^{x_{t+2}},\ldots,\omega^{x_s}$. Define

$$|\overline{\phi}(\overrightarrow{x};\alpha)\rangle=\overline{X}_1^{x_1}\overline{X}_2^{x_2}\ldots\overline{X}_t^{x_t}|\overline{\psi}(\overrightarrow{x}|_{[t+1,s]};\alpha)\rangle, \tag{16}$$

$\forall\alpha=1,2,\ldots,d^{n-s}$. Then $\{|\overline{\phi}(\overrightarrow{x};\alpha)\rangle\}_{\alpha=1}^{d^{n-s}}$ is an orthonormal basis for the simultaneous eigenspace of $\overline{Z}_1,\overline{Z}_2,\ldots,\overline{Z}_s$ corresponding to the eigenvalues $\omega^{x_1},\omega^{x_2},\ldots,\omega^{x_s}$. To see this, one only needs to realize that for $\forall i\in[1,t]$,

$$\begin{array}{l}\overline{Z}_i|\overline{\phi}(\overrightarrow{x};\alpha)\rangle\\ =\overline{Z}_i\overline{X}_1^{x_1}\overline{X}_2^{x_2}\ldots\overline{X}_t^{x_t}|\overline{\psi}(\overrightarrow{x}|_{[t+1,s]};\alpha)\rangle\\ =\omega^{x_i}\overline{X}_1^{x_1}\overline{X}_2^{x_2}\ldots\overline{X}_t^{x_t}\overline{Z}_i|\overline{\psi}(\overrightarrow{x}|_{[t+1,s]};\alpha)\rangle\\ =\omega^{x_i}\overline{X}_1^{x_1}\overline{X}_2^{x_2}\ldots\overline{X}_t^{x_t}|\overline{\psi}(\overrightarrow{x}|_{[t+1,s]};\alpha)\rangle\\ =\omega^{x_i}|\overline{\phi}(\overrightarrow{x};\alpha)\rangle,\end{array} \tag{17}$$

and $\forall i\in[t+1,s]$,

$$\begin{array}{l}\overline{Z}_i|\overline{\phi}(\overrightarrow{x};\alpha)\rangle\\ =\overline{Z}_i\overline{X}_1^{x_1}\overline{X}_2^{x_2}\ldots\overline{X}_t^{x_t}|\overline{\psi}(\overrightarrow{x}|_{[t+1,s]};\alpha)\rangle\\ =\overline{X}_1^{x_1}\overline{X}_2^{x_2}\ldots\overline{X}_t^{x_t}\overline{Z}_i|\overline{\psi}(\overrightarrow{x}|_{[t+1,s]};\alpha)\rangle\\ =\omega^{x_i}\overline{X}_1^{x_1}\overline{X}_2^{x_2}\ldots\overline{X}_t^{x_t}|\overline{\psi}(\overrightarrow{x}|_{[t+1,s]};\alpha)\rangle\\ =\omega^{x_i}|\overline{\phi}(\overrightarrow{x};\alpha)\rangle.\end{array} \tag{18}$$

Similarly, suppose $\{|\psi(\overrightarrow{x}|_{[t+1,s]};\alpha)\rangle\}_{\alpha=1}^{d^{n-s}}$ is an arbitrary orthonormal basis of the simultaneous eigenspace of

$Z_1, Z_2, \ldots, Z_t, Z_{t+1}, \ldots, Z_s$ corresponding to the eigenvalues $1, 1, \ldots, 1, \omega^{x_{t+1}}, \omega^{x_{t+2}}, \ldots, \omega^{x_s}$. Define

$$|\phi(\overrightarrow{x};\alpha)\rangle = X_1^{x_1} X_2^{x_2} \ldots X_t^{x_t} |\psi(\overrightarrow{x}|_{[t+1,s]};\alpha)\rangle, \quad (19)$$

$\forall \alpha = 1, 2, \ldots, d^{n-s}$. Then $\{|\phi(\overrightarrow{x};\alpha)\}_{\alpha=1}^{d^{n-s}}$ is an orthonormal basis for the simultaneous eigenspace of $Z_1, Z_2, \ldots, Z_s$ corresponding to the eigenvalues $\omega^{x_1}, \omega^{x_2}, \ldots, \omega^{x_s}$, $\forall \overrightarrow{x} = (x_1, x_2, \ldots, x_s) \in \mathbb{Z}_d^s$.

Define the following unitary operation

$$U = \sum_{\overrightarrow{x} \in \mathbb{Z}_d^s} \sum_{\alpha=1}^{d^{n-s}} |\overline{\phi}(\overrightarrow{x};\alpha)\rangle\langle\phi(\overrightarrow{x};\alpha)|. \quad (20)$$

From its definition, one can easily see that U is indeed unitary and

$$\overline{Z}_i = U Z_i U^\dagger, \ \forall i = 1, 2, \ldots, s. \quad (21)$$

Moreover, $\forall i = 1, 2, \ldots, t$, $\forall \overrightarrow{x} = (x_1, x_2, \ldots, x_s) \in \mathbb{Z}_d^s$, $\forall \alpha = 1, 2, \ldots, d^{n-s}$, we have

$$\begin{array}{l} \overline{X}_i |\overline{\phi}(\overrightarrow{x};\alpha)\rangle \\ = \overline{X}_i \overline{X}_1^{x_1} \overline{X}_2^{x_2} \ldots \overline{X}_t^{x_t} |\overline{\psi}(\overrightarrow{x}|_{[t+1,s]};\alpha)\rangle \\ = \overline{X}_1^{x_1} \overline{X}_2^{x_2} \ldots \overline{X}_i^{x_i \oplus 1} \ldots \overline{X}_t^{x_t} |\overline{\psi}(\overrightarrow{x}|_{[t+1,s]};\alpha)\rangle \\ = |\overline{\phi}(\overrightarrow{x} \oplus \overrightarrow{e}_i;\alpha)\rangle, \end{array} \quad (22)$$

and

$$\begin{array}{l} U X_i U^\dagger |\overline{\phi}(\overrightarrow{x};\alpha)\rangle \\ = U X_i |\phi(\overrightarrow{x};\alpha)\rangle \\ = U X_i X_1^{x_1} X_2^{x_2} \ldots X_t^{x_t} |\psi(\overrightarrow{x}|_{[t+1,s]};\alpha)\rangle \\ = U X_1^{x_1} X_2^{x_2} \ldots X_i^{x_i \oplus 1} \ldots X_t^{x_t} |\psi(\overrightarrow{x}|_{[t+1,s]};\alpha)\rangle \\ = U |\phi(\overrightarrow{x} \oplus \overrightarrow{e}_i;\alpha)\rangle \\ = |\overline{\phi}(\overrightarrow{x} \oplus \overrightarrow{e}_i;\alpha)\rangle, \end{array} \quad (23)$$

where $\overrightarrow{e}_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ (1 is the i-th element) and '$\oplus$' denotes addition modulo d. Since $\{\overline{\phi}(\overrightarrow{x};\alpha)\}_{\overrightarrow{x} \in \mathbb{Z}_d^s, 1 \le \alpha \le d^{n-s}}$ is an orthonormal basis of the n-qudit Hilbert space, two above equations actually tell us that

$$\overline{X}_i = U X_i U^\dagger, \ \forall i = 1, 2, \ldots, t. \quad (24)$$

Now the validity of this lemma follows immediately from Eqs.(21) and (24).

■

With the help of this lemma, we find that for a bipartition $\{T_1, T_2\}$, the structure of $S^{(T_2)}$ can influence the teleportation capacity of ρ_S with respect to this partition $\{T_1, T_2\}$, as the following theorem states:

Theorem 1 *Suppose $\{T_1, T_2\}$ is a bipartition of $[1, n]$. If there exist subgroups P_1 and P_2 of S such that*

$$\begin{array}{l} S^{(T_2)} = P_1^{(T_2)} P_2^{(T_2)}, \\ P_1^{(T_2)} \cong G_t^{(d)}, \\ P_2^{(T_2)} \cong \langle \gamma, Z_1^{a_1}, Z_2^{a_2}, \ldots, Z_s^{a_s}, X_1^{b_1}, X_2^{b_2}, \ldots, X_u^{b_u} \rangle, \end{array} \quad (25)$$

for some $t \ge 0$, $s \ge u \ge 0$, and $a_1, a_2, \ldots, a_s, b_1, b_2, \ldots, b_u \in \mathbb{Z}_d$, then t is an achievable teleportation capacity for ρ_S with respect to the partition $\{T_1, T_2\}$.

Proof: Suppose $|T_1| = m$ and $|T_2| = n-m$. By Eq.(25) we can find independent generators $g_1, g_2, \ldots, g_k$ of S such that

$$\begin{array}{ll} g_{2i-1} = R_{2i-1} \otimes \overline{Z}_i, & \forall 1 \le i \le t; \\ g_{2i} = R_{2i} \otimes \overline{X}_i, & \forall 1 \le i \le t; \\ g_{2t+i} = R_{2t+i} \otimes \overline{Z}_{t+i}^{a_i}, & \forall 1 \le i \le s; \\ g_{2t+s+i} = R_{2t+s+i} \otimes \overline{X}_{t+i}^{b_i}, & \forall 1 \le i \le u; \\ g_i = R_i \otimes I, & \forall 2t+s+u+1 \le i \le k. \end{array} \quad (26)$$

where $R_1, R_2, \ldots, R_k$ are some operators on the subsystem T_1, $\overline{Z}_1, \overline{Z}_2, \ldots, \overline{Z}_{t+s}, \overline{X}_1, \overline{X}_2, \ldots, \overline{X}_{t+u} \in G_{n-m}^{(d)}$ are operators on the subsystem T_2.

By lemma 1, we can find a unitary operator U acting on the subsystem T_2 such that

$$\begin{array}{ll} U \overline{Z}_i U^\dagger = Z_i, & \forall 1 \le i \le t+s, \\ U \overline{X}_i U^\dagger = X_i, & \forall 1 \le i \le t+u. \end{array} \quad (27)$$

Define

$$h_i = (I \otimes U) g_i (I \otimes U^\dagger), \quad (28)$$

$\forall i = 1, 2, \ldots, k$. Then we have

$$\begin{array}{ll} h_{2i-1} = R_{2i-1} \otimes Z_i, & \forall 1 \le i \le t; \\ h_{2i} = R_{2i} \otimes X_i, & \forall 1 \le i \le t; \\ h_{2t+i} = R_{2t+i} \otimes Z_{t+i}^{a_i}, & \forall 1 \le i \le s; \\ h_{2t+s+i} = R_{2t+s+i} \otimes X_{t+i}^{b_i}, & \forall 1 \le i \le u; \\ h_i = R_i \otimes I, & \forall 2t+s+u+1 \le i \le k. \end{array} \quad (29)$$

Suppose

$$T_2 = \{i_1, i_2, \ldots, i_{n-m}\} \quad (30)$$

with $i_1 < i_2 < \cdots < i_{n-m}$. One can see Eq.(29) implies $t \le n-m$. So define

$$\begin{array}{l} T_2' = \{i_1, i_2, \ldots, i_t\}, \\ T_2'' = \{i_{t+1}, i_{t+2}, \ldots, i_{n-m}\}. \end{array} \quad (31)$$

Since $g_1, g_2, \ldots, g_k$ mutually commute, by the definition of $h_1, h_2, \ldots, h_k$ we know that they also mutually commute. For $i = 1, 2, \ldots, t$, define

$$\begin{array}{ll} h'_{2i-1} & = R_{2i-1} \otimes h_{2i-1}^{(T_2')} \\ & = R_{2i-1} \otimes Z_i, \\ h'_{2i} & = R_{2i} \otimes h_{2i}^{(T_2')} \\ & = R_{2i} \otimes X_i, \end{array} \quad (32)$$

Then $h'_1, h'_2, \ldots, h'_{2t}$ are commuting operators on the subsystem $T_1 \cup T_2'$.

Now suppose Alice and Bob hold the subsystems T_1 and T_2 of ρ_S respectively. Assume Alice has a t-qudit system M and it is in an unknown state σ. We firstly propose a teleportation protocol, and then prove its validity. The protocol is as follows:

(1)Bob performs the unitary operation U on his subsystem T_2.

(2)Alice performs the projective measurement consisting of the projection operators $\{P(h_1', h_2', \ldots, h_{2t}'; \overrightarrow{x}) : \overrightarrow{x} \in \mathbb{Z}_d^{2t}\}$ on her T_1 subsystem of ρ_S and M. Then she tells the measurement outcome $\overrightarrow{x} = (x_1, x_2, \ldots, x_{2t})$ to Bob.

(3)Bob performs the unitary operation

$$V(\overrightarrow{x}) = \bigotimes_{i=1}^{t} (Z^{-x_{2i}} X^{x_{2i-1}}) \tag{33}$$

on the subsystem T_2'.

Now we prove that after this procedure, the state of the subsystem T_2' is exactly σ.

After step (1), one can see that ρ_S becomes $\rho_{S'} = P_{S'}/tr(P_{S'})$ where $S' = \langle h_1, h_2, \ldots, h_k \rangle$ and $P_{S'}$ is the projection operator onto the subspace stabilized by S'. By Eqs.(8) and (29), we have

$$\begin{aligned} P_{S'} &= \tfrac{1}{d^k} \prod_{i=1}^{k} (\sum_{j=0}^{d-1} h_i^j) \\ &= \tfrac{1}{d^k} \sum_{j_1,\ldots,j_k=0}^{d-1} \prod_{i=1}^{k} h_i^{j_i} \\ &= \tfrac{1}{d^k} \sum_{\overrightarrow{j} \in \mathbb{Z}_d^k} A(\overrightarrow{j}) \otimes B(\overrightarrow{j}) \otimes C(\overrightarrow{j}) \end{aligned} \tag{34}$$

where $\overrightarrow{j} = (j_1, j_2, \ldots, j_k)$, and

$$\begin{aligned} A(\overrightarrow{j}) &= R_1^{j_1} R_2^{j_2} \ldots R_k^{j_k}, \\ B(\overrightarrow{j}) &= \bigotimes_{i=1}^{t} (Z^{j_{2i-1}} X^{j_{2i}}), \\ C(\overrightarrow{j}) &= \bigotimes_{i=1}^{u} (Z^{a_i j_{2t+i}} X^{b_i j_{2t+s+i}}) \otimes \bigotimes_{i=u+1}^{s} Z^{a_i j_{2t+i}} \otimes I \end{aligned} \tag{35}$$

are operators on the subsystems T_1, T_2' and T_2'' respectively.

Then, by Eqs.(10) and (32), the projection operators of Alice's projective measurement in step (2) are

$$\begin{aligned} &P(h_1', h_2', \ldots, h_{2t}'; \overrightarrow{x}) \\ &= \tfrac{1}{d^{2t}} \prod_{i=1}^{2t} (\sum_{j=0}^{d-1} \omega^{-jx_i} h_i'^j) \\ &= \tfrac{1}{d^{2t}} \sum_{j_1,\ldots,j_{2t}=0}^{d-1} \prod_{i=1}^{2t} (\omega^{-j_i x_i} h_i'^{j_i}) \\ &= \tfrac{1}{d^{2t}} \sum_{\overrightarrow{j} \in \mathbb{Z}_d^{2t}} (\prod_{i=1}^{2t} \omega^{-j_i x_i}) D(\overrightarrow{j}) \otimes E(\overrightarrow{j}), \end{aligned} \tag{36}$$

where $\overrightarrow{j} = (j_1, j_2, \ldots, j_{2t})$, and

$$\begin{aligned} D(\overrightarrow{j}) &= R_1^{j_1} R_2^{j_2} \ldots R_{2t}^{j_{2t}}, \\ E(\overrightarrow{j}) &= \bigotimes_{i=1}^{t} (Z^{j_{2i-1}} X^{j_{2i}}) \end{aligned} \tag{37}$$

are operators acting on the subsystems T_1 and M respectively, $\forall \overrightarrow{x} = (x_1, x_2, \ldots, x_{2t}) \in \mathbb{Z}_d^{2t}$.

Since the density matrix of any t-qudit state can always be written as the linear combination of the generalized Pauli group elements $\{L(\overrightarrow{y}) \equiv \bigotimes_{i=1}^{t} (Z^{y_{2i-1}} X^{y_{2i}}) : \overrightarrow{y} = (y_1, y_2, \ldots, y_{2t}) \in \mathbb{Z}_d^{2t}\}$, we can assume that the unknown state σ is

$$\begin{aligned} \sigma &= \sum_{\overrightarrow{y} \in \mathbb{Z}_d^{2t}} \lambda_{\overrightarrow{y}} L(\overrightarrow{y}) \\ &= \sum_{\overrightarrow{y} \in \mathbb{Z}_d^{2t}} \lambda_{\overrightarrow{y}} \bigotimes_{i=1}^{t} (Z^{y_{2i-1}} X^{y_{2i}}) \end{aligned} \tag{38}$$

for some coefficients $\{\lambda_{\overrightarrow{y}}\}$. Then after Alice's measurement in step (2), if the measurement outcome is $\overrightarrow{x} = (x_1, x_2, \ldots, x_{2t})$, the state of the whole system becomes, up to a normalizing factor,

$$\begin{aligned} \widetilde{\rho} &= P(h_1', h_2', \ldots, h_{2t}'; \overrightarrow{x})(P_{S'} \otimes \sigma) P(h_1', h_2', \ldots, h_{2t}'; \overrightarrow{x}) \\ &= \tfrac{1}{d^{4t+k}} \sum_{\overrightarrow{j} \in \mathbb{Z}_d^k} \sum_{\overrightarrow{j}' \in \mathbb{Z}_d^{2t}} \sum_{\overrightarrow{j}'' \in \mathbb{Z}_d^{2t}} \sum_{\overrightarrow{y} \in \mathbb{Z}_d^{2t}} [\lambda_{\overrightarrow{y}} (\prod_{i=1}^{2t} \omega^{-x_i (j_i' + j_i'')}) \\ &\quad F(\overrightarrow{j}, \overrightarrow{j}', \overrightarrow{j}'') \otimes B(\overrightarrow{j}) \otimes C(\overrightarrow{j}) \otimes N(\overrightarrow{j}', \overrightarrow{j}'', \overrightarrow{y})] \end{aligned} \tag{39}$$

where $\overrightarrow{j} = (j_1, j_2, \ldots, j_k)$, $\overrightarrow{j}' = (j_1', j_2', \ldots, j_{2t}')$, $\overrightarrow{j}'' = (j_1'', j_2'', \ldots, j_{2t}'')$, and

$$\begin{aligned} &F(\overrightarrow{j}, \overrightarrow{j}', \overrightarrow{j}'') \\ &= D(\overrightarrow{j}') A(\overrightarrow{j}) D(\overrightarrow{j}'') \\ &= R_1^{j_1'} R_2^{j_2'} \ldots R_{2t}^{j_{2t}'} R_1^{j_1} R_2^{j_2} \ldots R_k^{j_k} R_1^{j_1''} R_2^{j_2''} \ldots R_{2t}^{j_{2t}''}, \\ &N(\overrightarrow{j}', \overrightarrow{j}'', \overrightarrow{y}) \\ &= E(\overrightarrow{j}') L(\overrightarrow{y}) E(\overrightarrow{j}'') \\ &= \bigotimes_{i=1}^{t} (Z^{j_{2i-1}'} X^{j_{2i}'} Z^{y_{2i-1}} X^{y_{2i}} Z^{j_{2i-1}''} X^{j_{2i}''}) \end{aligned} \tag{40}$$

are operators on the subsystems T_1 and M respectively.

Although Eqs.(39) and (40) seem very intricate, after tracing out the subsystems T_1, T_2'' and M, the reduced density matrix on the subsystem T_2' will become much simpler. Let us consider each summation term $F(\overrightarrow{j}, \overrightarrow{j}', \overrightarrow{j}'') \otimes B(\overrightarrow{j}) \otimes C(\overrightarrow{j}) \otimes N(\overrightarrow{j}', \overrightarrow{j}'', \overrightarrow{y})$.

Firstly, define

$$\begin{aligned} \Theta &= \{(j_{2t+1}, j_{2t+2}, \ldots, j_k) : \\ &\forall i = 1, 2, \ldots, s, \quad a_i j_{2t+i} \equiv 0 (mod \quad d); \\ &\forall i = 1, 2, \ldots, u, \quad b_i j_{2t+s+i} \equiv 0 (mod \quad d)\}. \end{aligned} \tag{41}$$

Then $tr(C(\overrightarrow{j})) \neq 0$ if and only if $(j_{2t+1}, j_{2t+2}, \ldots, j_k) \in \Theta$.

Secondly, $tr(N(\overrightarrow{j}', \overrightarrow{j}'', \overrightarrow{y})) \neq 0$ if and only if $\forall i = 1, 2, \ldots, 2t$, $y_i + j_i' + j_i'' \equiv 0 (mod\ d)$;

Thirdly, note that

$$\begin{aligned} &F(\overrightarrow{j}, \overrightarrow{j}', \overrightarrow{j}'') = \omega^{\xi(\overrightarrow{j}, \overrightarrow{j}', \overrightarrow{j}'')} R_1^{j_1' + j_1 + j_1''} R_2^{j_2' + j_2 + j_2''} \ldots \\ &R_{2t}^{j_{2t}' + j_{2t} + j_{2t}''} R_{2t+1}^{j_{2t+1}} R_{2t+2}^{j_{2t+2}} \ldots R_k^{j_k} \end{aligned} \tag{42}$$

for some $\xi(\overrightarrow{j}, \overrightarrow{j}', \overrightarrow{j}'') \in \mathbb{Z}_d$. Define

$$\begin{aligned}\Omega &= \{(j_{2t+1}, j_{2t+2}, \dots, j_k) : \exists \lambda \neq 0 \in \mathbb{C}, s.t. \\ &R_{2t+1}^{j_{2t+1}} R_{2t+2}^{j_{2t+2}} \dots R_k^{j_k} = \lambda I\}.\end{aligned} \quad (43)$$

Now we will prove $tr(F(\overrightarrow{j}, \overrightarrow{j}', \overrightarrow{j}'')) \neq 0$ if and only if $\forall i = 1, 2, \dots, 2t$, $j_i + j_i' + j_i'' \equiv 0 (mod\ d)$ and $(j_{2t+1}, j_{2t+2}, \dots, j_k) \in \Omega$. To prove this, one needs to realize that if $F(\overrightarrow{j}, \overrightarrow{j}', \overrightarrow{j}'') = \mu I$ for some μ, then it should commute with R_i, $\forall i = 1, 2, \dots, 2t$. Besides, since $g_1, g_2, \dots, g_k$ mutually commute, by Eqs.(26), we get

$$\begin{array}{ll} R_{2i-1}R_{2i} = \omega^{-1} R_{2i} R_{2i-1}, & \forall 1 \leq i \leq t; \\ R_{2i-1}R_{2i'-1} = R_{2i'-1}R_{2i-1}, & \forall 1 \leq i \neq i' \leq t; \\ R_{2i-1}R_{2i'} = R_{2i'}R_{2i-1}, & \forall 1 \leq i \neq i' \leq t; \\ R_{2i}R_{2i'} = R_{2i'}R_{2i}, & \forall 1 \leq i \neq i' \leq t; \\ R_i R_j = R_j R_i, & \forall 1 \leq i \leq 2t, \forall 2t+1 \leq j \leq k. \end{array} \quad (44)$$

So by Eq.(42)), $F(\overrightarrow{j}, \overrightarrow{j}', \overrightarrow{j}'')$ commutes with R_i, $\forall i = 1, 2, \dots, 2t$ if and only if $j_i + j_i' + j_i'' \equiv 0 (mod\ d)$, $\forall i = 1, 2, \dots, 2t$. In this case, Eq.(42) reduces into

$$F(\overrightarrow{j}, \overrightarrow{j}', \overrightarrow{j}'') = \omega^{\xi(\overrightarrow{j}, \overrightarrow{j}', \overrightarrow{j}'')} R_{2t+1}^{j_{2t+1}} R_{2t+2}^{j_{2t+2}} \dots R_k^{j_k}. \quad (45)$$

Then we have $tr(F(\overrightarrow{j}, \overrightarrow{j}', \overrightarrow{j}'')) \neq 0$ if and only if $(j_{2t+1}, j_{2t+2}, \dots, j_k) \in \Omega$.

Summarizing the above argument, we know that only when $\forall i = 1, 2, \dots, 2t$, $y_i = j_i \equiv (-j_i' - j_i'') (mod\ d)$ and $(j_{2t+1}, j_{2t+2}, \dots, j_k) \in \Theta \cap \Omega$, the corresponding term $F(\overrightarrow{j}, \overrightarrow{j}', \overrightarrow{j}'') \otimes B(\overrightarrow{j}) \otimes C(\overrightarrow{j}) \otimes N(\overrightarrow{j}', \overrightarrow{j}'', \overrightarrow{y})$ will not vanish after tracing out $F(\overrightarrow{j}, \overrightarrow{j}', \overrightarrow{j}'')$, $C(\overrightarrow{j})$ and $N(\overrightarrow{j}', \overrightarrow{j}'', \overrightarrow{y})$. Note that by the definition of Θ and Ω, for any $(j_{2t+1}, j_{2t+2}, \dots, j_k) \in \Theta \cap \Omega$,

$$\epsilon I = h_{2t+1}^{j_{2t+1}} h_{2t+2}^{j_{2t+2}} \dots h_k^{j_k} \quad (46)$$

for some $\epsilon \in \mathbb{C}$. Suppose a state $|\psi\rangle$ is stabilized by $S' = \langle h_1, h_2, \dots, h_k \rangle$. Then by Eq.(46) we obtain

$$\epsilon |\psi\rangle = h_{2t+1}^{j_{2t+1}} h_{2t+2}^{j_{2t+2}} \dots h_k^{j_k} |\psi\rangle = |\psi\rangle, \quad (47)$$

which is possible only if $\epsilon = 1$. So for any $(j_{2t+1}, j_{2t+2}, \dots, j_k) \in \Theta \cap \Omega$,

$$I = h_{2t+1}^{j_{2t+1}} h_{2t+2}^{j_{2t+2}} \dots h_k^{j_k} \quad (48)$$

Therefore, when $\forall i = 1, 2, \dots, 2t$, $y_i = j_i \equiv (-j_i' - j_i'') (mod\ d)$ and $(j_{2t+1}, j_{2t+2}, \dots, j_k) \in \Theta \cap \Omega$, we have

$$\begin{aligned} &\lambda_{\overrightarrow{y}} \Big(\prod_{i=1}^{2t} \omega^{-x_i(j_i'+j_i'')}\Big) F(\overrightarrow{j}, \overrightarrow{j}', \overrightarrow{j}'') \otimes N(\overrightarrow{j}', \overrightarrow{j}'', \overrightarrow{y}) \otimes C(\overrightarrow{j}) \\ &= \lambda_{\overrightarrow{y}} \prod_{i=1}^{2t} \omega^{x_i y_i} h_1^{j_1'} h_2^{j_2'} \dots h_{2t}^{j_{2t}'} h_1^{j_1} h_2^{j_2} \dots h_k^{j_k} h_1^{j_1''} h_2^{j_2''} \dots h_{2t}^{j_{2t}''} \\ &= \lambda_{\overrightarrow{y}} \prod_{i=1}^{2t} \omega^{x_i y_i} h_1^{j_1'+j_1+j_1''} h_2^{j_2'+j_2+j_2''} \dots h_{2t}^{j_{2t}'+j_{2t}+j_{2t}''} h_{2t+1}^{j_{2t+1}} \\ &h_{2t+2}^{j_{2t+2}} \dots h_k^{j_k} \\ &= \lambda_{\overrightarrow{y}} \prod_{i=1}^{2t} \omega^{x_i y_i} I. \end{aligned} \quad (49)$$

where the first equality makes use of Eqs.(29), (35) and (40), the second equality comes from the fact that $h_1, h_2, \dots, h_k$ mutually commute. So the state of the subsystem T_2' is, up to a normalizing factor,

$$\begin{aligned} \widetilde{\rho}^{(T_2')} &= tr_{T_1, T_2'', M}(\widetilde{\rho}) \\ &= \beta \sum_{\overrightarrow{y} \in \mathbb{Z}_d^{2t}} [\lambda_{\overrightarrow{y}} (\prod_{i=1}^{2t} \omega^{x_i y_i}) B(\overrightarrow{y})] \\ &= \beta \sum_{\overrightarrow{y} \in \mathbb{Z}_d^{2t}} [\lambda_{\overrightarrow{y}} (\prod_{i=1}^{2t} \omega^{x_i y_i}) \bigotimes_{i=1}^{t} (Z^{y_{2i-1}} X^{y_{2i}})], \end{aligned} \quad (50)$$

where β is some constant independent of $\lambda_{\overrightarrow{y}}$.

Finally, after step (3), the state of T_2' becomes, up to a normalizing factor,

$$\begin{aligned} &V(\overrightarrow{x}) \widetilde{\rho}^{(T_2')} V(\overrightarrow{x})^\dagger \\ &= \beta \sum_{\overrightarrow{y} \in \mathbb{Z}_d^{2t}} \lambda_{\overrightarrow{y}} (\prod_{i=1}^{2t} \omega^{x_i y_i}) \bigotimes_{i=1}^{t} (Z^{-x_{2i}} X^{x_{2i-1}} Z^{y_{2i-1}} X^{y_{2i}} \\ &X^{-x_{2i-1}} Z^{x_{2i}}) \\ &= \beta \sum_{\overrightarrow{y} \in \mathbb{Z}_d^{2t}} \lambda_{\overrightarrow{y}} \bigotimes_{i=1}^{t} (Z^{y_{2i-1}} X^{y_{2i}}) \\ &= \beta \sigma, \end{aligned} \quad (51)$$

where the first equality comes from Eqs.(33) and (50), and the second equality comes from Eq.(38). So after this protocol, the final state of T_2' is exactly the unknown t-qudit state σ.

■

Remark. It is worth noting that in the above proof the technique used for proving the validity the teleportation protocol is different from those used in most literatures. In most previous work, in order to prove that certain protocols really faithfully teleport an unknown state, authors usually first restricted the unknown state to be a pure state, then wrote both the previously shared entangled state and the unknown state in the vector form, and finally computed the effect of the protocol on the state vectors. The calculations were usually very complicated. In contrast, our approach here is to write the density matrices of the previously shared entangled state and the unknown state as linear combinations of generalized Pauli group elements and then take advantage of their attributes, especially their strong symmetry, to simplify the calculation. It is entirely possible that this technique could be applied to a wider class of states besides stabilizer states.

Although theorem 1 only deals with bipartitions, it becomes the foundation of the following theorem which can deal with general partition plans.

Theorem 2 *Suppose $\{T_1, T_2, \dots, T_{m+1}\}$ is a partition of $[1, n]$. If there exist subgroups $P_1, P_2, \dots, P_{m+1}$ of S such* *that*

$$\begin{array}{l} S^{(T_{m+1})} = \prod_{i=1}^{m+1} P_i^{(T_{m+1})}; \\ P_i^{(T_{m+1})} \cong G_{a_i}^{(d)}, \forall 1 \le i \le m; \\ P_i^{(T_{m+1}^C - T_i)} = \{\gamma^c I\}_{c \in \mathbb{Z}_{2d}}, \forall 1 \le i \le m; \\ P_{m+1}^{(T_{m+1})} \cong \langle \gamma, Z_1^{c_1}, Z_2^{c_2}, \ldots, Z_s^{c_s}, X_1^{d_1}, X_2^{d_2}, \ldots, X_u^{d_u} \rangle, \end{array} \tag{52}$$

for some $a_1, a_2, \ldots, a_m \ge 0$*,* $s \ge u \ge 0$*, and* $c_1, c_2, \ldots, c_s, d_1, d_2, \ldots, d_u \in \mathbb{Z}_d$*, then* $(a_1, a_2, \ldots, a_m)$ *is an achievable teleportation capacity for* ρ_S *with respect to the partition* $\{T_1, T_2, \ldots, T_{m+1}\}$*.*

Proof: Define

$$P = \prod_{i=1}^{m} P_i. \tag{53}$$

Then by Eq.(52) we obtain

$$P^{(T_{m+1})} \cong G_b^{(d)}, \tag{54}$$

where $b = \sum_{i=1}^m a_i$. So P and P_{m+1} satisfy the condition of theorem 1 with respect to the bipartition $\{T_{m+1}^C, T_{m+1}\}$. Consequently, if the subsystem $T_{m+1}^C = \bigcup_{i=1}^m T_i$ belongs to a single party Alice, she can faithfully teleport b unknown qudits to Bob who holds the subsystem T_{m+1}. And they can achieve this by performing the protocol presented in the proof of theorem 1. Actually, we are going to prove that under the given condition Eq.(52), Alice's projective measurement in step (2) in that protocol can be realized by LOCC with respect to the partition $\{T_1, T_2, \ldots, T_m\}$ (at the same time $a_1, a_2, \ldots, a_m$ of the b unknown qudits are also distributed along with $T_1, T_2, \ldots, T_m$ respectively).

Suppose $|T_i| = q_i$, $\forall i = 1, 2, \ldots, m+1$. Also, define $b_1 = 0$, $b_i = \sum_{j=1}^{i-1} a_j$, $\forall i = 2, 3, \ldots, m+1$.

By Eq.(52) we can find independent generators $g_1, g_2, \ldots, g_k$ of S such that $\forall i = 1, 2, \ldots, m$, $\forall j = 1, 2, \ldots, a_i$,

$$\begin{array}{l} g_{2b_i+2j-1} = I^{(T_1)} \otimes \cdots \otimes I^{(T_{i-1})} \otimes R_{2b_i+2j-1} \otimes I^{(T_{i+1})} \\ \otimes \cdots \otimes I^{(T_m)} \otimes \overline{Z}_{b_i+j}, \\ g_{2b_i+2j} = I^{(T_1)} \otimes \cdots \otimes I^{(T_{i-1})} \otimes R_{2b_i+2j} \otimes I^{(T_{i+1})} \\ \otimes \cdots \otimes I^{(T_m)} \otimes \overline{X}_{b_i+j}; \end{array} \tag{55}$$

$\forall i = 1, 2, \ldots, s$, $\forall j = 1, 2, \ldots, u$,

$$\begin{array}{l} g_{2b+i} = W_{2b+i} \otimes \overline{Z}_{b+i}^{c_i}, \\ g_{2b+s+j} = W_{2b+s+j} \otimes \overline{X}_{b+j}^{d_j}; \end{array} \tag{56}$$

$\forall i = 2b+s+u+1, 2b+s+u+2, \ldots, k$,

$$g_i = W_i \otimes I^{(T_{m+1})}, \tag{57}$$

where $I^{(T_i)}$ is the identity operator on the subsystem T_i, $\forall i = 1, 2, \ldots, m+1$; $R_{2b_i+1}, R_{2b_i+2}, \ldots, R_{2b_{(i+1)}}$ are some operators on the subsystem T_i, $\forall i = 1, 2, \ldots, m$; $\overline{Z}_1, \overline{Z}_2, \ldots, \overline{Z}_{b+s}, \overline{X}_1, \overline{X}_2, \ldots, \overline{X}_{b+u} \in G_{q_{m+1}}^{(d)}$ are operators on the subsystem T_{m+1}; $W_{2b+1}, W_{2b+2}, \ldots, W_k$ are some operators on the subsystem T_{m+1}^C.

By lemma 1, we can find a unitary operator U acting on T_{m+1} such that

$$\begin{array}{l} U\overline{Z}_iU^\dagger = Z_i, \quad \forall 1 \le i \le b+s; \\ U\overline{X}_jU^\dagger = X_j, \ \forall 1 \le j \le b+u. \end{array} \tag{58}$$

Define

$$h_i = (I \otimes U) g_i (I \otimes U^\dagger), \tag{59}$$

$\forall i = 1, 2, \ldots, 2b$. Then we have $\forall i = 1, 2, \ldots, m$, $\forall j = 1, 2, \ldots, a_i$,

$$\begin{array}{l} h_{2b_i+2j-1} = I^{(T_1)} \otimes \cdots \otimes I^{(T_{i-1})} \otimes R_{2b_i+2j-1} \otimes I^{(T_{i+1})} \\ \otimes \cdots \otimes I^{(T_m)} \otimes Z_{b_i+j}, \\ h_{2b_i+2j} = I^{(T_1)} \otimes \cdots \otimes I^{(T_{i-1})} \otimes R_{2b_i+2j} \otimes I^{(T_{i+1})} \\ \otimes \cdots \otimes I^{(T_m)} \otimes X_{b_i+j}. \end{array} \tag{60}$$

Since $g_1, g_2, \ldots, g_{2b}$ are commuting operators, by the definition of $h_1, h_2, \ldots, h_{2b}$, we know they are also commuting operators.

Suppose

$$T_{m+1} = \{i_1, i_2, \ldots, i_{q_{m+1}}\} \tag{61}$$

with $i_1 < i_2 < \cdots < i_{q_{m+1}}$. One can see Eq.(60) implies $b \le q_{m+1}$. So for $i = 1, 2, \ldots, m$, define

$$\begin{array}{l} T_i' = \{i_{b_i+1}, i_{b_i+2}, \ldots, i_{b_{(i+1)}}\}, \\ Q_i = T_i \cup T_i'. \end{array} \tag{62}$$

Then let

$$\begin{array}{l} T'' = \bigcup_{i=1}^m T_i' = \{i_1, i_2, \ldots, i_b\}, \\ T = \bigcup_{i=1}^m Q_i = (\bigcup_{i=1}^m T_i) \cup T''. \end{array} \tag{63}$$

Now for $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, a_i$, define

$$\begin{array}{ll} h'_{2b_i+2j-1} & = I^{(T_1)} \otimes \cdots \otimes I^{(T_{i-1})} \otimes R_{2b_i+2j-1} \otimes I^{(T_{i+1})} \\ & \otimes \cdots \otimes I^{(T_m)} \otimes h_{2b_i+2j-1}^{(T'')} \\ & = I^{(T_1)} \otimes \cdots \otimes I^{(T_{i-1})} \otimes R_{2b_i+2j-1} \otimes I^{(T_{i+1})} \\ & \otimes \cdots \otimes I^{(T_m)} \otimes Z_{b_i+j}, \\ h'_{2b_i+2j} & = I^{(T_1)} \otimes \cdots \otimes I^{(T_{i-1})} \otimes R_{2b_i+2j} \otimes I^{(T_{i+1})} \\ & \otimes \cdots \otimes I^{(T_m)} \otimes h_{2b_i+2j}^{(T'')} \\ & = I^{(T_1)} \otimes \cdots \otimes I^{(T_{i-1})} \otimes R_{2b_i+2j} \otimes I^{(T_{i+1})} \\ & \otimes \cdots \otimes I^{(T_m)} \otimes X_{b_i+j}, \end{array} \tag{64}$$

Then $h'_1, h'_2, \ldots, h'_{2b}$ are commuting operators on the subsystem T. Moreover, $\forall i = 1, 2, \ldots, m$, $h'_{2b_i+1}, h'_{2b_i+2}, \ldots, h'_{2b_{(i+1)}}$ only act nontrivially on the subsystem Q_i, i.e.

$$h_{2b_i+1}'^{(Q_i^C)} = h_{2b_i+2}'^{(Q_i^C)} = \cdots = h_{2b_{(i+1)}}'^{(Q_i^C)} = I. \tag{65}$$

Now for $i=1,2,\ldots,m$, $j=1,2,\ldots,a_i$, define

$$\begin{aligned} h''_{2b_i+2j-1} &= R_{2b_i+2j-1}\otimes h^{(T'_i)}_{2b_i+j-1} \\ &= R_{2b_i+2j-1}\otimes Z_j, \\ h''_{2b_i+2j} &= R_{2b_i+2j}\otimes h^{(T'_i)}_{2b_i+j} \\ &= R_{2b_i+2j}\otimes X_j. \end{aligned} \tag{66}$$

Then $h''_{2b_i+1}, h''_{2b_i+2}, \ldots, h''_{2b_{(i+1)}}$ are commuting operators on the subsystem Q_i. Furthermore, $\forall \overrightarrow{x}=(x_1,x_2,\ldots,x_{2b})\in\mathbb{Z}_d^{2b}$, $\forall \overrightarrow{j}=(j_1,j_2,\ldots,j_{2b})\in\mathbb{Z}_d^{2b}$,

$$\prod_{l=1}^{2b}(\omega^{-j_l x_l}h'^{j_l}_l)=\bigotimes_{i=1}^{m}[\prod_{l=2b_i+1}^{2b_{(i+1)}}(\omega^{-x_l}h''_l)^{j_l}]. \tag{67}$$

Consequently,

$$\begin{aligned} &P(h'_1,h'_2,\ldots,h'_{2b};\overrightarrow{x}) \\ &=\tfrac{1}{d^{2b}}\prod_{l=1}^{2b}(\sum_{j=0}^{d-1}\omega^{-jx_l}h'^j_l) \\ &=\tfrac{1}{d^{2b}}\sum_{j_1,\ldots,j_{2b}=0}^{d-1}\prod_{l=1}^{2b}(\omega^{-j_lx_l}h'^{j_l}_l) \\ &=\tfrac{1}{d^{2b}}\sum_{j_1,\ldots,j_{2b}=0}^{d-1}\bigotimes_{i=1}^{m}[\prod_{l=2b_i+1}^{2b_{(i+1)}}(\omega^{-x_l}h''_l)^{j_l}] \\ &=\tfrac{1}{d^{2b}}\bigotimes_{i=1}^{m}[\sum_{j_{2b_i+1},\ldots,j_{2b_{(i+1)}}=0}^{d-1}\prod_{l=2b_i+1}^{2b_{(i+1)}}(\omega^{-x_l}h''_l)^{j_l}] \\ &=\bigotimes_{i=1}^{m}[\tfrac{1}{d^{2a_i}}\prod_{l=2b_i+1}^{2b_{(i+1)}}\sum_{j=0}^{d-1}(\omega^{-x_l}h''_l)^{j}] \\ &=\bigotimes_{i=1}^{m}P(h''_{2b_i+1},h''_{2b_i+2},\ldots,h''_{2b_{(i+1)}};\overrightarrow{x}|_{[2b_i+1,2b_{(i+1)}]}), \end{aligned} \tag{68}$$

where $\overrightarrow{x}|_{[2b_i+1,2b_{(i+1)}]}=(x_{2b_i+1},x_{2b_i+2},\ldots,x_{2b_{(i+1)}})$. The first equality comes from Eq.(10), the third equality comes from Eq.(67), the fifth equality makes use of $b=\sum_{i=1}^m a_i$, and the last equality also comes from Eq.(10). So $P(h'_1,h'_2,\ldots,h'_{2b};\overrightarrow{x})$ is simply the tensor product of the projection operators $P(h''_{2b_i+1},h''_{2b_i+2},\ldots,h''_{2b_{(i+1)}};\overrightarrow{x}|_{[2b_i+1,2b_{(i+1)}]})$ on each subsystem Q_i.

Therefore, by making a little modification to the protocol in the proof of theorem 1, we get the protocol for our teleportation with respect to the partition $\{T_1,T_2,\ldots,T_{m+1}\}$ as follows:

(1)A_{m+1} performs the unitary operation U on the subsystem T_{m+1}.

(2)Suppose A_i has an unknown a_i-qudit state σ_i, $\forall i=1,2,\ldots,m$. A_i performs the projective measurement consisting of the projection operators $\{P(h''_{2b_i+1},h''_{2b_i+2},\ldots,h''_{2b_{(i+1)}};\overrightarrow{x})\}_{\overrightarrow{x}\in\mathbb{Z}_d^{2a_i}}$ on his T_i subsystem of ρ_S and σ_i, and then tells the measurement outcome $\overrightarrow{x}=(x_{2b_i+1},x_{2b_i+2},\ldots,x_{2b_{(i+1)}})$ to A_{m+1}, $\forall i=1,2,\ldots,m$.

(3)A_{m+1} performs the unitary operation

$$V(\overrightarrow{x})=\bigotimes_{i=1}^{b}(Z^{-x_{2i}}X^{x_{2i-1}}) \tag{69}$$

on the subsystem $\bigcup_{i=1}^m T'_i$.

Then by the proof of the theorem 1, we know that after this protocol, the final states of $T'_1,T'_2,\ldots,T'_m$ become $\sigma_1,\sigma_2,\ldots,\sigma_m$ respectively.

■

Remark 1. One can easily see that in the two protocols presented in the proofs of theorem 1 and 2, the receiver can actually perform the unitary operation U after receiving the senders' measurement outcomes, i.e. the order of step (1) and (2) can be altered.

Remark 2. One can see that our two theorems above do not require the state ρ_S to be a pure stabilizer state. When S is an incomplete stabilizer, ρ_S is a mixed state. In the subsequent section, we will also give concrete examples of mixed stabilizer states which are useful for perfect teleportation, even with respect to several different partition plans. Our argument mainly depends on the structure of the restrictions of S on each subsystem T_i. The purity of ρ_S is not an essential property that can greatly influence its teleportation capability.

IV. ILLUSTRATIONS

In this section we will analyze several states by using our theorems. In each example, the matrices X and Z are $X_{(d)}$ and $Z_{(d)}$ defined by Eq.(1) with the corresponding dimension d. We also use the notation X_j denotes the operation X acting on the jth qudit and similarly for Z_j.

We will consider three examples. The first example is re-examination of the standard teleportation protocol from our perspective. The second and third examples are detailed illustrations of how to find the achievable teleportation capacity and construct the corresponding protocol by utilizing our two theorems. The third example also proves the existence of mixed stabilizer states which are useful for perfect teleportation.

Example 1 *Let us begin with the standard teleportation protocol. Let*

$$|\Phi^+\rangle=\frac{1}{\sqrt{d}}\sum_{i=0}^{d-1}|ii\rangle \tag{70}$$

be the maximally entangled state in the $d\times d$ system. It is a stabilizer state and its stabilizer is $S=\langle g_1,g_2\rangle$, where

$$\begin{aligned} g_1&=Z_1^{-1}Z_2, \\ g_2&=X_1X_2. \end{aligned} \tag{71}$$

Consider the partition $\{\{1\},\{2\}\}$. We have $g_1^{(\{2\})}=Z$, $g_2^{(\{2\})}=X$ and consequently $S^{(\{2\})}\cong G_1^{(d)}$. So by theorem 1, if Alice and Bob hold the first and second qudits of $|\Phi^+\rangle$ respectively, then Alice can faithfully teleport an unknown qudit state to Bob. Moreover, in this special case, the protocol presented in the proof of theorem 1 becomes: Alice first performs the projective measurement in the basis of the simultaneous eigenstates of g_1,g_2 on her subsystem of $|\Phi^+\rangle$ and the unknown qudit; if her measurement *outcome corresponds to the eigenvalues ω^a, ω^b of g_1, g_2 for some $a, b \in \mathbb{Z}_d$, then Bob performs the unitary operation $Z^{-b}X^a$ on his qudit. One can easily see that this protocol is exactly the standard teleportation protocol.*

Example 2 *Consider a five-qutrit system, i.e. $d = 3$, $n = 5$. Define*

$$\begin{aligned} g_1 &= X_1X_2^2X_3Z_4Z_5, \\ g_2 &= Z_1^2Z_2I_3X_4I_5, \\ g_3 &= Z_1Z_2Z_3I_4X_5, \\ g_4 &= X_1X_2Z_3X_4Z_5^2, \\ g_5 &= I_1I_2Z_3^2X_4^2I_5. \end{aligned} \tag{72}$$

They are five independent commuting operators in $G_5'^{(3)}$. Then

$$S = \langle g_1, g_2, g_3, g_4, g_5 \rangle \tag{73}$$

is a complete stabilizer. Suppose $|\psi_S\rangle$ is the pure state stabilized by S. Then

$$\rho_S = |\psi_S\rangle\langle\psi_S| = \frac{1}{3^5}\prod_{i=1}^{5}(\sum_{j=0}^{2} g_i^j). \tag{74}$$

Consider the partition $\{T_1 = \{1, 2\}, T_2 = \{3, 4, 5\}\}$. We have

$$\begin{aligned} g_1^{(T_2)} &= X \otimes Z \otimes Z, \\ g_2^{(T_2)} &= I \otimes X \otimes I, \\ g_3^{(T_2)} &= Z \otimes I \otimes X, \\ g_4^{(T_2)} &= Z \otimes X \otimes Z^2, \\ g_5^{(T_2)} &= Z^2 \otimes X^2 \otimes I. \end{aligned} \tag{75}$$

One can check that we can write $g_1^{(T_2)} = \overline{Z}_1$, $g_2^{(T_2)} = \overline{X}_1$, $g_3^{(T_2)} = \overline{Z}_2$, $g_4^{(T_2)} = \overline{X}_2$, $g_5^{(T_2)} = \overline{Z}_3$. Let

$$\begin{aligned} P_1 &= \langle g_1, g_2, g_3, g_4 \rangle, \\ P_2 &= \langle g_5 \rangle. \end{aligned} \tag{76}$$

Then

$$\begin{aligned} S^{(T_2)} &= P_1^{(T_2)}P_2^{(T_2)}, \\ P_1^{(T_2)} &\cong G_2^{(3)}, \\ P_2^{(T_2)} &\cong \langle \gamma, Z \rangle, \end{aligned} \tag{77}$$

where $\gamma = e^{i\frac{\pi}{3}}$. So by theorem 1, if Alice and Bob hold the T_1 and T_2 subsystems of ρ_S respectively, then Alice can faithfully teleport an unknown two-qutrit state σ to Bob. We now show how to construct the corresponding teleportation protocol. By lemma 1 and its proof, we can find a unitary operation U acting on T_2 such that $Ug_1^{(T_2)}U^\dagger = Z_1$, $Ug_2^{(T_2)}U^\dagger = X_1$, $Ug_3^{(T_2)}U^\dagger = Z_2$, $Ug_4^{(T_2)}U^\dagger = X_2$, $Ug_5^{(T_2)}U^\dagger = Z_3$. Define

$$\begin{aligned} h_1 &= (I \otimes U)g_1(I \otimes U)^\dagger = X \otimes X^2 \otimes Z \otimes I \otimes I, \\ h_2 &= (I \otimes U)g_2(I \otimes U)^\dagger = Z^2 \otimes Z \otimes X \otimes I \otimes I, \\ h_3 &= (I \otimes U)g_3(I \otimes U)^\dagger = Z \otimes Z \otimes I \otimes Z \otimes I, \\ h_4 &= (I \otimes U)g_4(I \otimes U)^\dagger = X \otimes X \otimes I \otimes X \otimes I. \end{aligned} \tag{78}$$

Let $T_2' = \{3, 4\}$. Then define

$$\begin{aligned} h_1' &= X \otimes X^2 \otimes h_1^{(T_2')} \\ &= X \otimes X^2 \otimes Z \otimes I, \\ h_2' &= Z^2 \otimes Z \otimes h_2^{(T_2')} \\ &= Z^2 \otimes Z \otimes X \otimes I, \\ h_3' &= Z \otimes Z \otimes h_3^{(T_2')} \\ &= Z \otimes Z \otimes I \otimes Z, \\ h_4' &= X \otimes X \otimes h_4^{(T_2')} \\ &= X \otimes X \otimes I \otimes X. \end{aligned} \tag{79}$$

The protocol is as follows: (1)Bob performs the unitary operation U on the T_2 subsystem of ρ_S. (2)Alice performs the projective measurement in the basis of the simultaneous eigenstates of h_1', h_2', h_3', h_4' on her T_1 subsystem of ρ_S and σ. Suppose the measurement outcome corresponds to the eigenvalues $\omega^{x_1}, \omega^{x_2}, \omega^{x_3}, \omega^{x_4}$ of h_1', h_2', h_3', h_4', where $\omega = e^{i\frac{2\pi}{3}}$. She tells x_1, x_2, x_3, x_4 to Bob. (3)Bob performs the unitary operation $V = Z^{-x_2}X^{x_1} \otimes Z^{-x_4}X^{x_3}$ on the T_2' subsystem. Then after this procedure, Bob's T_2' subsystem becomes the state σ.

Example 3 *Consider an eight-qubit system. Define*

$$\begin{aligned} g_1 &= X_1Y_2I_3I_4I_5Z_6Y_7I_8, \\ g_2 &= X_1Z_2I_3I_4I_5X_6Y_7I_8, \\ g_3 &= I_1I_2Z_3Y_4Z_5I_6Y_7X_8, \\ g_4 &= I_1I_2Z_3I_4X_5I_6Y_7Z_8, \\ g_5 &= I_1I_2Z_3Z_4X_5Y_6X_7Y_8, \\ g_6 &= I_1I_2X_3X_4Z_5Y_6Z_7Y_8, \\ g_7 &= Z_1X_2I_3Z_4X_5I_6I_7I_8. \end{aligned} \tag{80}$$

They are seven commuting operators in $G_8'^{(2)}$. Let

$$S = \langle g_1, g_2, \ldots, g_7 \rangle. \tag{81}$$

Then the maximally mixed state over the subspace stabilized by S is

$$\rho_S = \frac{1}{2^8}\prod_{i=1}^{7}(I + g_i). \tag{82}$$

Consider the partition $\{T_1 = \{1, 2\}, T_2 = \{3, 4, 5\}, T_3 = \{6, 7, 8\}\}$. We have

$$\begin{aligned} g_1^{(T_3)} &= Z \otimes Y \otimes I, \\ g_2^{(T_3)} &= X \otimes Y \otimes I, \\ g_3^{(T_3)} &= I \otimes Y \otimes X, \\ g_4^{(T_3)} &= I \otimes Y \otimes Z, \\ g_5^{(T_3)} &= Y \otimes X \otimes Y, \\ g_6^{(T_3)} &= Y \otimes Z \otimes Y. \\ g_7^{(T_3)} &= I \otimes I \otimes I, \\ g_1^{(T_2)} &= g_2^{(T_2)} = I \otimes I \otimes I, \\ g_3^{(T_1)} &= g_4^{(T_1)} = g_5^{(T_1)} = g_6^{(T_1)} = I \otimes I. \end{aligned} \tag{83}$$

One can check that we can write $g_1^{(T_3)} = \overline{Z}_1$, $g_2^{(T_3)} = \overline{X}_1$, $g_3^{(T_3)} = \overline{Z}_2$, $g_4^{(T_3)} = \overline{X}_2$, $g_5^{(T_3)} = \overline{Z}_3$, $g_6^{(T_3)} = \overline{X}_3$. *Let*

$$\begin{array}{l} P_1 = \langle g_1, g_2\rangle, \\ P_2 = \langle g_3, g_4, g_5, g_6\rangle, \\ P_3 = \langle g_7\rangle. \end{array} \tag{84}$$

Then

$$\begin{array}{l} S^{(T_3)} = \prod_{i=1}^{3} P_i^{(T_3)}, \\ P_1^{(T_3)} \cong G_1^{(2)}, \\ P_2^{(T_3)} \cong G_2^{(2)}, \\ P_1^{(T_2)} = \{i^c I\otimes I\otimes I\}_{c\in\mathbb{Z}_4}, \\ P_2^{(T_1)} = \{i^c I\otimes I\}_{c\in\mathbb{Z}_4}, \\ P_3^{(T_3)} = \{i^c I\otimes I\otimes I\}_{c\in\mathbb{Z}_4}. \end{array} \tag{85}$$

So by theorem 2, $(1,2)$ *is an achievable teleportation capacity for* ρ_S *with respect to the partition* $\{\{1,2\},\{3,4,5\},\{6,7,8\}\}$. *In other words, supposing Alice, Bob and Charlie hold the subsystems* $\{1,2\}$, $\{3,4,5\}$ *and* $\{6,7,8\}$ *of* ρ_S *respectively, if Alice has an unknown qubit state* σ_1 *and Bob has an unknown two-qubit state* σ_2, *then they can simultaneously faithfully teleport* σ_1 *and* σ_2 *to Charlie. We now show how to construct the corresponding teleportation protocol. By lemma 1 and its proof, we can find a unitary operation* U *acting on* T_3 *such that* $Ug_1^{(T_3)}U^\dagger = Z_1$, $Ug_2^{(T_3)}U^\dagger = X_1$, $Ug_3^{(T_3)}U^\dagger = Z_2$, $Ug_4^{(T_3)}U^\dagger = X_2$, $Ug_5^{(T_3)}U^\dagger = Z_3$, $Ug_6^{(T_3)}U^\dagger = X_3$. *Define*

$$\begin{array}{l} h_1 = (I\otimes U)g_1(I\otimes U)^\dagger = X_1Y_2I_3I_4I_5Z_6I_7I_8, \\ h_2 = (I\otimes U)g_2(I\otimes U)^\dagger = X_1Z_2I_3I_4I_5X_6I_7I_8, \\ h_3 = (I\otimes U)g_3(I\otimes U)^\dagger = I_1I_2Z_3Y_4Z_5I_6Z_7I_8, \\ h_4 = (I\otimes U)g_4(I\otimes U)^\dagger = I_1I_2Z_3I_4X_5I_6X_7I_8, \\ h_5 = (I\otimes U)g_5(I\otimes U)^\dagger = I_1I_2Z_3Z_4X_5I_6I_7Z_8, \\ h_6 = (I\otimes U)g_6(I\otimes U)^\dagger = I_1I_2X_3X_4Z_5I_6I_7X_8. \end{array} \tag{86}$$

Let $T_1' = \{6\}, T_2' = \{7,8\}$. *Then define*

$$\begin{array}{ll} h_1'' & = X\otimes Y\otimes h_1^{(T_1')} \\ & = X\otimes Y\otimes Z, \\ h_2'' & = X\otimes Z\otimes h_2^{(T_1')} \\ & = X\otimes Z\otimes X, \\ h_3'' & = Z\otimes Y\otimes Z\otimes h_3^{(T_2')} \\ & = Z\otimes Y\otimes Z\otimes Z\otimes I, \\ h_4'' & = Z\otimes I\otimes X\otimes h_4^{(T_2')} \\ & = Z\otimes I\otimes X\otimes X\otimes I, \\ h_5'' & = Z\otimes Z\otimes X\otimes h_5^{(T_2')} \\ & = Z\otimes Z\otimes X\otimes I\otimes Z, \\ h_6'' & = X\otimes X\otimes Z\otimes h_6^{(T_2')} \\ & = X\otimes X\otimes Z\otimes I\otimes X. \end{array} \tag{87}$$

The protocol is as follows: (1)Charlie performs the unitary operation U *on the subsystem* T_3 *of* ρ_S. *(2.1)Alice performs the projective measurement consisting of the projection operators* $\{P(h_1'', h_2''; \overrightarrow{x}) : \overrightarrow{x} \in \mathbb{Z}_2^2\}$ *on her* T_1 *subsystem of* ρ_S *and* σ_1, *and then tells the measurement outcome* $\overrightarrow{x} = (x_1, x_2)$ *to Charlie; (2.2)Bob performs the projective measurement consisting of the projection operators* $\{P(h_3'', h_4'', h_5'', h_6''; \overrightarrow{x}) : \overrightarrow{x} \in \mathbb{Z}_2^4\}$ *on his* T_2 *subsystem of* ρ_S *and* σ_2, *and then tells the measurement outcome* $\overrightarrow{x} = (x_3, x_4, x_5, x_6)$ *to Charlie; (3)Charlie performs the unitary operation* $V = Z^{-x_2}X^{x_1}\otimes Z^{-x_4}X^{x_3}\otimes Z^{-x_6}X^{x_5}$ *on the subsystem* $T_1'\cup T_2'$. *After this procedure, Charlie's* T_1' *and* T_2' *subsystems become the states* σ_1 *and* σ_2 *respectively.*

Now consider another partition $\{T_1 = \{1,6\}, T_2 = \{3,8\}, T_3 = \{2,4,5,7\}\}$. *Define*

$$\begin{array}{l} g_5' = g_1g_2g_3g_4g_5 = -I_1X_2Z_3X_4Z_5I_6X_7I_8, \\ g_6' = g_1g_2g_6 = -I_1X_2X_3X_4Z_5I_6Z_7Y_8, \\ g_7' = g_1g_2g_7 = -Z_1I_2I_3Z_4X_5Y_6I_7I_8. \end{array} \tag{88}$$

Then

$$\begin{array}{ll} S & = \langle g_1, g_2, g_3, g_4, g_5, g_6, g_7\rangle \\ & = \langle g_1, g_2, g_3, g_4, g_5', g_6', g_7'\rangle. \end{array} \tag{89}$$

Moreover, we have

$$\begin{array}{l} g_1^{(T_3)} = Y\otimes I\otimes I\otimes Y, \\ g_2^{(T_3)} = Z\otimes I\otimes I\otimes Y, \\ g_3^{(T_3)} = I\otimes Y\otimes Z\otimes Y, \\ g_4^{(T_3)} = I\otimes I\otimes X\otimes Y, \\ g_5'^{(T_3)} = X\otimes X\otimes Z\otimes X, \\ g_6'^{(T_3)} = X\otimes X\otimes Z\otimes Z, \\ g_7'^{(T_3)} = I\otimes Z\otimes X\otimes I. \\ g_1^{(T_2)} = g_2^{(T_2)} = I\otimes I, \\ g_3^{(T_1)} = g_4^{(T_1)} = I\otimes I. \end{array} \tag{90}$$

One can check that we can write $g_1^{(T_3)} = \overline{Z}_1$, $g_2^{(T_3)} = \overline{X}_1$, $g_3^{(T_3)} = \overline{Z}_2$, $g_4^{(T_3)} = \overline{X}_2$, $g_5'^{(T_3)} = \overline{Z}_3$, $g_6'^{(T_3)} = \overline{X}_3$, $g_7'^{(T_3)} = \overline{Z}_4$. *Let*

$$\begin{array}{l} P_1 = \langle g_1, g_2\rangle, \\ P_2 = \langle g_3, g_4\rangle, \\ P_3 = \langle g_5', g_6', g_7'\rangle. \end{array} \tag{91}$$

Then

$$\begin{array}{l} S^{(T_3)} = \prod_{i=1}^{3} P_i^{(T_3)}, \\ P_1^{(T_3)} \cong P_2^{(T_3)} \cong G_1^{(2)}, \\ P_1^{(T_2)} = P_2^{(T_1)} = \{i^c I\otimes I\}_{c\in\mathbb{Z}_4}, \\ P_3^{(T_3)} \cong \langle i, Z_1, X_1, Z_2\rangle. \end{array} \tag{92}$$

Therefore, by theorem 2, $(1,1)$ *is an achievable teleportation capacity for* ρ_S *with respect to the partition* $\{\{1,6\},\{3,8\},\{2,4,5,7\}\}$. *In other words, supposing Alice, Bob and Charlie hold the subsystems* $\{1,6\}$, $\{3,8\}$ *and* $\{2,4,5,7\}$ *of* ρ_S *respectively, if Alice has an unknown qubit state* σ_1 *and Bob has an unknown qubit state* σ_2, *then they can simultaneously faithfully teleport σ_1 and σ_2 to Charlie. The reader can build the corresponding protocol through an analysis similar to the one above.*

Note that $S = \langle g_1, g_2, \ldots, g_7 \rangle$ is an incomplete stabilizer. So ρ_S is a mixed stabilizer state. But it is still useful for perfect teleportation with respect to at least two different partition plans.

V. CONCLUSION

In sum, we have studied the possibility of performing perfect many-to-one teleportation with a previously shared stabilizer state. We present two sufficient conditions for a stabilizer state to achieve a given nonzero teleportation capacity with respect to a given partition plan. The corresponding protocols are also explicitly constructed. Mixed stabilizer states are also found to be useful for perfect many-to-one teleportation. Our work provides a new perspective from the stabilizer formalism to view the standard teleportation protocol and also suggests a new technique to analyze the teleportation capability of multipartite entangled states.

We would like to point out several directions for future investigations. Firstly, we do not know whether the conditions of the two theorems are also necessary for ρ_S to achieve a nonzero capacity with respect to a given partition. We believe that it is the structure of the restrictions of the stabilizer S on each subsystem that determines teleportation capability of ρ_S. But it seems not easy to reach a thorough understanding. Secondly, we expect our techniques in the proof of theorem 1 to be extended to a wider class of entangled states besides stabilizer states. We think that as long as the considered state exhibits strong symmetry, our techniques can be readily applied. We hope our work can stimulate further research on the usefulness of a general multipartite entangled state for faithful teleportation.

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